Math. Log. Quart. 39 (1993) 323 – 337

Mathematical Logic
Quarterly

Hilbert's ε-Operator in Intuitionistic Type Theories

John L. Bell
Department of Philosophy, The University of Western Ontario,
London, Canada N6A 3K7[1)]

Abstract

We investigate Hilbert's ε-calculus in the context of intuitionistic type theories, that is, within certain systems of intuitionistic higher-order logic. We determine the additional deductive strength conferred on an intuitionistic type theory by the adjunction of closed ε-terms. We extend the usual topos semantics for type theories to the ε-operator and prove a completeness theorem. The paper also contains a discussion of the concept of "partially defined" ε-term.
MSC: 03B15, 03B20, 03G30.
Keywords: ε-calculus, Intuitionistic type theories.

1. Introduction

We investigate Hilbert's ε-calculus ([1], [6]) in the context of intuitionistic type theories, that is, within certain systems of intuitionistic higher-order logic. The paper is organized as follows: Section 2, which is chiefly expository in nature, contains a compressed account (following [2]) of the basic framework of type theory we shall employ, and of its semantics. In Section 3 we introduce the ε-operator. Noting that, in contrast with the classical case, the ε-calculus is in general *not* conservative over intuitionistic systems (see, e.g., [3]), we determine the additional deductive strength conferred on an intuitionistic type theory by the adjunction of (closed) ε-terms. In Section 4 we introduce and develop the concept of "partially defined" ε-terms. In Section 5 we extend the topos semantics for type theories to the ε-operator and prove a completeness theorem. Section 6 contains some examples establishing the independence of various concepts introduced in the paper. Finally, in Section 7 we explain why we have confined attention to *closed* ε-terms.

2. A framework for intuitionistic type theories

We summarize briefly the system presented in [2]. A *language* L *for intuitionistic type theory*, or a *language* for short, has the following ingredients:

[1)] I am grateful to JIM LAMBEK whose stimulating correspondence and paper [5] exerted a strong influence on the present work.

Basic symbols.

$\mathbf{1}$ (unit type),

$\mathbf{\Omega}$ (truth value type),

$\boldsymbol{S}, \boldsymbol{T}, \boldsymbol{U}, \ldots$ (ground types),

$\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, \ldots$ (function symbols).

Types. These are members of the smallest class containing $\mathbf{1}$, $\mathbf{\Omega}$ and the ground types and closed under *products* and *powers*; here the product of two types $\boldsymbol{A}$ and $\boldsymbol{B}$ is denoted by $\boldsymbol{A} \times \boldsymbol{B}$ and the power of a type $\boldsymbol{A}$ is denoted by $\mathbf{P}\boldsymbol{A}$.

Signatures. Each function symbol $\boldsymbol{f}$ is assigned a pair of types called its *signature*. Notation: $\boldsymbol{f} : \boldsymbol{A} \to \boldsymbol{B}$.

Terms and their associated **types** are specified as follows:

(i) $\#$ is a term of type $\mathbf{1}$, and for each type $\boldsymbol{A}$ we are given a list of variables $x, y, z, \ldots$ of that type;

(ii) terms are closed under the following operations (where $\tau : \boldsymbol{A}$ indicates that the term τ has type $\boldsymbol{A}$):

$$\langle\sigma,\tau\rangle : \boldsymbol{A} \times \boldsymbol{B} \quad \text{for } \sigma : \boldsymbol{A} \text{ and } \tau : \boldsymbol{B},$$

$$\boldsymbol{f}(\tau) : \boldsymbol{B} \quad \text{for } \tau : \boldsymbol{A} \text{ and } \boldsymbol{f} : \boldsymbol{A} \to \boldsymbol{B},$$

$$\{x : \alpha\} : \mathbf{P}\boldsymbol{A} \quad \text{for } x : \boldsymbol{A} \text{ and } \alpha : \mathbf{\Omega},$$

$$\sigma = \tau : \mathbf{\Omega} \quad \text{for } \sigma, \tau : \boldsymbol{A},$$

$$\sigma \in \tau : \mathbf{\Omega} \quad \text{for } \sigma : \boldsymbol{A} \text{ and } \tau : \mathbf{P}\boldsymbol{A}.$$

Formulas. These are the terms of type $\mathbf{\Omega}$.

Sentences are, as usual, formulas without free variables, i.e., variables x not appearing in a context of the form $\{x : \alpha\}$.

We use the letters α, β, γ to denote formulas and write $\alpha(x/\tau)$ for the result of substituting τ for x at each of the latter's free occurences in α.

Axioms and rules of inference. We adopt a sequent notation, writing $\Gamma|\alpha$ for the sequent composed of a finite set Γ of formulas and a formula α, and $|\alpha$ for $\emptyset|\alpha$. The *axioms* for L are, writing $\alpha \leftrightarrow \beta$ for $\alpha = \beta$,

$$|x = \# \quad (\text{with } x : \mathbf{1}),$$

$$x = y, \alpha(z/x)|\alpha(z/y) \quad (\text{with } x, y \text{ free for } z \text{ in } \alpha),$$

$$\langle x, y\rangle = \langle x', y'\rangle | x = x',$$

$$\langle x, y\rangle = \langle x', y'\rangle | y = y',$$

$$|x \in \{x : \alpha\} \leftrightarrow \alpha.$$

The *rules of inference* are

$$\frac{\Gamma|\alpha \quad \alpha, \Gamma|\beta}{\Gamma|\beta} \quad (\text{provided all free variables of } \alpha \text{ appear free in the conclusion}),$$

$$\frac{\Gamma|\alpha}{\beta, \Gamma|\alpha},$$

$$\frac{\Gamma|\alpha}{\Gamma(x/\tau)|\alpha(x/\tau)} \quad (\text{with } \tau \text{ free for } x \text{ in } \alpha \text{ and all the members of } \Gamma),$$

$$\frac{\Gamma|x \in \sigma \leftrightarrow x \in \tau}{\Gamma|\sigma = \tau} \quad (\text{provided } x \text{ is not free in conclusion}),$$

$$\frac{\alpha, \Gamma|\beta \quad \beta, \Gamma|\alpha}{\Gamma|\alpha \leftrightarrow \beta}.$$

