

Causal Sets and Frame-Valued Set Theory

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In spacetime physics any set \mathcal{C} of events—a *causal set*—is taken to be partially ordered by the relation \leq of *possible causation*: for $p, q \in \mathcal{C}$, $p \leq q$ means that q is in p 's future light cone. In her groundbreaking paper *The internal description of a causal set: What the universe looks like from the inside*, Fotini Markopoulou proposes that the causal structure of spacetime itself be represented by “sets evolving over \mathcal{C} ” —that is, in essence, by the topos $\mathit{Set}^{\mathcal{C}}$ of presheaves on \mathcal{C}^{op} . To enable what she has done to be the more easily expressed within the framework presented here, I will reverse the causal ordering, that is, \mathcal{C} will be replaced by \mathcal{C}^{op} , and the latter written as P —which will, moreover, be required to be no more than a *preordered* set. Specifically, then: P is a set of events preordered by the relation \leq , where $p \leq q$ is intended to mean that p is in q 's future light cone—that q *could* be the cause of p , or, equally, that p *could* be an effect of q . In that case, for each event p , the set $p \downarrow = \{q: q \leq p\}$ may be identified as the *causal future* of p , or the set of *potential effects* of p . In requiring that \leq be no more than a preordering—in dropping, that is, the antisymmetry of \leq —I am, in physical terms, allowing for the possibility that the universe is of Gödelian type, containing closed timelike lines.

Accordingly I fix a preordered set (P, \leq) , which I shall call the *universal causal set*. Markopoulou, in essence, suggests that viewing the universe “from the inside” amounts to placing oneself within the topos of presheaves $\mathit{Set}^{P^{\text{op}}}$. Here I am going to show how $\mathit{Set}^{P^{\text{op}}}$ may be effectively replaced by a certain model of intuitionistic set theory, with (I hope) illuminating results.

Let us suppose that we are given a relation \Vdash between events p and assertions φ : think of $p \Vdash \varphi$ as meaning that φ *holds* as a result of event p . Assume that the relation \Vdash is *persistent* in the sense that, if $p \Vdash \varphi$ and $q \leq p$, then $q \Vdash \varphi$: once an assertion holds, it continues to hold in the future. (The basic assertions we have in mind are of the form: “such and such is (or was) the case at such-and such a time (event)”.)

Given an assertion φ , the set $\llbracket \varphi \rrbracket = \{p: p \Vdash \varphi\}$ “measures” the degree or extent to which φ holds: the larger $\llbracket \varphi \rrbracket$ is, the “truer” φ is. In particular, when $\llbracket \varphi \rrbracket = P$, φ is “universally” or “absolutely” true, and when $\llbracket \varphi \rrbracket = \emptyset$, φ is “universally” or “absolutely” false. These $\llbracket \varphi \rrbracket$ may accordingly be thought of as “truth values”, with P corresponding to “absolute truth” and \emptyset to absolute falsity.

Because of the persistence property, each $\llbracket \varphi \rrbracket$ has the property of being “closed under potential effects”, or “causally closed”, that is, satisfies $p \in \llbracket \varphi \rrbracket$ and $q \leq p \rightarrow q \in \llbracket \varphi \rrbracket$. A subset of P with this property is called a *sieve*. Sieves serve as generalized “truth values” measuring the degree to which assertions hold. The set \widehat{P} of all sieves, or truth values has a natural logico-algebraic structure—that of a *complete Heyting algebra*, or *frame*. This concept is defined in the following way.

A *lattice* is a partially ordered set L with partial ordering \leq in which each two-element subset $\{x, y\}$ has a supremum or *join*—denoted by $x \vee y$ —and an infimum or *meet*—denoted by $x \wedge y$. A lattice L is *complete* if every subset X (including \emptyset) has a supremum or *join*—denoted by $\bigvee X$ —and an infimum or *meet*—denoted by $\bigwedge X$. Note that $\bigvee \emptyset = 0$, the least or *bottom* element of L , and $\bigwedge \emptyset = 1$, the largest or *top* element of L .

