

Choice Principles in Intuitionistic Set Theory

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In intuitionistic set theory, the law of excluded middle is known to be derivable from the standard version of the axiom of choice that every family of nonempty sets has a choice function. In this paper it is shown that each of a number of intuitionistically invalid logical principles, including the law of excluded middle, is, in intuitionistic set theory, equivalent to a suitably weakened version of the axiom of choice. Thus these logical principles may be viewed as choice principles.

We work in intuitionistic Zermelo-Fraenkel set theory **IST** (for a presentation, see [3], where it is called **ZF_I**). Let us begin by fixing some notation. For each set A we write PA for the power set of A , and QX for the set of *inhabited* subsets of A , that is, of subsets X of A for which $\exists x (x \in A)$. The set of functions from A to B is denoted by B^A ; the class of functions with domain A is denoted by $\text{Fun}(A)$. The empty set is denoted by 0 , $\{0\}$ by 1 , and $\{0, 1\}$ by 2 .

We tabulate the following *logical schemes*:

SLEM	$\alpha \vee \neg\alpha$	(α any sentence)
Lin	$(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$	(α, β any sentences)
Stone	$\neg\alpha \vee \neg\neg\alpha$	(α any sentence)
Ex	$\exists x[\exists x\alpha(x) \rightarrow \alpha(x)]$	($\alpha(x)$ any formula with at most x free)
Un	$\exists x[\alpha(x) \rightarrow \forall x\alpha(x)]$	($\alpha(x)$ any formula with at most x free)
Dis ¹	$\forall x[\alpha \vee \beta(x)] \rightarrow \alpha \vee \forall x\beta(x)$	(α any sentence, $\beta(x)$ any formula with at most x free)

Over intuitionistic logic, **Lin**, **Stone** and **Ex** are consequences of **SLEM**; and **Un** implies **Dis**. All of these schemes follow, of course, from the full law of excluded middle, that is **SLEM** for arbitrary formulas.

We formulate the following *choice principles*—here X is an arbitrary set and $\varphi(x,y)$ an arbitrary formula of the language of **IST** with at most the free variables x, y :

¹ **Dis** is equivalent, over intuitionistic predicate logic, to what is called in [4] the *higher dual distributive law*—

$$\mathbf{HDDL} \quad \forall x[\alpha(x) \vee \beta(x)] \rightarrow \exists x\alpha(x) \vee \forall x\beta(x).$$

$$\begin{aligned}
\mathbf{AC}_X & \quad \forall x \in X \exists y \varphi(x, y) \rightarrow \exists f \in \text{Fun}(X) \forall x \in X \varphi(x, fx) \\
\mathbf{AC}_X^* & \quad \exists f \in \text{Fun}(X) [\forall x \in X \exists y \varphi(x, y) \rightarrow \forall x \in X \varphi(x, fx)] \\
\mathbf{DAC}_X & \quad \forall f \in \text{Fun}(X) \exists x \in X \varphi(x, fx) \rightarrow \exists x \in X \forall y \varphi(x, y) \\
\mathbf{DAC}_X^* & \quad \exists f \in \text{Fun}(X) [\exists x \in X \varphi(x, fx) \rightarrow \exists x \in X \forall y \varphi(x, y)]
\end{aligned}$$

The first two of these are forms of the *axiom of choice* for X ; while classically equivalent, in **IST** \mathbf{AC}_X^* implies \mathbf{AC}_X , but not conversely. The principles \mathbf{DAC}_X and \mathbf{DAC}_X^* are *dual* forms of the axiom of choice for X : classically they are both equivalent to \mathbf{AC}_X and \mathbf{AC}_X^* , but in **IST** \mathbf{DAC}_X^* implies \mathbf{DAC}_X , and not conversely.

We also formulate what we shall call the *weak extensional selection principle*, in which $\alpha(x)$ and $\beta(x)$ are any formulas with at most the variable x free:

$$\mathbf{WESP} \quad \exists x \in 2 \alpha(x) \wedge \exists x \in 2 \beta(x) \rightarrow \exists x \in 2 \exists y \in 2 [\alpha(x) \wedge \beta(y) \wedge [\forall x \in 2 [\alpha(x) \leftrightarrow \beta(x)] \rightarrow x = y]].$$