These axioms and rules of inference yield a system of *natural deduction* in L. If is any collection of sequents in L, we say that the sequent $\Gamma|\alpha$ is *derivable* from and write $\Gamma \vdash_S \alpha$, provided there is a derivation of $\Gamma|\alpha$ using the basic axioms, t sequents in S, and the rules of inference. For $\Gamma \vdash_\emptyset \alpha$ we write $\Gamma \vdash \alpha$, and for $\emptyset \vdash_S$ we write $\vdash_S \alpha$. A *theory* in L is a collection of sequents closed under derivabili A theory in some typed intuitionistic language L will be called a *type theory*. S, T are type theories in languages L, L' with $S \subseteq T$, $\mathsf{L} \subseteq \mathsf{L}'$, then T is said to be *conservative extension* of S if, for any sequent $\Gamma|\alpha$ of L, we have $\Gamma \vdash_T \alpha \Rightarrow \Gamma \vdash_S \alpha$

Logical operators in L are defined as follows:

$$\text{true} \equiv \# = \#,$$

$$\alpha \wedge \beta \equiv \langle\alpha, \beta\rangle = \langle\text{true}, \text{true}\rangle,$$

$$\alpha \to \beta \equiv (\alpha \wedge \beta) \leftrightarrow \alpha,$$

$$\forall x \alpha \equiv \{x : \alpha\} = \{x : \text{true}\},$$

$$\text{false} \equiv \forall u.\, u = \text{true} \quad (\text{with } u : \mathbf{\Omega}),$$

$$\neg\alpha \equiv \alpha \to \text{false},$$

$$\alpha \vee \beta \equiv \forall u[(\alpha \to u \wedge \beta \to u) \to u] \quad (\text{with } u : \mathbf{\Omega} \text{ not in } \alpha, \beta),$$

$$\exists x \alpha \equiv \forall u[\forall x(\alpha \to u) \to u] \quad (\text{with } u : \mathbf{\Omega} \text{ not in } \alpha).$$

Other logical operators such as $\exists! x$ and set-theoretic terms such as $\{x\}$ can be int duced in the usual way. It can then be shown that the theorems of (free) higher-or intuitionistic logic are derivable in L (see [2], Ch. 3).

Convention. Whenever a formula is introduced as $\alpha(x, y, \ldots)$, we shall supp that all the free variables of α occur among $x, y, \ldots$.

The natural domains of interpretation for type theories are the *toposes* (see Ch. 2). These are categories possessing a terminal object 1, a subobject classifier finite products, and exponentials of the form Ω^A (which we shall write as $\mathrm{P}A$). *interpretation* I of L in a topos E is an assignment

· to each type $\boldsymbol{A}$ of an E-object $\boldsymbol{A}_I$ in such a way that $\mathbf{1}_I$=1, $\boldsymbol{\Omega}_I = \Omega$,
$\times \boldsymbol{B})_I = \boldsymbol{A}_I \times \boldsymbol{B}_I$, $(\mathbf{P}\boldsymbol{A})_I = \mathrm{P}(\boldsymbol{A}_I)$;

· to each function symbol $\boldsymbol{f} : \boldsymbol{A} \rightarrow \boldsymbol{B}$ of an E-arrow $\boldsymbol{f}_I : \boldsymbol{A}_I \rightarrow \boldsymbol{B}_I$.

In [2], Ch. 3, it is shown how to extend any interpretation I to arbitrary terms in ch a way that, if $\tau : \boldsymbol{B}$ has variables $x_1 : \boldsymbol{A}_1, \ldots, x_n : \boldsymbol{A}_n$, I assigns to τ an E-arrow $: (\boldsymbol{A}_1)_I \times \cdots \times (\boldsymbol{A}_n)_I \rightarrow \boldsymbol{B}_I$. Taking τ to be a formula leads to a notion of *validity* a sequent $\Gamma|\alpha$ under I, written $\Gamma \models_I \alpha$. I is said to be a *model* of S if $\Gamma \models_I \alpha$ ienever $\Gamma \vdash_S \alpha$.

Any type theory S determines a topos $\mathsf{C}(S)$, the *topos of S-sets and maps.* The jects of $\mathsf{C}(S)$ are all terms of type of the form $\mathbf{P}\boldsymbol{A}$ and the arrows all such terms iich are S-provably functional relations. There is a natural interpretation $C(S)$ of e language of S in $\mathsf{C}(S)$ which is also a model of S; in fact we have $\Gamma \vdash_S \alpha$ iff $\models_{C(S)} \alpha$. This fact leads to the *Basic Completeness Theorem for Type Theories*, mely

$$\Gamma \vdash_S \alpha \quad \text{iff} \quad \Gamma \models_I \alpha \text{ for every model } I \text{ of } S.$$

The procedure of associating a topos with a type theory can be reversed. Given topos E, we can associate with it a language L(E) called its *internal language*. ughly speaking, L(E) has the objects of E as types and the arrows of E as function mbols. There is a natural interpretation E of L(E) in E; the *theory* of E, Th(E), is e collection of sequents of L(E) valid under this interpretation. It is then the case at $\Gamma \vdash_{\mathrm{Th(E)}} \alpha$ iff $\Gamma \models_E \alpha$. For any topos E, the theory Th(E) can be shown to be *ell-termed*, i.e., if $\vdash_{\mathrm{Th(E)}} \exists! x\alpha(x)$, then $\vdash_{\mathrm{Th(E)}} \alpha(x/\tau)$ for some (closed) term τ.

