J. L. BELL

HILBERT'S ε-OPERATOR AND CLASSICAL LOGIC

1. INTRODUCTION: THE ε-AXIOM

In the course of his investigations into the foundations of mathematics, Hilbert came to regard the *axiom of choice* as an indispensable principle[1] and enlisted its support in his defence of classical mathematical reasoning against the attacks of the intuitionists. He refrained, however, from introducing the axiom overtly into his logical calculus: its set-theoretical character prevented it, one supposes, from possessing the formal transparency he demanded of a "logical" principle. Instead he adopted a postulate – *the logical ε-axiom*[2] – which, while closely akin in content to the axiom of choice, appeared both to possess a more evident formal simplicity and to reflect the more faithfully his professed conviction that (classical) mathematical practice requires the positing of what he termed *ideal elements*. To formulate this postulate he introduced, for each predicate $A(x)$ of his logical calculus, a term $\varepsilon x A$ or simply[3] ε_A which, intuitively, is intended to name an (indeterminate) object satisfying $A(x)$. Writing "$A(t)$" for the result of substituting "t" for each free occurrence of "x" in $A(x)$, the ε-axiom then reads:

$$(\varepsilon) \quad A(x) \rightarrow A(\varepsilon_A).$$

aIn any of the usual logical systems this is equivalent to

$$\exists x A(x) \leftrightarrow A(\varepsilon_A).$$

We may think of ε_A as naming an *ideal object* associated with A: all one knows about it is that, if anything satisfies A, it does.

Notice that since A may contain other free variables $y_1, y_2, \ldots,$ the identity of the ideal object ε_A depends, in general, on the values assigned to these variables. Thus ε_A may be regarded as the result of

Journal of Philosophical Logic **22**: 1 – 18, 1993.

having chosen, for each sequence of values of $y_1, y_2, \ldots,$ a value of x so that A is satisfied. That is, ε_A may be construed as a choice function, whose existence (in a set-theoretical framework) is justified by the axiom of choice.

Hilbert's chief purpose in formulating his ε-axiom was to enable the quantifiers to be defined in such a way as to avoid having to construe them as signifying conjunctions or disjunctions of infinite sets of propositions. Thus he defined[4] (assuming the ε-axiom)

$$\exists x A(x) \equiv A(\varepsilon_A)$$

$$\forall x A(x) \equiv A(\varepsilon_{\neg A}).$$

These definitions were used in turn to justify the logical principle he regarded as the lynch-pin of classical reasoning in mathematics, namely

$$\textbf{(Q)} \quad \neg \forall x A(x) \rightarrow \exists x \neg A(x).$$

Now it is a trivial matter to derive **(Q)** if we define $\exists x A(x)$ and $\forall x A(x)$ as above. However, while the definition of $\exists x A(x)$ seems perfectly acceptable, we may well ask: what justifies Hilbert's definition of $\forall x A(x)$? Clearly, it would be justified if one could derive

$$(*) \quad A(\varepsilon_{\neg A}) \rightarrow A(x).$$

Let us see how we might set about deriving (*), assuming (ε) as an axiom. All we know about $\varepsilon_{\neg A}$ is that[5]

$$\vdash_{\varepsilon} \neg A(x) \rightarrow \neg A(\varepsilon_{\neg A})$$

whence

$$\vdash_{\varepsilon} \neg\neg A(\varepsilon_{\neg A}) \rightarrow \neg\neg A(x).$$

Now if we were permitted to assume as an axiom, as of course we may classically, the *law of double negation*, viz.,

$$(\neg\neg) \quad \neg\neg A(x) \rightarrow A(x),$$

then (*) would immediately follow.

Hilbert actually assumes[6] $(\neg\neg)$ and so is justified in defining $\forall x A(x)$ as he does. However, it was pointed out (in 1925) by

Kolmogorov[7] that ($\neg\neg$) *already entails* (**Q**) in (intuitionistic) predicate logic. That is, for the purpose of deriving (**Q**), which for Hilbert was of prime importance, the assumption of ($\neg\neg$) renders the ε-axiom superfluous. It would seem natural, accordingly, to ask whether (**Q**) is derivable from the ε-axiom *without* assuming ($\neg\neg$), that is, within an intuitionistic framework. We answer this question in the negative in § 7.

2. SOME CONSEQUENCES OF THE ε-AXIOM

To begin with, we note that, although we shall see that (**Q**) is not derivable from (**ε**) intuitionistically, (**ε**) does imply the somewhat weaker scheme known as *Markov's principle*[8], viz.

$$(\mathbf{Mar})\quad \forall x[\neg\neg A(x) \rightarrow A(x)] \rightarrow$$
$$\rightarrow [\neg\forall x A(x) \rightarrow \exists x \neg A(x)].$$

Markov's principle asserts (**Q**) for *decidable* predicates $A(x)$.

