Finite Sets and Frege Structures[1]

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Call a family $\mathscr{F}$ of subsets of a set E *inductive* if $\varnothing \in \mathscr{F}$ and $\mathscr{F}$ is *closed under unions with disjoint singletons*, that is, if $\forall X \in \mathscr{F} \forall x \in E{-}X(X \cup \{x\} \in \mathscr{F}]$. A *Frege structure* is a pair (E, ν) with ν a map to E whose domain $\mathrm{dom}(\nu)$ is an inductive family of subsets of E such that $\forall X\,Y \in \mathrm{dom}(\nu)(\nu(X) = \nu(Y) \Leftrightarrow X \approx Y)$.[2] In [1] it is shown in a constructive setting that each Frege structure determines a subset which is the domain of a model of Peano's axioms. In this note we establish, within the same constructive setting, three facts. First, we show that the least inductive family of subsets of a set E is precisely the family of *decidable Kuratowski finite* subsets of E. Secondly, we establish that the procedure presented in [1] can be reversed, that is, any set containing the domain of a model of Peano's axioms determines a map which turns the set into a *minimal* Frege structure: here by a minimal Frege structure is meant one in which $\mathrm{dom}(\nu)$ is the least inductive family of subsets of E. And finally, we show that the procedures leading from minimal Frege structures to models of Peano's axioms and vice-versa are *mutually inverse*. It follows that the postulation of a (minimal) Frege structure is constructively equivalent to the postulation of a model of Peano's axioms.

All arguments will be formulated within constructive (intuitionistic) set theory[3].

1. Some definitions of finiteness. Fix a set E. By "set" "family", "singleton", etc. we shall mean "subset of E", "family of subsets of E", "singleton of E", etc. For a set X define

$K(X) \Leftrightarrow X$ is in every family containing $\varnothing$, all singletons, and closed under unions of pairs of its members. If $K(X)$, we shall say that X is *Kuratowski finite*.

$L(X) \Leftrightarrow X$ is in every family containing $\varnothing$ and closed under unions with singletons. If $L(X)$, we shall say that X is *finite*.

$M(X) \Leftrightarrow X$ is in every inductive family. If $M(X)$, we shall say that X is *strictly finite*.

$D(X) \Leftrightarrow$ for every $x, y \in X$, either $x = y$ or $x \neq y$. If $D(X)$, X is said to be *decidable*.

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[2] We write $X \approx Y$ for *there exists a bijection between X and Y.*

[3] This set theory may be taken to be intuitionistic Zermelo set theory or the local set theory presented in [2].

1.1. *Lemma.* $\forall X[\boldsymbol{M(X)} \Rightarrow \boldsymbol{L(X)}]$.
Proof. Obvious.

1.2. *Lemma.* $\forall X[\boldsymbol{K(X)} \Leftrightarrow \boldsymbol{L(X)}]$.
Proof. Clearly $\boldsymbol{L(X)} \Rightarrow \boldsymbol{K(X)}$. To prove the converse, it suffices to show that the family $\mathscr{L}$ = $\{X: \boldsymbol{L(X)}\}$ is closed under unions of pairs. To this end let $\Phi(U)$ be the property $\forall X \in \mathscr{L}\; U \cup X \in \mathscr{L}$. It suffices to show $\forall U[\boldsymbol{L(U)} \Rightarrow \Phi(U)]$. Clearly $\Phi(\varnothing)$. Assuming $\Phi(U)$ and $X \in \mathscr{L}$ we have $U \cup X \in \mathscr{L}$ and so $U \cup X \cup \{x\} \in \mathscr{L}$ for arbitrary x, whence $\Phi(U \cup \{x\})$. Hence $\forall U[\boldsymbol{L(U)} \Rightarrow \Phi(U)]$ and the result follows.