This principle asserts that, for any pair of instantiated properties of members of 2 , instances may be assigned to the properties in a manner that depends just on their extensions. **WESP** is a straightforward consequence of \mathbf{AC}_{Q2} . For taking $\varphi(u, y)$ to be $y \in u$ in \mathbf{AC}_{Q2} yields the existence of a function f with domain $Q2$ such that $fu \in u$ for every $u \in Q2$. Given formulas $\alpha(x)$, $\beta(x)$, and assuming the antecedent of **WESP**, the sets $U = \{x \in 2: \alpha(x)\}$ and $V = \{x \in 2: \beta(x)\}$ are members of $Q2$, so that $a = fU \in U$, and $b = fV \in V$, whence $\alpha(a)$ and $\beta(b)$. Also, if $\forall x \in 2 [\alpha(x) \leftrightarrow \beta(x)]$, then $U = V$, whence $a = b$; it follows then that the consequent of **WESP** holds.

We are going to show that each of the logical principles tabulated above is equivalent (over **IST**) to a choice principle. Starting at the top of the list, we have first:

- **WESP** and **SLEM** are equivalent over **IST**.

Proof. Assume **WESP**. Let σ be any sentence and define

$$\alpha(x) \equiv x = 0 \vee \sigma \quad \beta(x) \equiv x = 1 \vee \sigma.$$

With these instances of α and β the antecedent of **WESP** is clearly satisfied, so that there exist members a, b of 2 for which (1) $\alpha(a) \wedge \beta(b)$ and (2) $\forall x [[\forall x \in 2 [\alpha(x) \leftrightarrow \beta(x)] \rightarrow a = b]$. It follows from (1) that $\sigma \vee (a = 0 \wedge b = 1)$, whence (3) $\sigma \vee a \neq b$. And since clearly $\sigma \rightarrow \forall x \in 2 [\alpha(x) \leftrightarrow \beta(x)]$ we deduce from (2) that $\sigma \rightarrow a = b$, whence $a \neq b \rightarrow \neg\sigma$. Putting this last together with (3) yields $\sigma \vee \neg\sigma$, and **SLEM** follows.

For the converse, we argue informally. Suppose that **SLEM** holds. Assuming the antecedent of **WESP**, choose $a \in 2$ for which $\alpha(a)$. Now (using **SLEM**) define an element $b \in 2$ as follows. If $\forall x \in 2[\alpha(x) \leftrightarrow \beta(x)]$ holds, let $b = a$; if not, choose b so that $\beta(b)$. It is now easy to see that a and b satisfy $\alpha(a) \wedge \beta(b) \wedge [\forall x \in 2[\alpha(x) \leftrightarrow \beta(x)] \rightarrow a = b]$. **WESP** follows. ■

Remark. The argument for **WESP** \rightarrow **SLEM** is another “stripped down” version of Diaconescu’s theorem that, in a topos, the axiom of choice implies the law of excluded middle. The result may be compared with that of [2] to the effect that the presence of extensional ε -terms renders intuitionistic logic classical.

Next, we observe that, while **AC**₁ is (trivially) provable in **IST**, by contrast

- **AC**₁^{*} and **Ex** are equivalent over **IST**.

Proof. Assuming **AC**₁^{*}, take $\varphi(x,y) \equiv \alpha(y)$ in its antecedent. This yields an $f \in \text{Fun}(1)$ for which $\forall y \alpha(y) \rightarrow \alpha(f0)$, giving $\exists y[\exists y \alpha(y) \rightarrow \alpha(y)]$, i.e., **Ex**.

Conversely, define $\alpha(y) \equiv \varphi(0,y)$. Then, assuming **Ex**, there is b for which $\exists y \alpha(y) \rightarrow \alpha(b)$, whence $\forall x \in 1 \exists y \varphi(x,y) \rightarrow \forall x \in 1 \varphi(x,b)$. Defining $f \in \text{Fun}(1)$ by $f = \{(0,b)\}$ gives $\forall x \in 1 \exists y \varphi(x,y) \rightarrow \forall x \in 1 \varphi(x,fx)$, and **AC**₁^{*} follows. ■

Further, while **DAC**₁ is easily seen to be provable in **IST**, we have

- **DAC**₁^{*} and **Un** are equivalent over **IST**.

Proof. Given α , Define $\varphi(x,y) \equiv \alpha(y)$. Then, for $f \in \text{Fun}(1)$, $\exists x \in 1 \varphi(x,fx) \leftrightarrow \alpha(f0)$ and $\exists x \in 1 \forall y \varphi(x,y) \leftrightarrow \forall y \alpha(y)$. **DAC**₁^{*} then gives

$$\exists f \in \text{Fun}(1)[\alpha(f0) \rightarrow \forall y \alpha(y)],$$

from which **Un** follows easily.