To derive (**Mar**) from (**ε**) intuitionistically, we first employ the latter to obtain

$$\neg A(x) \vdash_{\varepsilon} \neg A(\varepsilon_{\neg A})$$

whence

$$\neg\neg A(\varepsilon_{\neg A}) \vdash_{\varepsilon} \neg\neg A(x).$$

Therefore

$$\forall x[\neg\neg A(x) \rightarrow A(x)],\ A(\varepsilon_{\neg A}) \vdash_{\varepsilon} A(x)$$

whence

$$\begin{aligned}
\forall x[\neg\neg A(x) \rightarrow A(x)],\quad A(\varepsilon_{\neg A}) &\vdash_{\varepsilon} \forall x A(x)\\
\forall x[\neg\neg A(x) \rightarrow A(x)]\quad &\vdash_{\varepsilon} A(\varepsilon_{\neg A}) \rightarrow \forall x A(x)\\
&\vdash_{\varepsilon} \neg\forall x A(x) \rightarrow \neg A(\varepsilon_{\neg A})\\
&\vdash_{\varepsilon} \neg\forall x A(x) \rightarrow \exists x \neg A(x)
\end{aligned}$$

as required.

Hilbert refers[9] to (**Q**) as the *law of excluded middle*, but is is more properly identified as a transfinite form of (one of) *de Morgan's laws*, viz.

$$(\mathbf{M})\quad \neg(B \wedge C) \rightarrow \neg B \vee \neg C.$$

(**M**) is not intuitionistically valid, but its dual form

$$\neg(B \vee C) \rightarrow \neg B \wedge \neg C$$

is.

It is straightforward matter to show that (**M**) follows from (**Q**) in any intuitionistic system I (in particular, Heyting arithmetic) which satisfies the following modest "decidability" condition: there is a constant **a** for which

$$(\mathbf{D})\quad \vdash_{\mathrm{I}} \forall x(x = \mathbf{a} \vee x \neq \mathbf{a}).$$

For suppose (**D**) holds; define

$$(*)\quad A(x) \equiv (x = \mathbf{a} \wedge B) \vee (x \neq \mathbf{a} \wedge C).$$

Then

$$\vdash_{\mathrm{I}} \forall x A(x) \leftrightarrow B \wedge C$$

and

$$\begin{aligned}\vdash_{\mathrm{I}} \exists x \neg A(x) &\leftrightarrow \exists x[\neg(x = \mathbf{a} \wedge B) \wedge \neg(x \neq \mathbf{a} \wedge C)]\\ &\leftrightarrow \exists x[(x = \mathbf{a} \rightarrow \neg B) \wedge (x \neq \mathbf{a} \rightarrow \neg C)\\ &\leftrightarrow \neg B \vee \neg C.\end{aligned}$$

From this we see immediately that (**Q**) $\vdash_{\mathrm{I}}$ (**M**).

More interestingly, we can obtain (**M**) directly from (ε), assuming that the intuitionistic system **I** in which derivation takes place satisfies, in addition to (**D**), the condition: there is a constant **b** for which $\vdash_{\mathrm{I}} \mathbf{a} \neq \mathbf{b}$. Let us denote the conjunction of these conditions by (**D**′). Suppose **I** satisfies (**D**′) and includes (ε) as an axiom. Define $A(x)$ as in (*) above. Then we have

$$\vdash_{\mathrm{I}} A(\mathbf{a}) \leftrightarrow B, \quad \vdash_{\mathrm{I}} A(\mathbf{b}) \leftrightarrow C$$

$$(**)\quad \vdash_{\mathrm{I}} \neg A(x) \leftrightarrow [(x = \mathbf{a} \rightarrow \neg B) \wedge (x \neq \mathbf{a} \rightarrow \neg C)].$$

Now

$$\neg A(\mathbf{a}) \vdash_{\mathrm{I}} \neg A(\varepsilon_{\neg A}) \quad \neg A(\mathbf{b}) \vdash_{\mathrm{I}} \neg A(\varepsilon_{\neg A}),$$

so

$$\neg B \vdash_{\mathrm{I}} \neg A(\varepsilon_{\neg A}) \quad \neg C \vdash_{\mathrm{I}} \neg A(\varepsilon_{\neg A})$$

whence

$$\neg\neg A(\varepsilon_{\neg A}) \vdash_{\mathrm{I}} \neg\neg B \quad \neg\neg A(\varepsilon_{\neg A}) \vdash_{\mathrm{I}} \neg\neg C.$$

It follows that

$$\neg\neg A(\varepsilon_{\neg A}) \vdash_{\mathrm{I}} \neg\neg B \wedge \neg\neg C,$$

so that, invoking the intuitionistically correct principle

$$\neg\neg B \wedge \neg\neg C \vdash \neg\neg(B \wedge C),$$

we obtain

$$\neg\neg A(\varepsilon_{\neg A}) \vdash_{\mathrm{I}} \neg\neg(B \wedge C).$$

Hence

$$\neg\neg\neg(B \wedge C) \vdash_{\mathrm{I}} \neg\neg\neg A(\varepsilon_{\neg A}).$$

But it is intuitionistically correct that

$$\vdash \neg\neg\neg B \leftrightarrow \neg B,$$

so we get

$$\neg(B \wedge C) \vdash_{\mathrm{I}} \neg A(\varepsilon_{\neg A}).$$

Hence, substituting "$\varepsilon_{\neg A}$" for "x" in (**) we obtain

$$(***) \quad \neg(B \wedge C) \vdash_{\mathrm{I}} (\varepsilon_{\neg A} = \mathbf{a} \rightarrow \neg B) \wedge (\varepsilon_{\neg A} \neq \mathbf{a} \rightarrow \neg C).$$

But, by **(D)**,

$$\vdash \varepsilon_{\neg A} = \mathbf{a} \vee \varepsilon_{\neg A} \neq \mathbf{a},$$

and this, together with (***) gives

$$\neg(B \wedge C) \vdash_{\mathrm{I}} \neg B \vee \neg C,$$

as required.