1.3. *Lemma.* $\forall X[\boldsymbol{M(X)} \Rightarrow \boldsymbol{D(X)}]$.
Proof. Obviously $\boldsymbol{D(\varnothing)}$. If $\boldsymbol{D(X)}$ and $x \notin X$, clearly $\boldsymbol{D(X \cup \{x\})}$. The result follows.
1.4. *Lemma.* $\forall X[\boldsymbol{M(X)} \Rightarrow \forall a[\boldsymbol{D(X \cup \{a\})} \Rightarrow (a \in X \text{ or } a \notin X)]]$.
Proof. Write $\Phi(X)$ for the condition following the first implication. Clearly $\Phi(\varnothing)$. Suppose that $\Phi(X)$ and $x \notin X$. If $\boldsymbol{D(X \cup \{x\} \cup \{a\})}$, then $\boldsymbol{D(X \cup \{a\})}$, so, since $\Phi(X)$, either $a \in X$ or $a \notin X$. Since $\boldsymbol{D(X \cup \{x\} \cup \{a\})}$, it follows that $a = x$ or $a \neq x$. Hence
$$(a \in X \;\&\; a = x) \text{ or } (a \notin X \;\&\; a = x) \text{ or } (a \in X \;\&\; a \neq x) \text{ or } (a \notin X \;\&\; a \neq x),$$
The first three disjuncts each imply $a \in X \cup \{x\}$, and the last disjunct means $a \notin X \cup \{x\}$. Accordingly $a \in X \cup \{x\}$ *or* $a \notin X \cup \{x\}$. We conclude that $\Phi(X \cup \{x\})$ and the result follows.

1.5. *Lemma.* $\forall X[\boldsymbol{L(X)} \;\&\; \boldsymbol{D(X)} \Rightarrow \boldsymbol{M(X)}]$.
Proof. We need to show $\forall X[\boldsymbol{L(X)} \Rightarrow \Phi(X)]$, where $\Phi(X)$ is $\boldsymbol{D(X)} \Rightarrow \boldsymbol{M(X)}$. Clearly $\Phi(\varnothing)$. Assume $\Phi(X)$ and $\boldsymbol{D(X \cup \{a\})}$. Then $\boldsymbol{D(X)}$, so, since $\Phi(X)$, it follows that $\boldsymbol{M(X)}$. Since $\boldsymbol{D(X \cup \{a\})}$, 1.4 gives $a \in X$ or $a \notin X$. In either case we deduce that $\boldsymbol{M(X \cup \{a\})}$. Hence $\Phi(X \cup \{a\})$, and the result follows.

From these lemmas we immediately infer

1.6. *Theorem.* For any set E, the families of strictly finite, decidable finite, and decidable Kuratowski finite subsets coincide. ∎

2. Frege structures from models of Peano's axioms. Let $(N, s, 0)$ be a model of Peano's axioms; we use letters m, n, p as variables ranging over N. We write $<$ for the usual (constructively definable) strict order relation on N (see, e.g., Prop. 7.5 of [2]). Define $g: N \to PN$ by $g(n) = \{m: m < n\}$; then g satisfies (and in fact can be defined by) the equations

$$g(0) = \varnothing \qquad g(sn) = g(n) \cup \{n\}.$$

In order to prove the next Lemma we require the following fact, which is proved constructively as Lemma 3 of [1]:

(*)   *for any sets X, Y, and any x, y such that $x \notin X$, $y \notin Y$, if $X \cup \{x\} \approx Y \cup \{y\}$,*

   then $X \approx Y$.

2.1. *Lemma*. For all m, n, $g(m) \approx g(n) \Leftrightarrow m = n$.

Proof. Write $\Phi(n)$ for $\forall m[g(m) \approx g(n) \Leftrightarrow m = n]$. Then clearly $\Phi(0)$. If $\Phi(n)$ and $g(m) \approx g(sn) = g(n) \cup \{n\}$, then $m \neq 0$ so that $m = sp$ for some p. Hence

$$g(p) \cup \{p\} = g(sp) = g(m) \approx g(sn) = g(n) \cup \{n\}.$$

Since $p \notin g(p)$ and $n \notin g(n)$, (*) above implies that $g(p) \approx g(n)$, so, since $\Phi(n)$, $p = n$ and $m = sp = sn$. Hence $\Phi(sp)$, and the result follows by induction.