Conversely, given φ , define $\alpha(y) \equiv \varphi(0,y)$. Then from **Un** we infer that there exists b for which $\alpha(b) \rightarrow \forall y \alpha(y)$, i.e. $\varphi(0,b) \rightarrow \forall y \varphi(0,y)$. Defining $f \in \text{Fun}(1)$ by $f = \{(0,b)\}$ then gives $\varphi(0,f0) \rightarrow \exists x \in 1 \forall y \varphi(x,y)$, whence $\exists x \in 1 \varphi(x,fx) \rightarrow \exists x \in 1 \forall y \varphi(x,y)$, and **Un** follows. ■

Next, while **AC**₂ is easily proved in **IST**, by contrast we have

- **DAC**₂ and **Dis** are equivalent over **IST**.

Proof. The antecedent of **DAC**₂ is equivalent to the assertion

$$\forall f \in \text{Fun}(2)[\varphi(0, f0) \vee \varphi(1, f1)],$$

which, in view of the natural correlation between members of $\text{Fun}(2)$ and ordered pairs, is equivalent to the assertion

$$\forall y \forall y' [\varphi(0, y) \vee \varphi(1, y')].$$

The consequent of **DAC**₂ is equivalent to the assertion

$$\forall y \in Y \varphi(0, y) \vee \forall y' \in Y \varphi(1, y')$$

So **DAC**₂ itself is equivalent to

$$\forall y \forall y' [\varphi(0, y) \vee \varphi(1, y')] \rightarrow \forall y \varphi(0, y) \vee \forall y' \varphi(1, y').$$

But this is obviously equivalent to the scheme

$$\forall y \forall y' [\alpha(y) \vee \beta(y')] \rightarrow \forall y \alpha(y) \vee \forall y' \beta(y'),$$

where y does not occur free in β , nor y' in α . And this last is easily seen to be equivalent to **Dis**. ■

Now consider **DAC**₂^{*}. This is quickly seen to be equivalent to the assertion

$$\exists z \exists z' [\varphi(0, z) \vee \varphi(1, z')] \rightarrow \forall y \varphi(0, y) \vee \forall y' \varphi(1, y'),$$

i.e. to the assertion, for arbitrary $\alpha(x)$, $\beta(x)$, that

$$\exists z \exists z' [\alpha(z) \vee \beta(z')] \rightarrow \forall y \alpha(y) \vee \forall y' \beta(y').$$

This is in turn equivalent to the assertion, for any sentence α ,

$$\exists y [\alpha \vee \beta(y) \rightarrow \alpha \vee \forall y \beta(y)] \quad (*)$$

Now (*) obviously entails **Un**. Conversely, given **Un**, there is b for which $\beta(b) \rightarrow \forall y \beta(y)$.

Hence $\alpha \vee \beta(b) \rightarrow \alpha \vee \forall y \beta(y)$, whence (*). So we have shown that

- Over **IST**, **DAC**₂^{*} is equivalent to **Un**, and hence also to **DAC**₁^{*}.

In order to provide choice schemes equivalent to **Lin** and **Stone** we introduce

$$\mathbf{ac}_X^* \quad \exists f \in 2^X [\forall x \in X \exists y \in 2 \varphi(x, y) \rightarrow \forall x \in X \varphi(x, fx)]$$

$$\mathbf{wac}_X^* \quad \exists f \in 2^X [\forall x \in X \exists y \in 2 \varphi(x, y) \rightarrow \forall x \in X \varphi(x, fx)] \text{ provided } \vdash_{\mathbf{IST}} \forall x [\varphi(x, 0) \rightarrow \neg \varphi(x, 1)]$$

Clearly \mathbf{ac}_X^* is equivalent to

$$\exists f \in 2^X [\forall x \in X [\varphi(x, 0) \vee \varphi(x, 1)] \rightarrow \forall x \in X \varphi(x, fx)]$$

and similarly for \mathbf{wac}_X^* .

Then

- Over **IST**, \mathbf{ac}_1^* and \mathbf{wac}_1^* are equivalent, respectively, to **Lin** and **Stone**.