3. THE PRINCIPLE OF EXTENSIONALITY AND THE LAW OF EXCLUDED MIDDLE

We have seen that Markov's principle and de Morgan's law are derivable from the ε-axiom. As we shall show in § 7, however, **(Q)** is *not* so derivable, and therefore neither is $(\neg\neg)$, nor the latter's equivalent the (genuine) *law of excluded middle*, viz.

$$\textbf{(LEM)} \quad A(x) \vee \neg A(x).$$

This prompts the question of whether a strengthened version of the ε-axiom suffices to yield **(LEM)** (and hence also **(Q)**). To put it another way, would the provision of further information concerning the behaviour of the ideal objects ε_A ensure the derivability of **(LEM)** and classical logic? We shall presently answer this question in the affirmative.

Since the ε_A have been introduced *intensionally*, that is, by the form of the defining predicate A, we do not yet possess a useful sufficient *identity* condition for them. One such condition is suggested by Hilbert's use of the ε_A to define the existential quantifier (see § 1). The meaning of $\exists x A(x)$ is classically determined by the *extension* of $A(x)$, i.e., the class of objects satisfying $A(x)$: $\exists x A(x)$ may be construed as meaning "the extension of $A(x)$ is non-empty". But, assuming the ε-axiom, the meaning of $\exists x A(x)$ is determined by the identity of ε_A: here $\exists x A(x)$ may be construed as meaning "ε_A satisfies A". Thus, in determining the meaning of $\exists x A(x)$, the role of ε_A is to serve as a surrogate for the extension of $A(x)$. In that case, it would not be unreasonable to suppose that the identity of ε_A is completely determined by the extension of $A(x)$. That is, the *extensional equivalence* of predicates should be sufficient to yield the coincidence of the associated ideal objects. (Thus, for example, the ideal man would coincide with the ideal featherless biped.)

All this may be formulatd symbolically as the following *extensionality principle for ideal objects*[10]

$$\textbf{(Ext)} \quad \forall x[A(x) \leftrightarrow B(x)] \rightarrow \varepsilon_A = \varepsilon_B.$$

We are going to show that the addition of **(Ext)** to (ε) suffices to yield **(LEM)**. In fact, only substantially weakened versions of (ε) and **(Ext)** are required for the argument to go through.

Let **T** be an intuitionistic theory formulated in a first order language $\mathscr{L}$. We assume that $\mathscr{L}$ contains constants **0**, **1** and $\vdash_{\mathbf{T}} \mathbf{0} \neq \mathbf{1}$.

LEMMA[11]. *Given a formula $A(x)$ of $\mathscr{L}$ define the formulas $B(x, y)$, $C(x, y)$ by*

$$B(x, y) \equiv y = \mathbf{0} \vee A(x)$$

$$C(x, y) \equiv y = \mathbf{1} \vee A(x).$$

Suppose that there are in $\mathscr{L}$ terms sx, tx such that

(1) $$\vdash_{\mathbf{T}} B(x, sx) \wedge C(x, tx)$$

(2) $$A(x) \vdash_{\mathbf{T}} sx = tx.$$

Then

$$\vdash_{\mathbf{T}} A(x) \vee \neg A(x).$$

Proof. From (1)

$$\vdash_{\mathbf{T}} [sx = \mathbf{0} \vee A(x)] \wedge [tx = \mathbf{1} \vee A(x)]$$

whence, by distributivity,

$$\vdash_{\mathbf{T}} [sx = \mathbf{0} \wedge tx = \mathbf{1}] \vee A(x)$$

so that

(3) $$\vdash_{\mathbf{T}} sx \neq tx \vee A(x).$$

But, from (2),

$$sx \neq tx \vdash_{\mathbf{T}} \neg A(x)$$

and the conclusion follows from (3). ∎

THEOREM. *Suppose that to each formula $B(x, y)$ of $\mathscr{L}$ such that $\vdash_{\mathbf{T}} \exists y B(x, y)$ it is possible to assign a term $\varepsilon_B x$ in such a way that*

(1′) $$\vdash_{\mathbf{T}} B(x, \varepsilon_B x)$$

(2′) $$\vdash_{\mathbf{T}} \forall y[B(x, y) \leftrightarrow C(x, y)] \rightarrow \varepsilon_B x = \varepsilon_C x.$$

Then for any formula $A(x)$,

$$\vdash_{\mathbf{T}} A(x) \vee \neg A(x).$$

Proof. Defining $B(x, y)$, $C(x, y)$ as in the statement of the Lemma, it is readily seen that $\vdash_{\mathbf{T}} \exists y B(x, y)$ and $\vdash_{\mathbf{T}} \exists y C(x, y)$. Then taking sx, tx as $\varepsilon_B x$, $\varepsilon_C x$ respectively, (1′) above yields (1) and (2′) yields (2). The conclusion now follows from the Lemma. ■

Let us summarize. Call a predicate $B(y)$ **T**-*exampled* if $\vdash_{\mathbf{T}} \exists y B(y)$ and call a term t such that $\vdash_{\mathbf{T}} B(t)$ an *exemplar* for B. Call **T** *extensionally saturated* if an exemplar can be assigned to any **T**-exampled predicate in a manner that depends only on its extension. The Theorem then asserts that the law of excluded middle holds in any extensionally saturated theory. As an immediate consequence, (ε) *and* (**Ext**) *together yield* (**LEM**).