A *Heyting algebra* is a lattice L with top and bottom elements such that, for any elements $x, y \in L$, there is an element—denoted by $x \Rightarrow y$ —of L such that, for any $z \in L$,

$$z \leq x \Rightarrow y \text{ iff } z \wedge x \leq y.$$

Thus $x \Rightarrow y$ is the *largest* element z such that $z \wedge x \leq y$. So in particular, if we write $\neg x$ for $x \Rightarrow 0$, then $\neg x$ is the largest element z such that $x \Rightarrow z = 0$: it is called the *pseudocomplement* of x . A *Boolean algebra* is a Heyting algebra in which $\neg\neg x = x$ for all x , or equivalently, in which $x \vee \neg x = 1$ for all x .

If we think of the elements of a (complete) Heyting algebra as “truth values”, then $0, 1, \wedge, \vee, \neg, \Rightarrow, \bigvee, \bigwedge$ represent “true”, “false”, “and”, “or”, “not” and “implies”, “there exists” and “for all”, respectively. The laws satisfied by these operations in a general Heyting algebra correspond to those of *intuitionistic logic*. In Boolean algebras the counterpart of the law of excluded middle also holds.

A basic fact about *complete* Heyting algebras is that the following identity holds in them:

$$(*) \quad x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i)$$

And conversely, in any complete lattice satisfying (*), defining the operation \Rightarrow by $x \Rightarrow y = \bigvee\{z: z \wedge x \leq y\}$ turns it into a Heyting algebra.

In view of this result a complete Heyting algebra is frequently defined to be a complete lattice satisfying (*). A complete Heyting algebra is briefly called a *frame*.

In the frame $\widehat{P} \leq$ is \subseteq , joins and meets are just set-theoretic unions and intersections, and the operations \Rightarrow and \neg are given by

$$I \Rightarrow J = \{p : I \cap p \downarrow \subseteq J\} \quad \neg I = \{p : I \cap p \downarrow = \emptyset\}.$$

Frames do duty as the “truth-value algebras” of the (current) *language of mathematics*, that is, *set theory*. To be precise, associated with each frame H is a structure V^H —the *universe of H -valued sets*—with the following features.

- Each of the members of V^H —the *H -sets*—is a map from a subset of V^H to H .
- Corresponding to each sentence σ of the language of set theory (with names for all elements of V^H) is an element $\llbracket \sigma \rrbracket = \llbracket \sigma \rrbracket^H \in H$ called its *truth value in V^H* . These “truth values” satisfy the following conditions. For $a, b \in V^H$,

$$\begin{aligned} \llbracket b \in a \rrbracket &= \bigvee_{c \in \text{dom}(a)} \llbracket b = c \rrbracket \wedge a(c) & \llbracket b = a \rrbracket &= \bigvee_{c \in \text{dom}(a) \cup \text{dom}(b)} (\llbracket c \in b \rrbracket \leftrightarrow \llbracket c \in a \rrbracket) \\ \llbracket \sigma \wedge \tau \rrbracket &= \llbracket \sigma \rrbracket \wedge \llbracket \tau \rrbracket, \text{ etc.} \\ \llbracket \exists x \varphi(x) \rrbracket &= \bigvee_{a \in V^H} \llbracket \varphi(a) \rrbracket \\ \llbracket \forall x \varphi(x) \rrbracket &= \bigwedge_{a \in V^H} \llbracket \varphi(a) \rrbracket \end{aligned}$$

A sentence σ is *valid*, or *holds*, in V^H , written $V^H \models \sigma$, if $\llbracket \sigma \rrbracket = 1$, the top element of H .

- The axioms of intuitionistic Zermelo-Fraenkel set theory are valid in V^H . In this sense V^H is an *H -valued model* of *IZF*. Accordingly the category \mathcal{Set}^H of sets constructed within V^H is a topos: in fact \mathcal{Set}^H can be shown to be equivalent to the topos of canonical sheaves on H .
- There is a canonical embedding $x \mapsto \hat{x}$ of the usual universe V of sets into V^H satisfying

$$\llbracket u \in \hat{x} \rrbracket = \bigvee_{y \in x} \llbracket u = \hat{y} \rrbracket \text{ for } x \in V, u \in V^{(H)}$$

$$x \in y \leftrightarrow V^{(H)} \models \hat{x} \in \hat{y}, \quad x = y \leftrightarrow V^{(H)} \models \hat{x} = \hat{y} \text{ for } x, y \in V$$

$$\varphi(x_1, \dots, x_n) \leftrightarrow V^{(H)} \models \varphi(\hat{x}_1, \dots, \hat{x}_n) \text{ for } x_1, \dots, x_n \in V \text{ and restricted } \varphi$$

(Here a formula φ is *restricted* if all its quantifiers are restricted, i.e. can be put in the form $\forall x \in y$ or $\exists x \in y$.)