Proof. Let α and β be sentences, and define $\varphi(x, y) \equiv x = 0 \wedge [(y = 0 \wedge \alpha) \vee (y = 1 \wedge \beta)]$. Then $\alpha \leftrightarrow \varphi(0, 0)$ and $\beta \leftrightarrow \varphi(0, 1)$, and so $\forall x \in 1 [\varphi(x, 0) \vee \varphi(x, 1)] \leftrightarrow \varphi(0, 0) \vee \varphi(0, 1) \leftrightarrow \alpha \vee \beta$. Therefore

$$\begin{aligned} \exists f \in 2^1 [\forall x \in 1 [\varphi(x, 0) \vee \varphi(x, 1)] \rightarrow \forall x \in 1 \varphi(x, fx)] &\leftrightarrow \exists f \in 2^1 [\alpha \vee \beta \rightarrow \varphi(0, f0)] \\ &\leftrightarrow [\alpha \vee \beta \rightarrow \varphi(0, 0)] \vee [\alpha \vee \beta \rightarrow \varphi(0, 1)] \\ &\leftrightarrow [\alpha \vee \beta \rightarrow \alpha] \vee [\alpha \vee \beta \rightarrow \beta] \\ &\leftrightarrow \beta \rightarrow \alpha \vee \alpha \rightarrow \beta. \end{aligned}$$

This yields $\mathbf{ac}_1^* \rightarrow \mathbf{Lin}$. For the converse, define $\alpha \equiv \varphi(0, 0)$ and $\beta \equiv \varphi(0, 1)$ and reverse the argument.

To establish the second stated equivalence, notice that, when $\varphi(x, y)$ is defined as above, but with β replaced by $\neg\alpha$, it satisfies the provisions imposed in \mathbf{wac}_1^* . As above, that principle gives $(\neg\alpha \rightarrow \alpha) \vee (\alpha \rightarrow \neg\alpha)$, that is, $\neg\alpha \vee \neg\neg\alpha$. So **Stone** follows from \mathbf{wac}_1^* . Conversely, suppose that φ meets the condition imposed in \mathbf{wac}_1^* . Then from $\varphi(0, 0) \rightarrow \neg\varphi(0, 1)$ we deduce $\neg\neg\varphi(0, 0) \rightarrow \neg\varphi(0, 1)$; now, assuming **Stone**, we have $\neg\varphi(0, 0) \vee \neg\neg\varphi(0, 0)$, whence $\neg\varphi(0, 0) \vee \neg\varphi(0, 1)$. Since $\neg\varphi(0, 0) \rightarrow [\varphi(0, 0) \rightarrow \varphi(0, 1)]$ and $\neg\varphi(0, 1) \rightarrow [\varphi(0, 1) \rightarrow \varphi(0, 0)]$ we deduce $[\varphi(0, 0) \rightarrow \varphi(0, 1)] \vee [\varphi(0, 1) \rightarrow \varphi(0, 0)]$. From the argument above it now follows that $\exists f \in 2^1 [\forall x \in 1 [\varphi(x, 0) \vee \varphi(x, 1)] \rightarrow \forall x \in 1 \varphi(x, fx)]$. Accordingly \mathbf{wac}_1^* is a consequence of **Stone**.

In conclusion, we show how certain of the principles we have introduced can be derived in the presence of *term-forming operators*.

The ε - and τ -*operators* are term-forming operators yielding, for formulas $\alpha(x)$, terms $\varepsilon_x\alpha$ and $\tau_x\alpha$ in which the variable x is no longer free; they are introduced in conjunction with the axioms—the ε - and τ -*schemes*:

$$\exists x\alpha(x) \rightarrow \alpha(\varepsilon x\alpha) \quad \alpha(\tau x\alpha) \rightarrow \forall x\alpha(x).$$

It is an easy matter to derive **Un** from the τ -scheme when τ is merely allowed to act on formulas with at most one free variable. When τ 's action is extended to formulas with two free variables, the τ -scheme applied in **IST** yields the full dual axiom of choice $\forall X \mathbf{DAC}_X^*$. For under these conditions we have, for any formula $\varphi(x, y)$,

$$\forall x \in X [\varphi(x, \tau_y \varphi(x, y)) \rightarrow \forall y \varphi(x, y)] \quad (*)$$

Let $t \in \text{Fun}(X)$ be the map $x \mapsto \tau_y \varphi(x, y)$. Assuming that $\forall f \in Y^X \exists x \in X \varphi(x, fx)$, let $a \in X$ satisfy $\varphi(a, ta)$. We deduce from (*) that $\forall y \in Y \varphi(a, y)$, whence $\exists x \in X \forall y \in Y \varphi(x, y)$. The dual axiom of choice follows.