If **T** is pure intuitionistic predicate calculus or Heyting arithmetic, it is well known[12] that any **T**-exampled predicate has an exemplar. Thus it is not the mere availability of exemplars, but the stipulation that they can be chosen extensionally, which yields the law of excluded middle. In short, *not existence, but extensional existence, yields classical logic.*

4. THE PRINCIPLE OF EXTENSIONALITY VS. THE PRINCIPLE OF BIVALENCE

It is instructive to contrast the derivation of the law of excluded middle we have just presented with what may be regarded as the orthodox one. Excluded middle is usually derived from the *principle of bivalence* which asserts that any statement must be either true or false. (Note that, although the principle of bivalence clearly implies excluded middle, the converse fails.) Now since the (metalinguistic) use of "either . . . or" in its formulation makes the principle of bivalence essentially a strengthened version of excluded middle, deriving the latter from it is tantamount to a *petitio principii.* Nonetheless, one may still ask whether the principle of bivalence is *itself* derivable from more fundamental principles.

It is usually held that asserting the principle of bivalence is constitutive of a realist or objective attitude toward the meaning of statements. Thus, while it is recognized that, for instance, an epistemic construal of statements leads to violations of the principle of bivalence, it is (at least implicitly) held that the realist construal *implies* bivalence. However, if we are not prepared to accept this assertion *tout court*, then it would seem that we are entitled to ask: what is it about the structure of reality that warrants the claim that statements, construed as referring to that reality, take just *two* truth values ("true" and "false")? Is there an answer to this question that is not circular? One might, for example, *define*, as Frege did, the objective reference of a statement to be one of two truth vlaues: but there seems to be nothing in the concept of "objective" compelling one to select the number "two" here. Another approach would be to argue that the existence of just two truth values is the *simplest* (non-degenerate) possibility. While this can scarcely be denied, it cannot yield the required conclusion without the further assumption that reality in fact possesses the appropriate simplicity. However, this again begs the question unless one supplies *both* a deeper analysis of the concept of simplicity, *and* a cogent argument that reality possesses this attribute.

The point I wish to emphasize is that while, as far as I am aware, the principle of bivalence has (as yet) no convincing derivation from *realist* principles, the Theorem above shows that, by contrast, its cousin the law of excluded middle *can* be noncircularly derived from *ontological* principles. For (ε) is clearly ontological in nature, while (**Ext**) acts as an adjunct supplying conditions for the existents postulated in (ε). In addition, neither (ε) nor (**Ext**) contains explicit occurrences of the logical operations ($\vee$, $\neg$) appearing in (**LEM**), so the derivation of the latter from these principles may justly be called non-circular.

A further point to be noted is that the extensionality principle does not actually assert the *existence* of extensions[13] of predicates, so in positing it we are not introducing completed infinities (which of course Hilbert himself was trying to avoid). It may be objected, however, that the universal quantifier $(\forall x)$ on the left hand side of the arrow in (**Ext**) already presupposes the existence of a domain over which x varies and that such a domain is, in general, infinite. Without denying the justice of this claim, we observe that the proof of the Lemma shows that, in deriving (**LEM**), the full strength of (**Ext**) is *not* required. In fact, we need only assume that the bound variable ranges over the (finite) set $\{0, 1\}$, so merely necessitating the introduction of ideal elements of this set. To obtain excluded middle, we need then just suppose that these ideal elements of $\{0, 1\}$ can be specified extensionally. This seems a modest assumption, which in particular involves no deployment of completed infinities.

5. A MATHEMATICAL CONSEQUENCE OF THE EXTENSIONALITY PRINCIPLE

It is a tenet of the intuitionistic school that all functions from reals to reals are continuous[14]. We shall show how **(Ext)** and **(ε)** can be used to construct a *discontinuous* such function in an elegant way.