We shall need the following fundamental result (proved in [5] for a somewhat fferent system of type theory than the present one). Call a type theory S *witnessed* for any formula $\alpha(x)$, $\vdash_S \exists x\alpha$ implies $\vdash_S \alpha(x/\tau)$ for some (closed) term τ.

Theorem 2.1. *Any type theory has a conservative witnessed extension.*

Proof. We first perform the following general construction. Let Σ be a set of rmulas of L such that each $\alpha \in \Sigma$ has at most one free variable $x_\alpha : \boldsymbol{A}_\alpha$. We shall op the subscript "α" from x_α thus writing "x" for "x_α", "x_1" for "x_{α_1}", etc. Let Σ) the language obtained from L by adding for each $\alpha \in \Sigma$ a new function symbol $: \mathbf{1} \rightarrow \boldsymbol{A}_\alpha$; write c_α for $\boldsymbol{c}_\alpha(\#)$. Then c_α is a new closed term of type $\boldsymbol{A}_\alpha$ which we iall call the *indeterminate associated with α*. Let $S(\Sigma)$ be the theory in $\mathsf{L}(\Sigma)$ whose cioms are those of S together with all sequents of the form $|\beta(x_1/c_{\alpha_1}, \ldots, x_n/c_{\alpha_n})$, here $\alpha_1, \ldots, \alpha_n \in \Sigma$ and $\alpha_1, \ldots, \alpha_n \vdash_S \beta$. We remark that $\vdash_{S(\Sigma)} \alpha(x/c_\alpha)$, whence $_{S(\Sigma)} \exists x\alpha(x)$ for any $\alpha \in \Sigma$.

Claim. *If $\vdash_S \exists x\alpha(x)$ for all $\alpha \in \Sigma$, then $S(\Sigma)$ is a conservative extension of S.*

We sketch a proof of this claim. Suppose that $\Gamma|\gamma$ is a sequent of L and that $\vdash_{S(\Sigma)} \gamma$. Let $\mathcal{P}$ be a derivation in $S(\Sigma)$ of $\Gamma|\gamma$. Let $c_{\alpha_1}, \ldots, c_{\alpha_n}$ be the indetermi-ates occurring in $\mathcal{P}$ and let $y_1, \ldots, y_n$ be variables not appearing in $\mathcal{P}$ such that y_i id c_{α_i} have the same type for $i = 1, \ldots, n$. For each sequent $\Delta|\delta$ of $\mathsf{L}(\Sigma)$ let $\Delta'|\delta'$ e obtained by replacing each c_{α_i} by y_i. Now let $\mathcal{P}'$ be obtained from $\mathcal{P}$ be replac-g each of its sequents $\Delta|\delta$ by the sequent $\alpha_1(x_1/y_1), \ldots, \alpha_n(x_n/y_n), \Delta'|\delta'$. Then $'$ is a derivation of the sequent $\alpha_1(x_1/y_1), \ldots, \alpha_n(x_n/y_n), \Gamma|\gamma$ in the theory S aug-iented by some axioms of the form $\alpha_1(x_1/y_1), \ldots, \alpha_n(x_n/y_n)|\beta(x_1/y_1, \ldots, x_n/y_n)$,

where $|\beta(x_1/c_{\alpha_1}, \ldots, x_n/c_{\alpha_n})$ is an axiom of $S(\Sigma)$, i.e., where $\alpha_1, \ldots, \alpha_n \vdash_S \beta$. It follows that $\mathcal{P}'$ yields a derivation in S of $\alpha_1(x_1/y_1), \ldots, \alpha_n(x_n/y_n), \Gamma|\gamma$. Then

$$\exists y_1\alpha_1(x_1/y_1) \wedge \cdots \wedge \exists y_n\alpha_n(x_n/y_n), \Gamma \vdash_S \gamma.$$

So if $\vdash_S \exists x_i\alpha_i(x_i)$ for $i = 1, \ldots, n$, we get $\Gamma \vdash_S \gamma$ as required. □

Now, to prove the Theorem, take Σ to be the collection of formulas $\alpha(x)$ of L such that $\vdash_S \exists x\alpha(x)$. Put $S' = S(\Sigma)$, $\mathsf{L}'=\mathsf{L}(\Sigma)$. It follows from the Claim above that S' is a conservative extension of S. Define recursively $S_0 = S$, $S_{n+1} = (S_n)'$, $\mathsf{L}_0=\mathsf{L}$, $\mathsf{L}_{n+1} = (\mathsf{L}_n)'$ and put $T = \bigcup_{n\in\omega} S_n$ and $\mathsf{M} = \bigcup_{n\in\omega} \mathsf{L}_n$. It follows immediately by induction that T is a conservative extension of S in the language M. And finally, T is witnessed. For any given M-formula $\alpha(x)$ is in L_n for some n. If c_α is the indeterminate associated with α, we have $\vdash_{S_{n+1}} \alpha(x/c_\alpha)$, so a *fortiori* $\vdash_T \alpha(x/c_\alpha)$, completing the proof. □

3. Type theories with the ε-operator

Let S be a type theory in a language L. A type $\boldsymbol{A}$ of L is said to be *S-inhabited* if $\vdash_S \exists x.x = x$ with $x : \boldsymbol{A}$. Clearly any type of the form $\mathbf{P}\boldsymbol{A}$ is S-inhabited.

The *ε-language* $\mathsf{L}_\varepsilon(S)$ *over* S is obtained by adding to L the symbol ε and the new term forming clause

· for any formula $\alpha(x)$ with x of S-inhabited type, $\varepsilon_x\alpha$ is a (closed) term of the same type as x.