We observe that $V^{(2)}$ is essentially just the usual universe of sets.

It follows from the last of these assertions that the canonical representative \hat{H} of H is a Heyting algebra in $V^{(H)}$. A particularly important H -set is the H -set Φ_H defined by

$$\text{dom}(\Phi_H) = \{\hat{a} : a \in H\}, \quad \Phi_H(\hat{a}) = a \text{ for } a \in H.$$

Then $V^{(H)} \models \Phi_H \subseteq \hat{H}$. Also, for any $a \in H$ we have $\llbracket \hat{a} \in \Phi_H \rrbracket = a$, and in particular, for any sentence σ , $\llbracket \sigma \rrbracket = \llbracket \llbracket \sigma \rrbracket \in \Phi_H \rrbracket$. Thus

$$V^{(H)} \models \sigma \leftrightarrow V^{(H)} \models \llbracket \sigma \rrbracket \in \Phi_H;$$

in this sense Φ_H represents the “true” sentences in $V^{(H)}$. Φ_H is called the *canonical truth set* in $V^{(H)}$.

Now let us return to our causal set P . The topos $\mathcal{S}et^{(\hat{P})}$ of sets in $V^{(\hat{P})}$ is, as I have observed, equivalent to the topos of canonical sheaves on \hat{P} , which is itself, as is well known, equivalent to the topos $\mathcal{S}et^{P^{op}}$ of presheaves on P . My proposal is then, that we work in $V^{(\hat{P})}$ rather than, as did Markopoulou, within $\mathcal{S}et^{P^{op}}$. That is, describing what the universe looks like “from the inside” will amount to reporting the view from $V^{(\hat{P})}$. For simplicity let me write H for \hat{P} .

The “truth value” $\llbracket \sigma \rrbracket$ of a sentence σ in $V^{(H)}$ is a sieve of events in P , and it is natural to think of the events in $\llbracket \sigma \rrbracket$ as those at which σ “holds”. So one introduces the *forcing* relation \Vdash_P in $V^{(H)}$ between sentences and elements of P by

$$p \Vdash_P \sigma \leftrightarrow p \in \llbracket \sigma \rrbracket.$$

This satisfies the standard so-called Kripke rules, viz.,

- $p \Vdash_P \varphi \wedge \psi \leftrightarrow p \Vdash_P \varphi \ \& \ p \Vdash_P \psi$
- $p \Vdash_P \varphi \vee \psi \leftrightarrow p \Vdash_P \varphi \ \text{or} \ p \Vdash_P \psi$
- $p \Vdash_P \varphi \rightarrow \psi \leftrightarrow \forall q \leq p [q \Vdash_P \varphi \rightarrow q \Vdash_P \psi]$
- $p \Vdash_P \neg \varphi \leftrightarrow \forall q \leq p \ q \not\Vdash_P \varphi$
- $p \Vdash_P \forall x \varphi \leftrightarrow p \Vdash_P \varphi(a)$ for every $a \in V^{(\hat{P})}$
- $p \Vdash_P \exists x \varphi \leftrightarrow p \Vdash_P \varphi(a)$ for some $a \in V^{(\hat{P})}$.