Let **T** be an intuitionistic first-order theory formalizing the elementary algebra of the real numbers, and consider the formulas

$$A(x, y) \equiv y = \mathbf{0} \vee (x = \mathbf{0} \wedge y = \mathbf{1})$$

$$B(x, y) \equiv y = \mathbf{1} \vee (x = \mathbf{0} \wedge y = \mathbf{0}).$$

Clearly

$$\vdash_{\mathbf{T}} \exists y A(x, y) \wedge \exists y B(x, y).$$

Assuming (ε) is an axiom of T, there are functions (i.e., terms) f, g such that

$$\vdash_{\mathbf{T}} A(x, fx) \wedge B(x, gx),$$

i.e.,

$$(*) \quad \begin{cases} \vdash_{\mathbf{T}} fx = \mathbf{0} \vee (x = \mathbf{0} \wedge fx = \mathbf{1}) \\ \vdash_{\mathbf{T}} gx = \mathbf{1} \vee (x = \mathbf{0} \wedge gx = \mathbf{0}). \end{cases}$$

Observe that it was, strictly speaking, unnecessary to invoke **(ε)** merely to obtain f, g satisfying (*), for we could clearly have taken $fx \equiv \mathbf{0}$ and $gx \equiv \mathbf{1}$ identically. But now suppose we insist that **(Ext)** holds in T and that accordingly we may choose f and g so that they depend *extensionally* on A and B, i.e.,

$$\forall y[A(x, y) \leftrightarrow B(x, y)] \vdash_{\mathbf{T}} fx = gx.$$

In particular, we would then have

$$x = \mathbf{0} \vdash_{\mathbf{T}} fx = gx$$

i.e.,

$$(**) \quad \vdash_{\mathbf{T}} f(\mathbf{0}) = g(\mathbf{0}).$$

From (*) it follows that

$$(***) \quad \vdash_{\mathbf{T}} (fx = \mathbf{0} \wedge gx = \mathbf{1}) \vee x = \mathbf{0}.$$

Now define $h \equiv 1 - (f - g)^2$. Then, by (***)

$$\vdash_{\mathbf{T}} hx = \mathbf{0} \vee x = \mathbf{0},$$

whence

$$\vdash_{\mathbf{T}} x \neq \mathbf{0} \rightarrow hx = \mathbf{0}.$$

But $\vdash_{\mathbf{T}} h(\mathbf{0}) = \mathbf{1}$ from (**). We conclude that h is a δ-function, and hence discontinuous.

The argument here may be paraphrased as follows, using set-theoretic notation. Suppose we have obtained two functions f, g such that

$$\text{graph}\,(f) \subseteq \{\langle x, 0\rangle : x \in \mathbb{R}\} \cup \{\langle 0, 1\rangle\}$$

$$\text{graph}\,(g) \subseteq \{\langle x, 1\rangle : x \in \mathbb{R}\} \cup \{\langle 0, 0\rangle\}.$$

The principle of extensionality applied to the definitions of f and g implies (as above) that f and g coincide at 0. This in turn implies that the graph of f or the graph of g has a "jump" at 0, although we do not know which. However, this does not matter because the function $h \equiv 1 - (f - g)^2$ has a jump at 0 in either case.

We may summarize this by asserting that, in a world in which all maps are continuous (e.g., the smooth topos), some existents must be conceived as being given solely *intensionally*.

6. HILBERT'S τ-OPERATOR

The ε-operator was not the first device invented by Hilbert to justify the use of classical reasoning in mathematics. For in 1923 he introduced what amounts to a dual form of the ε-operator, *the τ-operator*, which was governed by a principle he called the *Transfinite Axiom*[15]:

$$(\textbf{Trans}) \quad A(\tau_A) \rightarrow A(x).$$

That is, for each predicate $A(x)$, τ_A is an object which, if it satisfies $A(x)$, then anything does. One surmises that Hilbert abandoned the τ-operator in favour of the ε-operator because the latter (together with its governing ε-axiom) is more closely related to the axiom of choice, and consequently, more easily justified.

The τ-operator has the merit that it can be used to derive (**Q**). To obtain (**Q**) from (**Trans**), note that from the latter we can derive

$$\neg\forall x A(x) \rightarrow \neg A(\tau_A);$$

but

$$\neg A(\tau_A) \vdash \exists x \neg A(x)$$

and (**Q**) follows. Thus in an existential sense the τ-operator is stronger than the ε-operator (although (**Trans**) alone will not yield (**ŁEM**): see § 7).

It is interesting to examine a specific case which the strength of the τ-operator is made evident. In § 5 of Kolmogorov's 1925 paper he considers some examples of mathematical propositions unprovable without the use of the law of exluded middle. One such proposition is the following: every point not belonging to a given closed set (in $\mathbb{R}$) is contained in an interval disjoint from the set. Let us see how this can be derived from (**Trans**). Given a closed set C and a point p, let I be a variable ranging over intervals containing p, and let $\Phi(I)$ be the formula

$$\exists x(x \in I \wedge x \in C).$$

Then since C is closed, we have

$$(*) \quad \vdash \forall I \Phi(I) \rightarrow p \in C.$$

Now let I_0 be τ_Φ; then I_0 is an interval and

$$\vdash \Phi(I_0) \rightarrow \forall I \Phi(I).$$

This together with (*) yields

$$\vdash \Phi(I_0) \rightarrow p \in C$$

whence

$$\vdash p \notin C \rightarrow \neg\Phi(I_0).$$

So, if $p \notin C$, I_0 is an interval containing p disjoint from C.

It is clear that the ε-operator will not yield this result directly.