Of course, all occurences of x in $\varepsilon_x\alpha$ are regarded as being bound occurences. Note also that *iterated ε-terms* may be formed in $\mathsf{L}_\varepsilon(S)$. For example, $\mathsf{L}_\varepsilon(S)$ contains the term $\varepsilon_x\alpha(x, y/\varepsilon_y\beta)$ provided x and y have S-inhabited types.

The *ε-extension* S_ε of S is the theory in $\mathsf{L}_\varepsilon(S)$ obtained from S by adding as axioms the sequents

$$\exists x\alpha|\alpha(x/\varepsilon_x\alpha)$$

for any $\mathsf{L}_\varepsilon(S)$-formula $\alpha(x)$ with x of S-inhabited type.

We are going to determine the deductive strength of S_ε relative to S. Let S^* be the theory in L obtained from S by adding as axioms the sequents

$$|\exists x(\exists x\alpha \rightarrow \alpha)$$

for all formulas $\alpha(x)$ with x of S-inhabited type.

Theorem 3.1. *For any sequent $\Gamma|\beta$ of* L, *$\Gamma \vdash_{S_\varepsilon} \beta$ iff $\Gamma \vdash_{S^*} \beta$.*

Proof. For the "$\Leftarrow$"-direction, we note that $\exists x\alpha(x) \vdash_{S_\varepsilon} \alpha(x/\varepsilon_x\alpha)$, whence $\vdash_{S_\varepsilon} \exists x\alpha \rightarrow \alpha(x/\varepsilon_x\alpha)$, so $\vdash_{S_\varepsilon} \exists x(\exists x\alpha \rightarrow \alpha)$ by existential generalization.

Conversely, let T be a conservative witnessed extension of S^* as provided by Theorem 2.1. Then for any L-formula $\alpha(x)$ with x of S-inhabited type we have $\vdash_T \exists x\alpha \rightarrow \alpha(x/\tau)$ for some closed term τ of the language $\mathsf{L}(T)$ of T. For each such formula $\alpha(x)$ choose such a τ and denote it by $\tau_{\alpha,x}$. Note that the sequent $\exists x\alpha|\alpha(x/\tau_{\alpha,x})$ is then derivable from T. We now define a type preserving map $\tau \mapsto \tau^*$ recursively as follows:

$$(\#)^* = \#, \qquad x^* = x,$$
$$\boldsymbol{f}(\tau)^* = \boldsymbol{f}(\tau^*), \qquad \langle\sigma,\tau\rangle^* = \langle\sigma^*,\tau^*\rangle, \qquad \{x:\alpha\}^* = \{x:\alpha^*\},$$
$$(\sigma=\tau)^* = (\sigma^*=\tau^*), \quad (\sigma\in\tau)^* = (\sigma^*\in\tau^*) \quad (\varepsilon_x\alpha)^* = \tau_{\alpha^*,x}.$$

It is readily checked (by induction) that this definition is coherent, type preserving, and commutes with logical operations (i.e. $(\alpha\wedge\beta)^* = \alpha^*\wedge\beta^*$, etc.).

We now claim that, for any sequent $\Gamma|\beta$ of $\mathsf{L}_\varepsilon(S)$,

$$(*) \qquad \Gamma\vdash_{S_\varepsilon}\beta \Rightarrow \Gamma^*\vdash_T\beta^*,$$

where, if Γ is $\{\alpha_1,\dots,\alpha_n\}$, Γ^* is $\{\alpha_1^*,\dots,\alpha_n^*\}$. To prove (*), start with a derivation $\mathcal{P}$ of $\Gamma|\beta$ in S_ε. Replace each sequent $\Delta|\delta$ in $\mathcal{P}$ by the sequent $\Delta^*|\delta^*$, thus obtaining a new list $\mathcal{P}^*$ of sequents in $\mathsf{L}(T)$. This process carries basic axioms to basic axioms, axioms of S to axioms of S, and application of a rule of inference to an application of the same rule of inference, and any S_ε-axiom $\exists x\alpha|\alpha(x/\varepsilon_x\alpha)$ to the sequent $\exists x\alpha^*|\alpha^*(x/\tau_{\alpha^*,x})$ which, as we have observed above, is derivable in T. Therefore $\mathcal{P}^*$ is a derivation of $\Gamma^*|\beta^*$ in T, which establishes (*).

Finally, if $\Gamma|\beta$ is a sequent of L, $\Gamma^*|\beta^*$ is the same as $\Gamma|\beta$, so that $\Gamma\vdash_{S_\varepsilon}\beta$ implies $\Gamma\vdash_T\beta$ by (*) and $\Gamma\vdash_{S^*}\beta$ by the fact that T is a conservative extension of S^*. □

This result has an illuminating corollary, to state which we require some definitions.

A type theory T in a language $\mathsf{L}(T)$ (possibly containing ε-terms) is said to be *classical* if $\vdash_T \forall u.u\vee\neg u$ with $u:\mathbf{\Omega}$; *stable* if $\vdash_T \exists x(\exists x\alpha\rightarrow\alpha)$ for any $\mathsf{L}(T)$-formula $\alpha(x)$ with x of T-inhabited type; *Hilbertian* if, for any $\mathsf{L}(T)$-formula $\alpha(x)$ with x of T-inhabited type, there is a closed $\mathsf{L}(T)$-term τ of the same type as x such that $\exists x\alpha\vdash_T\alpha(x/\tau)$.

It is readily shown that any classical theory is stable, and a theory is Hilbertian if and only if it is witnessed and stable. We shall show in Section 6 that the first implication cannot be reversed, and that the Hilbertian property is genuinely stronger both than that of being witnessed and that of being stable.