Define the set $K \in V^{(H)}$ by $\text{dom}(K) = \{\hat{p} : p \in P\}$ and $K(\hat{p}) = p \downarrow$. Then, in $V^{(H)}$, K is a subset of \hat{P} and for $p \in P$, $\llbracket \hat{p} \in K \rrbracket = p \downarrow$. K is the counterpart in $V^{(\hat{P})}$ of Markopoulou’s evolving set *Past*. (\hat{P} , incidentally, is the $V^{(H)}$ - counterpart of the constant presheaf on P with value P —which Markopoulou calls *World*.) The fact that, for any $p, q \in P$ we have

$$(*) \quad q \Vdash_P \hat{p} \in K \leftrightarrow q \leq p$$

may be construed as asserting that *the events in the causal future of a given event are precisely those forcing (the canonical representative of) that event to be a member of K* . Or, equally, *the events in the causal past of a*

given event are precisely those forced by that event to be a member of K . For this reason we shall call K the *causal set* in $V^{(H)}$.

If we identify each $p \in P$ with $p \downarrow \in H$, P may then be regarded as a subset of H so that, in $V^{(H)}$, \hat{P} is a subset of \hat{H} . It is not hard to show that $V^{(H)} \models K = \Phi_H \cap \hat{P}$. Moreover, it can be shown that, for any sentence σ , $\llbracket \sigma \rrbracket = \llbracket \exists p \in K. p \leq \llbracket \sigma \rrbracket \rrbracket$, so that, with moderate abuse of notation,

$$V^{(H)} \models [\sigma \leftrightarrow \exists p \in K. p \Vdash \sigma].$$

That is, in $V^{(H)}$, a sentence holds precisely when it is forced to do so at some “causal past stage” in K . This establishes the centrality of K —and, correspondingly, that of the “evolving” set *Past*—in determining the truth of sentences “from the inside”, that is, inside the universe $V^{(H)}$.

Markopoulou also considers the *complement* of *Past*—i.e., in the present setting, the $V^{(H)}$ -set $\neg K$ for which $\llbracket \hat{p} \in \neg K \rrbracket = \llbracket p \notin K \rrbracket = \neg p \downarrow = \{q : \forall r \leq q. r \not\leq p\}$. Markopoulou calls (*mutatis mutandis*) the events in $\neg p \downarrow$ those *beyond p 's causal horizon*, in that no observer at p can ever receive “information” from any event in $\neg p \downarrow$. Since clearly we have

$$(†) \quad q \Vdash_P \hat{p} \in \neg K \leftrightarrow q \in \neg p \downarrow,$$

it follows that *the events beyond the causal horizon of an event p are precisely those forcing (the canonical representative of) p to be a member of $\neg K$* . In this sense $\neg K$ reflects, or “measures” the causal structure of P .

In this connection it is natural to call $\neg \neg p \downarrow = \{q : \forall r \leq q \exists s \leq r. s \leq p\}$ the *causal horizon* of p : it consists of those events q for which an observer placed at p could, in its future, receive information from any event in the future of an observer placed at q . Since

$$q \Vdash_P \hat{p} \in \neg \neg K \leftrightarrow q \in \neg \neg p \downarrow,$$

it follows that *the events within the causal horizon of an event are precisely those forcing (the canonical representative of) p to be a member of $\neg\neg K$.*

It is easily shown that $\neg K$ is *empty* (i.e. $V^{(H)} \models \neg K = \emptyset$) if and only if P is *directed downwards*, i.e., for any $p, q \in P$ there is $r \in P$ for which $r \leq p$ and $r \leq q$. This holds in the case, considered by Markopoulou, of *discrete Newtonian time evolution*—in the present setting, the case in which P is the opposite \mathbb{N}^{op} of the totally ordered set \mathbb{N} of natural numbers. Here the corresponding complete Heyting algebra H is the family of all downward-closed sets of natural numbers. In this case the H -valued set K representing *Past* is *neither finite nor actually infinite* in $V^{(H)}$.

To see this, observe first that, for any natural number n , we have $\llbracket \neg(\hat{n} \in \neg K) \rrbracket = \mathbb{N}$. It follows that $V^{(H)} \models \neg\neg \forall n \in \hat{\mathbb{N}}. n \in K$. But, working in $V^{(H)}$, if $\forall n \in \hat{\mathbb{N}}. n \in K$, then K is not finite, so if K is finite, then $\neg \forall n \in \hat{\mathbb{N}}. n \in K$, and so $\neg\neg \forall n \in \hat{\mathbb{N}}. n \in K$ implies the non-finiteness of K .