In the case of the ε -operator, the number of free variables in the formulas on which the operator is allowed to act is an even more sensitive matter. If ε is allowed to act only on formulas with at most one free variable (so yielding only closed terms), the corresponding ε -scheme applied in **IST** is easily seen to yield both **Ex** and \mathbf{ac}_1^* , and so also **Lin**. But it is (in essence) shown in [1] that, if only closed ε -terms are admitted, **SLEM** is not derivable, and so therefore neither is **WESP**. The situation changes dramatically when ε is permitted to operate on formulas with *two* free variables. For then from the corresponding ε -scheme it is easy to derive \mathbf{AC}_X for all sets X , and in particular \mathbf{AC}_{Q_2} , and hence also **SLEM**.

I have found three ways of strengthening, or modifying, the single-variable ε -scheme so as to enable it to yield **SLEM**. The first, presented originally in [2], is to add to the ε -scheme *Ackermann's Extensionality Principle*, viz.

$$\forall x [\alpha(x) \leftrightarrow \beta(x)] \rightarrow \varepsilon x\alpha = \varepsilon x\beta.$$

From these **WESP** is easily derived, and so, *a fortiori*, **SLEM**.

The second approach is to take the ε -axiom in the (classically equivalent) form

$$(*) \quad \alpha(\varepsilon x\alpha) \vee \forall x \neg \alpha(x).$$

From this we can intuitionistically derive **SLEM** as follows:

Given a sentence β , define $\alpha(x)$ to be the formula

$$(x = 0 \wedge \beta) \vee (x = 1 \wedge \neg\beta).$$

Then from (*) we get

$$[(\varepsilon_x \alpha = 0 \wedge \beta) \vee ((\varepsilon_x \alpha = 1 \wedge \neg \beta))] \vee \forall x \neg [(x = 0 \wedge \beta) \vee (x = 1 \wedge \neg \beta)],$$

which implies

$$[\beta \vee \neg \beta] \vee [\forall x \neg (x = 0 \wedge \beta) \wedge \forall x \neg (x = 1 \wedge \neg \beta)],$$

whence

$$[\beta \vee \neg \beta] \vee [\neg \beta \wedge \neg \neg \beta],$$

winding up with

$$\beta \vee \neg \beta.$$

The third method is to allow ε to act on *pairs* of formulas, each with a *single* free variable. Here, for each pair of formulas $\alpha(x)$, $\beta(x)$ we introduce the “relativized” ε -term $\varepsilon_x \alpha / \beta$ and the “relativized” ε -axioms

$$(1) \exists x \beta(x) \rightarrow \beta(\varepsilon_x \alpha / \beta) \quad (2) \exists x [\alpha(x) \wedge \beta(x)] \rightarrow \alpha(\varepsilon_x \alpha / \beta).$$

That is, $\varepsilon_x \alpha / \beta$ may be thought of as an individual that satisfies β if anything does, and which in addition satisfies α if anything satisfies both α and β . Notice that the usual ε -term $\varepsilon_x \alpha$ is then $\varepsilon_x \alpha / x = x$. In the classical ε -calculus $\varepsilon_x \alpha / \beta$ may be defined by taking

$$\varepsilon_x \alpha / \beta = \varepsilon_y [[y = \varepsilon_x (\alpha \wedge \beta) \wedge \exists x (\alpha \wedge \beta)] \vee [y = \varepsilon_x \beta \wedge \neg \exists x (\alpha \wedge \beta)]].$$

But the relativized ε -scheme is not derivable in the intuitionistic ε -calculus since it can be shown to imply **SLEM**. To see this, given a formula γ define

$$\alpha(x) \equiv x = 1 \quad \beta(x) \equiv x = 0 \vee \gamma.$$

Write a for $\varepsilon_x \alpha / \beta$. Then we certainly have $\exists x \beta(x)$, so (1) gives $\beta(a)$, i.e.

$$(3) \quad a = 0 \vee \gamma$$

Also $\exists x (\alpha \wedge \beta) \leftrightarrow \gamma$, so (2) gives $\gamma \rightarrow \alpha(a)$, i.e.

$$\gamma \rightarrow a = 1,$$

whence

$$a \neq 1 \rightarrow \neg \gamma,$$

so that

$$a = 0 \rightarrow \neg\gamma.$$

And the conjunction of this with (3) gives $\gamma \vee \neg\gamma$, as claimed.

References

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