7. A MODEL THEORY FOR THE INTUITIONISTIC ε-CALCULUS

Let $\mathscr{L}$ be a first-order language with equality which for simplicity we will assume has just one unary predicate symbol P. The language $\mathscr{L}_\varepsilon$ has an additional symbol ε and an additional term forming scheme:

$$\varepsilon x\varphi \text{ is a term whenever } \varphi \text{ is a formula.}$$

We assume the usual axioms and rules of inference for first-order intuitionistic logic in $\mathscr{L}_\varepsilon$, together with the axiom scheme:

$$(\boldsymbol{\varepsilon}) \quad \exists x\varphi \rightarrow \varphi(x/\varepsilon x\varphi).$$

$\mathscr{L}_\varepsilon$, together with these axioms and rules of inference, will be called the *intuitionistic ε-calculus*. We write "$\vdash_{\overline{\varepsilon}} \varphi$" for "$\varphi$ is provable in this calculus".

Now let $L = \langle L, \leqslant \rangle$ be an inversely well-ordered set, i.e., such that every non-empty subset X has a largest element which we shall denote by $\bigvee X$. It follows that L has a largest element 1; we shall also assume that L has a least element 0. It follows that every subset X of L has a infimum $\bigwedge X$. L can then be easily shown to be a *complete Heyting algebra*, i.e., a complete lattice satisfying the distributive law

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} x \wedge y_i.$$

We can then define the operations $\Rightarrow$, $\Leftrightarrow$, $*$ on L by

$$x \Rightarrow y = \bigvee \{z : z \wedge x \leqslant y\}$$

$$x \Leftrightarrow y = (x \Rightarrow y) \wedge (y \Rightarrow x)$$

$$x^* = x \Rightarrow 0.$$

Let L be an inversely well-ordered set with least element, let M be a non-empty set, and let e be a choice function for the power set $\mathscr{P}M$ of M, that is, e is a map $\mathscr{P}M - \{\emptyset\} \rightarrow M$ such that $e(X) \in X$ for all $X \neq \emptyset$ in $\mathscr{P}M$. Define the map $\bar{e}$: $L^M \rightarrow M$ (where L^M is the set of all functions $M \rightarrow L$) as follows: for each $f \in L^M$, the set $f[M] =$

$\{f(m): m \in M\}$ has a largest element $l = \vee f[M]$. We put

$$\bar{e}(f) = e(f^{-1}(\{l\})).$$

Notice that then

$$(*) \quad f(\bar{e}(f)) = l = \vee f[M].$$

An $\mathscr{L}_\varepsilon$*-structure* is a system of the form

$$\mathfrak{M} = \langle M, L, e, eq_{\mathfrak{M}}, P_{\mathfrak{M}} \rangle$$

where

M is a non-empty set
L is an inversely well-ordered set with 0
e is a choice function for $\mathscr{P}M$
$P_{\mathfrak{M}}: M \to L$
$eq_{\mathfrak{M}}: M \times M \to L$ satisfies the equality laws:
$eq_{\mathfrak{M}}(m, m) = 1$, $eq_{\mathfrak{M}}(m, n) \wedge eq_{\mathfrak{M}}(n, p) \leqslant eq_{\mathfrak{M}}(m, p)$
$eq_{\mathfrak{M}}(m, n) = eq_{\mathfrak{M}}(n, m)$
$eq_{\mathfrak{M}}(m, n) \wedge P_{\mathfrak{M}}(m) \leqslant P_{\mathfrak{M}}(n)$.

Given an $\mathscr{L}_\varepsilon$-structure $\mathfrak{M}$ and a map α from the set Var of variables of $\mathscr{L}$ to M (which we shall call a *valuation* in M) we define, for each formula φ and each term t of $\mathscr{L}_\varepsilon$, the *value*

$$[\![\varphi]\!]^\alpha_{\mathfrak{M}} \in L \quad [t]^\alpha_{\mathfrak{M}} \in M$$

of φ and t under α in $\mathfrak{M}$ recursively as follows:

$$[x]^\alpha_{\mathfrak{M}} = \alpha(x) \quad \text{for} \quad x \in \text{Var}$$

$$[\![Pt]\!]^\alpha_{\mathfrak{M}} = P_{\mathfrak{M}}([t]^\alpha_{\mathfrak{M}})$$

$$[\![t = u]\!]^\alpha_{\mathfrak{M}} = eq_{\mathfrak{M}}([t]^\alpha_{\mathfrak{M}}, [u]^\alpha_{\mathfrak{M}})$$

$$[\![\varphi \wedge \psi]\!]^\alpha_{\mathfrak{M}} = [\![\varphi]\!]^\alpha_{\mathfrak{M}} \wedge [\![\psi]\!]^\alpha_{\mathfrak{M}}$$

$$[\![\varphi \vee \psi]\!]^\alpha_{\mathfrak{M}} = [\![\varphi]\!]^\alpha_{\mathfrak{M}} \vee [\![\psi]\!]^\alpha_{\mathfrak{M}}$$

$$[\![\varphi \to \psi]\!]^\alpha_{\mathfrak{M}} = [\![\varphi]\!]^\alpha_{\mathfrak{M}} \Rightarrow [\![\psi]\!]^\alpha_{\mathfrak{M}}$$

$$[\![\varphi \leftrightarrow \psi]\!]^\alpha_{\mathfrak{M}} = [\![\varphi]\!]^\alpha_{\mathfrak{M}} \Leftrightarrow [\![\psi]\!]^\alpha_{\mathfrak{M}}$$

$$[\![\neg \varphi]\!]^\alpha_{\mathfrak{M}} = ([\![\varphi]\!]^\alpha_{\mathfrak{M}})^*$$

$$[\![\exists x\varphi]\!]^{\alpha}_{\mathfrak{M}} = \bigvee_{m\in M} [\![\varphi]\!]^{\alpha(x/m)}_{\mathfrak{M}}$$

$$[\![\forall x\varphi]\!]^{\alpha}_{\mathfrak{M}} = \bigwedge_{m\in M} [\![\varphi]\!]^{\alpha(x/m)}_{\mathfrak{M}},$$

where $\alpha(x/m)$ is the map which coincides with α except possibly at x, where it assigns value m.