But, in $V^{(H)}$, K is not actually infinite. For (again working in $V^{(H)}$), if K were actually infinite (i.e., if there existed an injection of $\hat{\mathbb{N}}$ into K), then the statement

$$\forall x \in K \exists y \in K. x > y$$

would also have to hold in $V^{(H)}$. But calculating that truth value gives:

$$\begin{aligned}
& \llbracket \forall x \in K \exists y \in K. x > y \rrbracket \\
&= \bigcap_{m \in \mathbb{N}^{op}} [m \downarrow \Rightarrow \bigcup_{n \in \mathbb{N}^{op}} n \downarrow \cap \llbracket \hat{m} > \hat{n} \rrbracket] \\
&= \bigcap_m [m \downarrow \Rightarrow \bigcup_{n < m} n \downarrow] \\
&= \bigcap_m [m \downarrow \Rightarrow (m+1) \downarrow] \\
&= \bigcap_m (m+1) \downarrow = \emptyset
\end{aligned}$$

So $\forall x \in K \exists y \in K. x > y$ is false in $V^{(H)}$ and therefore K is not actually infinite. In sum, the causal set K in is *potentially, but not actually infinite*.

In order to formulate an observable causal *quantum theory* Markopoulou considers the possibility of introducing a *causally evolving algebra of observables*. This amounts to specifying a presheaf \mathcal{A} of C^* -algebras on P , which, in the present framework, corresponds to specifying a set \mathcal{A} in $V^{(H)}$ satisfying

$$V^{(H)} \models \mathcal{A} \text{ is a } C^*\text{-algebra.}$$

The “internal” C^* -algebra \mathcal{A} is then subject to the intuitionistic internal logic of $V^{(H)}$: *any* theorem concerning C^* -algebras—provided only that it be constructively proved—automatically applies to \mathcal{A} . Reasoning with \mathcal{A} is more direct and simpler than reasoning with \mathcal{A} .

This same procedure of “internalization” can be performed with any causally evolving object: each such object of type \mathcal{I} corresponds to a set S in $V^{(H)}$ satisfying

$$V^{(H)} \models S \text{ is of type } \mathcal{I}.$$

Internalization may also be applied in the case of the presheaves *Antichains* and *Graphs* considered by Markopoulou. Here, for each event p , *Antichains*(p) consists of all sets of causally unrelated events in *Past*(p), while *Graphs*(p) is the set of all graphs supported by elements of *Antichains*(p). In the present framework *Antichains* is represented by the $V^{(H)}$ -set $Anti = \{ X \subseteq \hat{P} : X \text{ is an antichain} \}$ and *Graphs* by the $V^{(H)}$ -set $Grph$

= $\{G: \exists X \in A . G \text{ is a graph supported by } A\}$. Again, both *Anti* and *Grph* can be readily handled using the internal intuitionistic logic of $V^{(H)}$.

Cover schemes or *Grothendieck topologies* may be used to force certain conditions to prevail in the associated models. (This corresponds to the process of *sheafification*.) A *cover scheme* on P is a map \mathbf{C} assigning to each $p \in P$ a family $\mathbf{C}(p)$ of subsets of $p \downarrow = \{q: q \leq p\}$, called *(C-)covers of p* , such that, if $q \leq p$, any cover of p can be sharpened to a cover of q , i.e.,

$$S \in \mathbf{C}(p) \ \& \ q \leq p \rightarrow \exists T \in \mathbf{C}(q) [\forall t \in T \exists s \in S (t \leq s)].$$

A cover S of an event p may be thought of as a “sampling” of the events in p ’s causal future, a “survey” of p ’s potential effects, in short, a *survey of p* . Using this language the condition immediately above becomes: *for any survey S of a given event p , and any event q which is a potential effect of p , there is a survey of q each event in which is the potential effect of some event in S .*

There are three naturally defined cover schemes on P we shall consider. First, each sieve A in P determines two cover schemes \mathbf{C}_A and \mathbf{C}^A defined by

$$S \in \mathbf{C}_A(p) \leftrightarrow p \in A \cup S \qquad S \in \mathbf{C}^A(p) \leftrightarrow p \downarrow \cap A \subseteq S$$

If $p \in A$, any part of p ’s causal future thus counts as a \mathbf{C}_A –survey of p , and any part of p ’s causal future extending the common part of that future with A counts as a \mathbf{C}^A –survey of p . Notice that then $\emptyset \in \mathbf{C}_A(p) \leftrightarrow p \in A$ and $\emptyset \in \mathbf{C}^A(p) \leftrightarrow p \downarrow \cap A = \emptyset$.