We can now state and prove our promised

Corollary 3.2. *The following are equivalent for any type theory S:*

(i) *S is stable;*

(ii) *S_ε is a conservative extension of S;*

(iii) *S has a conservative Hilbertian extension.*

Proof. (i)⇒(ii) is an immediate consequence of Theorem 3.1, since it is evident that S is stable iff $S=S^*$. The proof of (iii)⇒(i) is left as an easy exercise to the reader. Finally, for (ii)⇒(iii), suppose that S_ε is a conservative extension of S. Then any S_ε-inhabited type is S-inhabited, since no new types are added in the passage from L to $\mathsf{L}_\varepsilon(S)$. So for any $\mathsf{L}_\varepsilon(S)$-formula $\alpha(x)$, if x is of S_ε-inhabited type, it is also of S-inhabited type, so that $\varepsilon_x\alpha$ is a term of $\mathsf{L}_\varepsilon(S)$ and the sequent $\exists x\alpha|\alpha(x/\varepsilon_x\alpha)$ is an axiom of S_ε. Therefore $\exists x\alpha\vdash_{S_\varepsilon}\alpha(x/\varepsilon_x\alpha)$, and it follows that S_ε is a Hilbertian extension of S. Hence (iii). □

Remarks. (1) It follows in particular that any classical theory satisfies (ii) an (iii) of the Corollary; for first-order theories this fact is well-known (see [6]). (2) Th implication (i)⇒(iii) of the Corollary is asserted in [5] for a somewhat different syste of type theory.

4. The partial ε-operator

In intuitionistic type theories there is a straightforward way of formulating a notio of "partially defined" term which, as we shall see, leads naturally to the concept o *partial ε-operator.*

Now although there is no explicit provision for partially defined terms in ou present framework, we can produce an acceptable surrogate as follows. Suppose were a "partially defined" term of some type $\boldsymbol{A}$ in a type theory S. Consider th extension U of the property of being equal to ξ. Then U would be a closed term o type $\mathbf{P}\boldsymbol{A}$ satisfying the condition

(a) $\vdash_S \forall x\in U\forall y\in U.x=y$

(but in general *not* $\vdash_S \exists x.x\in U$, since ξ is only *partially* defined). A closed ter $U:\mathbf{P}\boldsymbol{A}$ satisfying (a) is called an ***A**-singleton* (over S); this notion will be taken a representing within type theories the concept of partially defined term.

Let us see what happens when we replace ordinary terms by partially define terms in some of our previous definitions. For example, consider the concept of bein Hilbertian. For partially defined terms it would read

(b) for any formula $\alpha(x)$, there is a partially defined term ξ such tha $\exists x\alpha\vdash_S\alpha(x/\xi)$.

(Note that we were able to drop the condition that the type of x be inhabited sinc we are now only concerned with terms that are *partially* defined.) But $\alpha(x/\xi)\equiv$ $\exists x(x=\xi\wedge\alpha(x))$ and if U is the singleton arising as the extension of the property o being equal to ξ, we have $\vdash\exists x(x=\xi\wedge\alpha)\leftrightarrow\exists x\in U.\alpha$. So our condition (b) become

(b′) for any formula $\alpha(x)$ with $x:\boldsymbol{A}$, there is an $\boldsymbol{A}$-singleton U such tha $\exists x\alpha\vdash_S\exists x\in U.\alpha$.

A type theory S satisfying (b′) for all types $\boldsymbol{A}$ will be called *partially Hilbertian.*

Turning now to the concept of stability, we introduce the defined predicate

$$\mathrm{sing}(u)\equiv\forall x\in u\forall y\in u.x=y, \text{ where } u:\mathbf{P}\boldsymbol{A}.$$

Note that a closed term $U:\mathbf{P}\boldsymbol{A}$ is then an $\boldsymbol{A}$-singleton iff $\vdash_S \mathrm{sing}(U)$. A type theor S is now said to be *partially stable* if for any formula $\alpha(x)$, we have

$$\vdash_S \exists u[\mathrm{sing}(u)\wedge\exists x\alpha\rightarrow\exists x\in u.\alpha].$$

We can now assert the

Proposition 4.1. *For any type theory S we have*

(i) *if S is Hilbertian, then S is partially Hilbertian;*

(ii) *if S is stable, then S is partially stable,*

(iii) *if S is well-termed, then S is partially Hilbertian if and only if S is witnesse and partially stable.*

Proof. We prove (i); the proof of (ii) is similar and that of (iii) easy. Suppose that S is Hilbertian and let $\alpha(x)$ be a formula with $x : \mathbf{A}$. Define $\beta(u) \equiv \exists x[u = \{x\} \wedge \alpha(x)]$ with $u : \mathbf{PA}$. There is a closed term $V : \mathbf{PA}$ such that $\exists u\beta(u) \vdash \beta(u/V)$. Let $U = \{x : V = \{x\}\}$; then U is an $\mathbf{A}$-singleton and we have

$$\begin{aligned} \exists x\alpha(x) &\vdash_S \exists u\beta(u) \\ &\vdash_S \beta(u/V) \\ &\vdash_S \exists x[V = \{x\} \wedge \alpha(x)] \\ &\vdash_S \exists x[x \in U \wedge \alpha(x)], \end{aligned}$$

as required. □

We can now introduce, by analogy with ε, the *partial ε-operator*, which we shall denote by π.

The *partial ε-language* $\mathsf{L}_\pi(S)$ over S is obtained by adding to the language L of S the symbol π and the term-forming clause

· for any formula $\alpha(x)$ with $x : \mathbf{A}$, $\pi_x\alpha$ is a (closed) term of type $\mathbf{PA}$.