Finally,

$$[\varepsilon x\varphi]^{\alpha}_{\mathfrak{M}} = \bar{e}(h_{\varphi}),$$

where $h_{\varphi}: M \to L$ is defined by

$$h_{\varphi}(m) = [\![\varphi]\!]^{\alpha(x/m)}_{\mathfrak{M}}.$$

A formula φ is said to be $\mathfrak{M}$-*valid* if $[\![\varphi]\!]^{\alpha}_{\mathfrak{M}} = 1$ for every valuation α in M, and ε-*valid*, written $\models_{\varepsilon} \varphi$, if φ is $\mathfrak{M}$-valid for all $\mathfrak{M}$. We can now prove the

ε-Soundness Theorem.

$$\vdash_{\varepsilon} \varphi \Rightarrow \models_{\varepsilon} \varphi$$

for any formula φ.

Proof. The axioms and rules of inference for intuitionistic logic are well known[16] to be valid in any complete Heyting algebra valued structure, in particular any $\mathscr{L}_{\varepsilon}$-structure. So to prove the result we need only verify that the ε-axiom is valid in any $\mathscr{L}_{\varepsilon}$-structure.

To do this, we observe that

$$\bigvee h_{\varphi}[M] = \bigvee_{m\in M} [\![\varphi]\!]^{\alpha(x/m)}_{\mathfrak{M}} = [\![\exists x\varphi]\!]^{\alpha}_{\mathfrak{M}}$$

and

$$\begin{aligned}\bigvee h_{\varphi}[M] = h_{\varphi}(\bar{e}(h_{\varphi})) &= h_{\varphi}([\varepsilon x\varphi]^{\alpha}_{\mathfrak{M}}) \\ &= [\![\varphi]\!]^{\alpha(x/[\varepsilon x\varphi]^{\alpha}_{\mathfrak{M}})}_{\mathfrak{M}} \\ &= [\![\varphi(x/\varepsilon x\varphi)]\!]^{\alpha}_{\mathfrak{M}}.\end{aligned}$$

Therefore

$$\begin{aligned}&[\![\exists x\varphi \leftrightarrow \varphi(x/\varepsilon x\varphi)]\!]^{\alpha}_{\mathfrak{M}} \\ &\quad = [\![\exists x\varphi]\!]^{\alpha}_{\mathfrak{M}} \Leftrightarrow [\![\varphi(x/\varepsilon x\varphi)]\!]^{\alpha}_{\mathfrak{M}} \\ &\quad = 1,\end{aligned}$$

as required. ∎

We observe parenthetically that the converse of this result, which would constitute a completeness theorem is, unfortunately, false. For it is not hard to see that the sentence

$$\varepsilon x(x = x) \;=\; \varepsilon x(x \neq x)$$

is ε-valid; but it is known[17] that this sentence is not deducible even in the classical ε-calculus. (The provision of a complete semantics for the intuitionistic ε-calculus is an open problem.)

COROLLARY. *The following schemes are not provable in the intuitionistic ε-calculus*:

(i) $\neg \forall x \varphi \rightarrow \exists x \neg \varphi$

(ii) $\varphi \vee \neg \varphi$.

Proof. (i) By the Soundness Theorem, it suffices to find an $\mathscr{L}_\varepsilon$-structure $\mathfrak{M}$, a formula φ of $\mathscr{L}_\varepsilon$ and a valuation α in M such that, in L,

$$\left(\bigwedge_{m \in M} [\![\varphi]\!]_{\mathfrak{M}}^{\alpha(x/m)}\right)^* \Rightarrow \bigvee_{m \in M} ([\![\varphi]\!]_{\mathfrak{M}}^{\alpha(x/m)})^* \neq 1.$$

A necessary and sufficient condition for this is that

$$(*) \quad \left(\bigwedge_{m \in M} [\![\varphi]\!]_{\mathfrak{M}}^{\alpha(x/m)}\right)^* \nleq \bigvee_{m \in M} ([\![\varphi]\!]_{\mathfrak{M}}^{\alpha(x/m)})^*.$$

First we speicfy L. L is to be the order topology on the set Neg of negative integers. That is, L consists of $\emptyset$ together with all subsets of Neg of the form

$$\{x \in \mathrm{Neg} : x \leqslant -n\} \;=\; (\leftarrow, -n]$$

for $n \in \omega$, ordered by inclusion. It is easy to check that L is an inversely well-ordered set.