Next, we have the *dense cover scheme* **Den** given by:

$$S \in \mathbf{Den}(p) \leftrightarrow \forall q \leq p \exists s \in S \exists r \leq s (r \leq q):$$

That is, S is a dense survey of p provided that for every potential effect q of p there is an event in S with a potential effect in common with q .

Given a cover scheme \mathbf{C} on P , a sieve I will be said to *encompass* an element $p \in P$ if I includes a \mathbf{C} -cover of p . Thus a sieve I encompasses p if it contains all the events in some survey of p . Call I *\mathbf{C} -closed* if it contains every element of P that it encompasses, i.e. if

$$\exists S \in \mathbf{C}(p) (S \subseteq I) \rightarrow p \in I.$$

The set $\widehat{\mathbf{C}}$ of all \mathbf{C} -closed sieves in P , partially ordered by inclusion, can be shown to be a frame—the frame *induced* by \mathbf{C} —in which the operations of meet and \Rightarrow coincide with those of \widehat{P} . Passing from $V^{(P)}$ to $V^{(\widehat{\mathbf{C}})}$ is the process of *sheafification*: essentially, it amounts to replacing the forcing relation \Vdash_P in $V^{(P)}$ by the new forcing relation $\Vdash_{\widehat{\mathbf{C}}}$ in $V^{(\widehat{\mathbf{C}})}$. For atomic sentences σ these are related by

$$p \Vdash_{\widehat{\mathbf{C}}} \sigma \leftrightarrow \exists S \in \mathbf{C}(p) \forall s \in S. s \Vdash_P \sigma;$$

i.e., p *\mathbf{C} -forces the truth of a sentence just the truth of that sentence is P -forced by every event in some \mathbf{C} -survey of p .*

The frame induced by the dense cover scheme \mathbf{Den} in P turns out to be a complete Boolean algebra B . For the corresponding causal set K_B in V^B we find that

$$\begin{aligned} q \Vdash_B \widehat{p} \in K_B &\leftrightarrow q \in \neg\neg p \downarrow \\ &\leftrightarrow q \text{ is in } p\text{'s causal horizon.} \end{aligned}$$

Comparing this with (*) above, we see that moving to the universe $V^{(B)}$ —“Booleanizing” it, so to speak—amounts to replacing causal futures by causal horizons. When P is linearly ordered, as for example in the case of Newtonian time, the causal horizon of any event coincides with the whole of P , B is the two-element Boolean algebra $\mathbf{2}$, and $V^{(B)}$ reduces to the universe V of “static” sets. In this case, then, the effect of “Booleanization” is to render the universe timeless.

The universes associated with the cover schemes \mathbf{C}^A and \mathbf{C}_A seem also to have a rather natural physical meaning. Consider, for instance, the case in which A is the sieve $p\downarrow$ —the causal future of p . In the associated universe $V^{(\widehat{\mathbf{C}}^A)}$ the corresponding causal set K^A satisfies, for every event q

$$q \Vdash_{\widehat{\mathbf{C}}^A} \widehat{p} \in K^A.$$

Comparing this with (*), we see that in $V^{(\widehat{\mathbf{C}}^A)}$ that every event has been “forced” into p ’s causal future: in short, that p now marks the “beginning” of the universe as viewed from inside $V^{(\widehat{\mathbf{C}}^A)}$.

Similarly, we find that the causal set K_A in the universe $V^{(\widehat{\mathbf{C}}_A)}$ satisfies, for every event q ,

$$q \Vdash_{\widehat{\mathbf{C}}_A} \widehat{p} \in \neg K_A;$$

a comparison with (†) above reveals that, in $V^{(\widehat{\mathbf{C}}_A)}$, every event has been “forced” beyond p ’s causal horizon. In effect, p has become a *singularity*.