Now suppose that E is a set such that $N \subseteq E$. Define

$$\nu = \{(X,n) \in PE \times N: X \approx g(n)\}.$$

2.2. *Lemma.* $\mathrm{dom}(\nu)$ is the family of strictly finite subsets of E.

Proof.   We need to show that $\mathrm{dom}(\nu)$ is the least inductive family. First, $\mathrm{dom}(\nu)$ is inductive. For it clearly contains $\varnothing$ and, if $X \in \mathrm{dom}(\nu)$, then $X \approx g(n)$ for some n. If $x \notin X$, then $X \cup \{x\} \approx g(n) \cup \{n\} = g(sn)$, whence $X \cup \{x\} \in \mathrm{dom}(\nu)$. And $\mathrm{dom}(\nu)$ is the least inductive family. For suppose that $\mathscr{F}$ is any inductive family. For each n let $\mathscr{H}_n = \{X: X \approx g(n)\}$. We claim that $\mathscr{H}_n \subseteq \mathscr{F}$ for all n. For obviously $\mathscr{H}_0 = \{\varnothing\} \subseteq \mathscr{F}$. Now suppose that $\mathscr{H}_n \subseteq \mathscr{F}$. If $X \approx g(sn)$, then $X \approx g(n) \cup \{n\}$, so for some $x \in X$ (which may be taken to be the image of n under the bijection $g(n) \cup \{n\} \approx X$), we have $X - \{x\} \approx g(n)$. It follows that $X - \{x\} \in \mathscr{H}_n \subseteq \mathscr{F}$, and so $X = (X - \{x\}) \cup \{x\} \in \mathscr{F}$. The claim now follows by induction; accordingly $\mathrm{dom}(\nu)$, as the union of all the $\mathscr{H}_n$, is included in $\mathscr{F}$. Therefore $\mathrm{dom}(\nu)$ is the least inductive family and the Lemma is proved.

2.3. *Lemma*. ν is a function and $X \approx g(\nu(X))$ for all $X \in \mathrm{dom}(\nu)$.

Proof. Suppose that $(X,n) \in \nu$ and $(X,m) \in \nu$. Then $X \approx g(n)$ and $X \Phi g(m)$ whence $g(m) \approx g(n)$ and so $m = n$ by Lemma 2.1. The remaining claim is obvious.

2.4. *Lemma.* For all $X, Y \in \mathrm{dom}(\nu)$, $X \approx Y \Leftrightarrow \nu(X) = \nu(Y)$.

    Proof. We have, using the previous Lemma, $\nu(X) = \nu(Y) \Leftrightarrow g(\nu(X)) \approx g(\nu(Y)) \Leftrightarrow$ $X \approx Y$.

    Lemmas 2.2 - 2.4 establish

    2.5. *Theorem. (E, ν)* is a minimal Frege structure. ∎

    (E,ν) is called the minimal Frege structure *associated* with E and *(N, s, 0)*.

    Finally, we show that the processes of deriving models of Peano's axioms from minimal Frege structures and *vice-versa* are mutually inverse.

    Suppose that we are given a minimal Frege structure *(E,μ)*. As shown in [1], the associated model *(N,s,0)* of Peano's axioms is obtained in the following way. First, the family $\mathbb{N}$ is defined as the least subfamily of $\mathrm{dom}(\mu)$ containing $\varnothing$ and such that, if $X \in \mathbb{N}$ and $\mu(X) \notin X$, then $X \cup \{\mu(X)\} \in \mathbb{N}$: it is shown that $\mu(X) \notin X$ for all $X \in \mathbb{N}$. The associated model *(N,s,0)* of Peano's axioms is then defined by $N = \{\mu(X) : X \in \mathbb{N}\}$, $s(\mu(X)) = \mu(X \cup \mu(X)\})$, and $0 = \mu(\varnothing)$.