The *partial ε-extension* S_π of S is the theory in $\mathsf{L}_\pi(S)$ obtained from S by adding as axioms the sequents

$$|\text{sing}(\pi_x\alpha) \quad \text{and} \quad \exists x\alpha|(\exists x \in \pi_x\alpha)\alpha$$

for all $\mathsf{L}_\pi(S)$-formulas $\alpha(x)$. Clearly S_π is partially Hilbertian.

Now let $S^\wedge$ be the theory in L obtained by adding to S all sequents

$$|\exists u[\text{sing}(u) \wedge (\exists x\alpha \to \exists x \in u.\alpha)]$$

for all $\alpha(x)$. Clearly $S^\wedge$ is partially stable, and S_π is an extension of $S^\wedge$.

In essentially the same way as we proved Theorem 3.1, we obtain the

Theorem 4.2. *For any sequent $\Gamma|\beta$ of* L,

$$\Gamma \vdash_{S_\pi} \beta \quad \textit{iff} \quad \Gamma \vdash_{S^\wedge} \beta.$$

Proof. We give a sketch. Let T be a conservative witnessed extension of $S^\wedge$ as guaranteed by Theorem 2.1. For each $\alpha(x)$ choose a closed term $U_{\alpha,x} : \mathbf{PA}$ in $\mathsf{L}(T)$ so that

$$\vdash_S \text{sing}(U_{\alpha,x}) \wedge (\exists x\alpha \to \exists x \in U_{\alpha,x}.\alpha).$$

Define a translation $\tau \mapsto \tau^\wedge$ of the terms of $\mathsf{L}_\pi(S)$ to those of $\mathsf{L}(T)$ as in the proof of Theorem 3.1, except that now $(\pi_x\alpha)^\wedge = U_{\alpha^\wedge,x}$. The proof now proceeds like that of Theorem 3.1. □

Corollary 4.3. *The following are equivalent for any type theory S:*

(i) *S is partially stable;*

(ii) *S_π is a conservative extension of S;*

(iii) *S has a conservative partially Hilbertian extension.* □

5. Interpreting the ε-operator in a topos

We shall call a topos E *Hilbertian* if any diagram of the form $X \rightarrowtail A \twoheadrightarrow 1$ in E can be expanded to a commutative diagram of the form

$$\begin{array}{ccccccc} X & \twoheadrightarrow & U & \rightarrowtail & 1 & & \\ & & \downarrow & & \downarrow & & \\ & & X & \rightarrowtail & A & \twoheadrightarrow & 1 \end{array}$$

Hilbertian toposes will turn out to be the appropriate structures for interpreting ε-operators in type theories.

Note first that, in a Hilbertian topos, every subobject of 1 is projective. For this it suffices to show that for the canonical epi-mono factorization $X \twoheadrightarrow U \rightarrowtail 1$ of any $X \to 1$ there is an arrow $U \to X$. Now if $X^\sim$ is the partial map classifier of X (see [4], Ch. 1), we have a diagram $X \rightarrowtail X^\sim \twoheadrightarrow 1$. Using the Hilbertian property, we expand this to

$$\begin{array}{ccccccc} X & \twoheadrightarrow & U & \rightarrowtail & 1 & & \\ & & \downarrow & & \downarrow & & \\ & & X & \rightarrowtail & X^\sim & \twoheadrightarrow & 1 \end{array}$$

and the assertion follows.

The proof of the following proposition is routine and, accordingly, omitted (see [2], 4.32).

Proposition 5.1. *Let* E *be a topos. Then we have:*

(i) $\text{Th}(\mathsf{E})$ *is witnessed iff* 1 *is projective in* E;

(ii) $\text{Th}(\mathsf{E})$ *is partially Hilbertian iff every subobject of* 1 *is projective in* E;

(iii) $\text{Th}(\mathsf{E})$ *is Hilbertian iff* E *is Hilbertian.*

Let E be a Hilbertian topos. For each object X let

$$X \twoheadrightarrow U_X \overset{u_X}{\rightarrowtail} 1$$

be the canonical epi-mono factorization of $X \to 1$. For any $A \twoheadrightarrow 1$, choose for each monic $m : X \rightarrowtail A$ arrows $\varepsilon(m) : 1 \to A$ and $\pi(m) : U_X \to X$ such that the following diagram commutes:

$$(*)\qquad \begin{array}{ccc} U_X & \xrightarrow{\;u_X\;} & 1 \\ \pi(m)\big\downarrow & & \big\downarrow \varepsilon(m) \\ X & \xrightarrow{\;m\;} & A \end{array}$$

We shall call $\varepsilon(m)$ the *canonical element of A determined by the monic m*. We shall assume that each Hilbertian topos has been assigned canonical elements $\varepsilon(m)$ for all monics m.

Now let S be a type theory in a language L and let I be an interpretation of L in a Hilbertian topos E. We extend I to an interpretation of the ε-language $\mathsf{L}_\varepsilon(S)$ as follows. For any arrow $u : A \to \Omega$, let $\overline{u}$ be the monic (unique up to isomorphism) to A classified by u, i.e. such that the following diagram is a pullback:

$$\begin{array}{ccc} \bullet & \longrightarrow & 1 \\ \overline{u}\big\downarrow & & \big\downarrow \text{true} \\ A & \xrightarrow{\;u\;} & \Omega \end{array}$$

Then for any formula $\alpha(x)$ with the type $\boldsymbol{A}$ of x S-inhabited we define

$$(\varepsilon_x\alpha)_I = \varepsilon(\overline{\alpha_I}) : 1 \to A.$$

Proposition 5.2. *If I is a model of S, then I, as extended, is a model of* S_ε.