We further specify:

M = the set ω of natural numbers
$eq_{\mathfrak{M}}(m, n)$ = Neg if $m = n$
$= \emptyset$ if $m \neq n$
$P_{\mathfrak{M}}(n) = (\leftarrow, -n]$

$$e(X) = \text{least element of } X(\subseteq \omega)$$
$$\alpha(x_n) = n \text{ for } n \in \omega$$

where x_n is the n^{th} variable of $\mathscr{L}$.

Now let φ be Px. Then

$$[\![\varphi]\!]_{\mathfrak{M}}^{\alpha(x/m)} = (\leftarrow, -m]$$
$$([\![\varphi]\!]_{\mathfrak{M}}^{\alpha(x/m)})^* = (\leftarrow, -m]^* = \emptyset$$
$$\bigwedge_{m\in M} [\![\varphi]\!]_{\mathfrak{M}}^{\alpha(x/m)} = \bigwedge_{m\in M} (\leftarrow, -m] = \emptyset$$
$$\bigvee_{m\in M} ([\![\varphi]\!]_{\mathfrak{M}}^{\alpha(x/m)})^* = \emptyset$$
$$\left(\bigwedge_{m\in M} [\![\varphi]\!]_{\mathfrak{M}}^{\alpha(x/m)}\right)^* = \emptyset^* = \text{Neg}$$

Since Neg $\not\subseteq \emptyset$, we see that (*) holds, completing the proof of (i). (ii) follows from (i), since **(Q)** is a consequence of **(LEM)**. ■

It is easy to extend the proof of the Corollary to show that neither (i) nor (ii) is deducible in the intuitionistic ε-calculus even when we add as an axiom the assumption that $\mathbf{0} \neq \mathbf{1}$ (or, indeed, principles **(D)** or **(D′)**). Therefore *the principle of extensionality is an indispensible assumption in the derivation of* **(LEM)**, *and also of* **(Q)**.

Finally, we remark that a similar model theory can be built for the τ-symbol and a soundness theorem proved, only using well-ordered sets in place of inversely well-ordered sets. The independence of the law of excluded middle (but not, as we have seen, of **(Q)**) in the corresponding τ-calculus then follows.

ACKNOWLEDGMENTS

I am grateful to William Demopoulos and David de Vidi for helpful discussions on the contents of this paper.

NOTES

[1] See, e.g. [7], where in §4.8 Hilbert is quoted as follows: "The essential idea on which the axiom of choice is based constitutes a general logical principle which, even for the first elements of mathematical inference, is indispensable".

[2] See especially [3], [4].

[3] My notation differs inessentially from Hilbert's.

[4] See [4]. It should be emphasized that for the technical purposes of this paper the quantifiers are regarded as being given *in addition* to the ε-operator, and are *not* regarded as being defined in terms of it.

[5] We write "$\vdash_\varepsilon \varphi$" for "$\varphi$ is deducible from (ε) in the intuitionistic predicate calculus".

[6] Hilbert explicitly assumes (¬¬) in [3] and [4].

[7] Kolmogorov shows that (¬¬) implies (Q) in §4 of [5]. He also questions Hilbert's adoption of (¬¬): see footnote 4 of [5].

[8] See, e.g. [1], Ch. III, §1.

[9] In section IV of [4].

[10] I am not sure who first formulated the extensionality principle, but it is usualy attributed to Ackermann: see the introduction to [6].

[11] This lemma is a "stripped down" version of a result of Diaconescu in topos theory: within a topos, the axiom of choice implies the law of excluded middle. (See Thm. 1.1 of Ch. VIII of [1].)

[12] See, e.g. [2] or [8].

[13] It is worth pointing out here that postulating the existence of extensions of predicates does not suffice to yield (**LEM**), for there exist perfectly respectable intuitionistic versions of *set theory* in which (**LEM**) fails: see, e.g., [1]. In such set theories, the "stripped down" version of Diaconescu's theorem (see footnote 11) may be stated: *if the power set of* $\{0, 1\}$ *has a choice function, then the law of excluded middle holds.*

[14] See, e.g., [1].

[15] See §4.8 of [7].

[16] See, e.g., [8].

[17] See §5 of Ch. I of [6].

REFERENCES

[1] Beeson, M. J., *Foundations of Constructive Mathematics*. Springer-Verlag, Berlin, 1985.

[2] Dummett, M., *Elements of Intuitionism*. Clarendon Press, Oxford, 1977.

[3] Hilbert, D., 'On the infinite' (1925). In: van Heijenoort [9].

[4] Hilbert, D., 'The foundations of mathematics' (1928). In van Heijenoort [9].

[5] Kolmogorov, A. N., 'On the principle of excluded middle' (1925). In: van Heijenoort [9].

[6] Leisenring, A. C., *Mathematical Logic and Hilbert's ε-Symbol*. Gordon & Breach, New York, 1969.

[7] Moore, G. H., *Zermelo's Axiom of Choice*. Springer-Verlag, New York, 1982.

[8] Rasiowa, H. and Sikorski, R., *The Mathematics of Metamathematics*. PWN, Warsaw, 1963.

[9] van Heijenoort, J. (ed). *From Frege to Gödel*. Harvard University Press, Cambridge, 1967.

Department of Philosophy,
University of Western Ontario,
Talbot College,
London, Canada,
N6A 3KY.