    We observe that since *(E,μ)* is minimal, for any $X \in \mathrm{dom}(\mu)$ there is a (unique) $X^* \in \mathbb{N}$ for which $X \approx X^*$, and so $\mu(X) = \mu(X^*)$. To prove this, it suffices to show that the set of $X \in \mathrm{dom}(\mu)$ with this property contains $\varnothing$ and is closed under unions with disjoint singletons. The first claim is obvious. If $X \in \mathrm{dom}(\mu)$, $x \notin X$, and $X \approx X^*$ with $X^* \in \mathbb{N}$, then

$$X \cup \{x\} \approx X^* \cup \{\mu(X^*)\} \in \mathbb{N},$$

since, as observed above, $\mu(X^*) \notin X^*$. This establishes the second claim, and the observation.

    Now let *(E,ν)* be the minimal Frege structure associated with the model *(N,s,0)* of Peano's axioms in turn associated with *(E,μ)*. We claim that $\mu = \nu$. To prove this it suffices to show that

$$(**) \quad X \approx g(\mu(X)) \text{ for all } X \in \mathbb{N},$$

where $\mathbb{N}$ is defined as above. For then, by Lemma 2.3, we will have $g(\nu(X)) \approx X \approx g(\mu(X))$ and so $\mu(X) = \nu(X)$ by Lemma 2.1. This last equality for all $X \in \mathbb{N}$ in turn yields $\mu(Y) = \nu(Y)$ for all $Y \in \mathrm{dom}(\mu) = \mathrm{dom}(\nu)$. For then, by our observation above, $\mu(Y) = \mu(Y^*) = \nu(Y^*) = \nu(Y)$.

    So it only remains to prove (**). It is clearly satisfied by $\varnothing$. If $X \approx g(\mu(X))$ with $X \in \mathbb{N}$, then, since $\mu(X) \notin X$,

$$X \cup \{\mu(X)\} \approx g(\mu(X)) \cup \{\mu(X)\} = g(s\mu(X)).$$

(**) now follows from the definition of $\mathbb{N}$. So our claim that $\mu = \nu$ is established.

Conversely, suppose we are given a set E and a model *(N, s, 0)* of Peano's axioms with $N \subseteq E$. Let *(E,ν)* be the associated minimal Frege structure. We note first that, for any $n \in N$, we have $\nu(g(n)) = n$, where, as before, $g(n) = \{m: m < n\}$. For by Lemma 2.3, $g(n) \approx g(\nu(g(n)))$, so that, by Lemma 2.1., $n = g(\nu(n))$. Now let *(N*,s*,0*)* be the model of Peano's axioms associated with the Frege structure *(E,ν)*. We claim that *(N,s,0)* and *(N*,s*,0*)* are identical.

First, $N^* = \{\nu(X): X \in \mathbb{N}^*\}$, where $\mathbb{N}^*$ is the least subfamily of dom(ν) containing $\varnothing$ and such that $X \in \mathbb{N}^*$ and $\nu(X) \notin X$ implies $X \cup \{\nu(X)\} \in \mathbb{N}^*$. Using the fact that $\nu(g(n)) = n$ for all $n \in N$, it is easily shown that $\mathbb{N}^* = \{g(n): n \in N\}$. Thus $N^* = \{\nu(X): X \in \mathbb{N}^*\} = \{\nu(g(n)): n \in N\} = \{n: n \in N\} = N$. Finally $0^* = \nu(0) = \nu(g(0)) = 0$ and

$$s^*(n) = s^*(\nu(g(n))) = \nu(g(n) \cup \{\nu(g(n))\}) = \nu(g(n) \cup \{n\}) = \nu(g(sn)) = sn,$$

so that $s^* = s$.

Thus we have established that the two processes are mutually inverse.

References

[1] Bell, John L. *Frege's Theorem in a Constructive Setting. Journal of Symbolic Logic,* vol. 64, no. 2, 1999

[2] Bell, John L. *Toposes and Local Set Theories: An Introduction.* Oxford University Press, 1988.