Proof. It suffices to show that the sequent $\exists x\alpha|\alpha(x/\varepsilon_x\alpha)$ is valid under I and for this it suffices (by, e.g., [2], 2.14) to show that there is an arrow g making the following diagram commute:

$$\begin{array}{ccc} U & \xrightarrow{\;(\exists x\alpha)_I\;} & 1 \\ g\big\downarrow & & \big\downarrow (\varepsilon_x\alpha)_I \\ X & \xrightarrow{\;\overline{\alpha_I}\;} & A \end{array}$$

But the commutativity of the diagram (*) above shows that we may take g to be $\pi(\overline{\alpha_I})$. □

In general, interpretations of $\mathsf{L}_\varepsilon(S)$ in Hilbertian toposes validate more than just the theorems of S_ε. For example, let us call two formulas $\alpha(x)$, $\beta(y)$ (S)-*similar* if (i) x and y are of the same S-inhabited type, (ii) x is free for y in β, (iii) y is free for x in α and (iv) $\vdash_S \forall x[\alpha(x) \leftrightarrow \beta(y/x)]$. It is then easy to see that any interpretation in a Hilbertian topos assigns the *same* value to $\varepsilon_x\alpha$ and $\varepsilon_y\beta$ whenever $\alpha(x)$ and $\beta(y)$ are similar. This being the case, what is the theory common to all Hilbertian toposes? We proceed to identify it.

Given a theory T in some language L, let $\mathsf{L}^\wedge(\mathsf{T})$ be obtained by adding to L all ε-terms $\varepsilon_x\alpha$ *not already present in* L for formulas $\alpha(x)$ with x of T-inhabited type. Let T^+ be the theory in $\mathsf{L}^\wedge_\varepsilon(T)$ obtained from T by adding as axioms the sequents

$$\exists x\alpha|\alpha(x/\varepsilon_x\alpha) \quad \text{for } new\ \varepsilon\text{-terms } \varepsilon_x\alpha,$$

$$|\varepsilon_x\alpha = \varepsilon_y\beta \quad \text{for all } (T\text{-}) \text{ similar formulas } \alpha(x), \beta(y).$$

Now define recursively $T_0 = S$, $T_{n+1} = (T_n)^+$, $\mathsf{L}_0 = \mathsf{L}$, $\mathsf{L}_{n+1} = \mathsf{L}^\wedge_\varepsilon(T_n)$. Let $S^\sim_\varepsilon = \bigcup_{n\in\omega} T_n$, $\mathsf{L}^\sim_\varepsilon(S) = \bigcup_{n\in\omega}\mathsf{L}_n$. We call $S^\sim_\varepsilon$ the *strong ε-extension of S*.

Lemma 5.3. $S^\sim_\varepsilon$ *is Hilbertian and* $\vdash_{S^\sim_\varepsilon} \varepsilon_x\alpha = \varepsilon_y\beta$ *for any* $S^\sim_\varepsilon$*-similar formulas* $\alpha(x)$, $\beta(y)$.

Proof. If $\alpha(x)$ is a formula of $\mathsf{L}^\sim_\varepsilon(S)$ with x of $S^\sim_\varepsilon$-inhabited type, then $\vdash_{T_n} \exists x.x = x$ for some n; let n_0 be the least such n. Then $\exists\alpha|\alpha(x/\varepsilon_x\alpha)$ is an axiom of T_{n_0+1} so that $\exists x\alpha \vdash_{S^\sim_\varepsilon} \alpha(x/\varepsilon_x\alpha)$. Therefore $S^\sim_\varepsilon$ is Hilbertian. Now let $\alpha(x)$, $\beta(y)$ be $S^\sim_\varepsilon$-similar formulas. Then there is a least n such that $\alpha(x)$ and $\beta(y)$ are T_n-similar. So $|\varepsilon_x\alpha = \varepsilon_y\beta$ is an axiom of T_{n+1} and hence derivable in $S^\sim_\varepsilon$. □

Any model I of S in a Hilbertian topos can be extended to a model (also denoted by I) of $S^\sim_\varepsilon$ by iterating the procedure of extending I to a models of S_ε; we leave the details to the reader.

All this leads to

Theorem 5.4 (ε-Completeness Theorem). *For any sequent* $\Gamma|\alpha$ *of* $\mathsf{L}^\sim_\varepsilon(S)$ *(and hence, a fortiori, of* $\mathsf{L}_\varepsilon(S)$*),* $\Gamma \vdash_{S^\sim_\varepsilon} \alpha$ *iff* $\Gamma \models_I \alpha$ *for every Hilbertian model I of S.*

Proof. We give just a sketch, as the details are tedious but routine.

$\Rightarrow$. Show by induction on n that $\Gamma \vdash_{T_n} \alpha$ implies $\Gamma \models_I \alpha$ for every Hilbertian model I of S.

$\Leftarrow$. Consider the topos $\mathsf{C}(S^\sim_\varepsilon)$. It is Hilbertian, for if $\boldsymbol{A}$ is an $S^\sim_\varepsilon$-inhabited type and $X = \{x : \alpha\}$ with $x : \boldsymbol{A}$, the following diagram commutes, where A is $\{x : x = x\}$ with $x : \boldsymbol{A}$ and i_X is the insertion map.

$$\begin{array}{ccc} \{u \in 1 : \exists x\alpha\} & \longrightarrow & 1 \\ \# \mapsto \varepsilon_x\alpha \big\downarrow & & \big\downarrow \# \mapsto \varepsilon_x\alpha \\ X & \xrightarrow{\;i_X\;} & A \end{array}$$

Now choose canonical elements in $\mathsf{C}(S^\sim_\varepsilon)$ in such a way that $\varepsilon(i_X)$ is always the map $1 \to A$ with value $\varepsilon_x\alpha$. Lemma 5.3 implies that this definition is coherent (it would not be if we had not replaced S_ε with $S^\sim_\varepsilon$). One can now show (along the lines of the proof of 3.28 of [2]) that $\Gamma \vdash_{S^\sim_\varepsilon} \alpha$ iff $\Gamma \models_{C(S^\sim_\varepsilon)} \alpha$. Since $C(S^\sim_\varepsilon)$ is a Hilbertian model of S, the implication follows. □

Remark. It is possible to establish similar results relating partial ε-theories and "partially Hilbertian" toposes, i.e., those in which every subobject of 1 is projective. We omit the details.

6. Mostly counterexamples

We establish the nonequivalence of the various properties of type theories we have introduced. First, we require the following

Proposition 6.1. *Let S be a stable theory in a language with a natural number system* $(\boldsymbol{N}, s, 0)$ (see [2], Ch. 7). *Then for any sentences β, γ we have*

$$\vdash_S (\beta \to \gamma) \vee (\gamma \to \beta).$$

Proof. Write 1 for $s0$ and let x, y be variables of type $\boldsymbol{N}$. It is well known (see [2], 7.6) that $\boldsymbol{N}$ is *decidable*, i.e. $\vdash_S \forall x \forall y(x = y \vee x \neq y)$. Now define

$$\alpha(x) \equiv (x = 0 \wedge \beta) \vee (x \neq 0 \wedge \gamma).$$

Then $\vdash_S \alpha(0) \leftrightarrow \beta$ and $\vdash_S \alpha(1) \leftrightarrow \gamma$, so that

$$\vdash_S \exists x \alpha(x) \leftrightarrow \beta \vee \gamma.$$

Also $(x \neq 0 \wedge \alpha(x)) \vdash_S \gamma$, whence

$$(*) \qquad x \neq 0 \vdash_S \alpha(x) \to \gamma.$$

So we have

$$\begin{aligned} \exists x \alpha(x) \to \alpha(x) &\vdash_S \beta \vee \gamma \to \alpha(x) \\ &\vdash_S (\beta \vee \gamma \to \alpha(x)) \wedge (x = 0 \vee x \neq 0) \\ &\vdash_S [(\beta \vee \gamma \to \alpha(x)) \wedge x = 0] \vee [(\beta \vee \gamma \to \alpha(x)) \wedge x \neq 0] \\ &\vdash_S (\beta \vee \gamma \to \alpha(0)) \vee [(\beta \vee \gamma \to \alpha(x)) \wedge (\alpha(x) \to \gamma)], \text{ by } (*) \\ &\vdash_S (\beta \vee \gamma \to \beta) \vee (\beta \vee \gamma \to \gamma) \\ &\vdash_S (\gamma \to \beta) \vee (\beta \to \gamma). \end{aligned}$$

Therefore

$$\exists x(\exists x \alpha(x) \to \alpha(x)) \vdash_S (\beta \to \gamma) \vee (\gamma \to \beta).$$

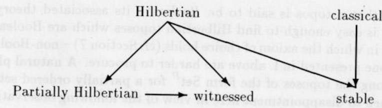
So if S is stable the required conclusion follows immediately. □

Corollary 6.2. *Any stable theory S in a language with a natural number system satisfies (an equivalent of) de Morgan's law, viz., for any sentence β, $\vdash_S \neg\beta \vee \neg\neg\beta$.* □

Remark. The proof of Proposition 6.1 is easily adapted to yield the following strengthening of a result of §2 of [3], viz.:

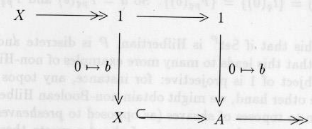
Let T be a theory in the first-order intuitionistic ε-calculus such that, for some constants 0, 1, *we have* $\vdash_T \forall x(x = 0 \vee x \neq 0) \wedge 0 \neq 1$. *Then for any sentences β, γ,* $\vdash_T (\beta \to \gamma) \vee (\gamma \to \beta)$. □

Now we can establish the following diagram relating the various properties we have introduced, where the arrows indicate irreversible implications.



It suffices, in view of what we already know, to establish the following nonimplications.

I. *Hilbertian $\not\Rightarrow$ classical.* For this it suffices to produce a Hilbertian topos which is not classical. To this end, let M be a monoid consisting of two elements $1, e$ such that $e^2 = e$. Then since M is not a group, the topos Set^M of M-sets is not classical. But it is Hilbertian. For let X be a non-empty sub-M-set of an M-set A. Choose $a \in X$ and let $b = e \cdot a \in X$. Then the diagram



commutes, yielding the desired conclusion.

II. *Classical $\not\Rightarrow$ witnessed.* For this we need only exhibit a Boolean topos which is not witnessed. An example is the topos of G-sets, where G is a group with at least 2 elements.

III. *Witnessed $\not\Rightarrow$ partially Hilbertian.* It suffices to produce a topos in which 1 is, but not all of its subobjects are, projective. To this end, let P be the ordered set $(\omega, \geq)$ augmented by a least element $*$. Consider the topos $\mathsf{E} = \mathsf{Set}^P$ of sets varying over P. Since P has a least element, 1 is projective in E. But not every subobject of 1 is projective in E. For consider the E-object X defined by $X(n) = \omega - \{0, \ldots, n-1\}$, $X(*) = \emptyset$, X_{mn} = insertion map $X(m) \hookrightarrow X(n)$ for $m \geq n$. Let U be the subobject of 1 in E defined by $U(n) = 1$ for all $n \in \omega$, $U(*) = \emptyset$. Then the evident arrow $X \to U$ in E is epic but there is no E-arrow $U \to X$.

IV. *Partially Hilbertian $\not\Rightarrow$ stable.* It suffice, by Corollary 6.2, to produce a topos E in which every subobject of 1 is projective but in which the law

$$(*) \qquad \neg\beta \vee \neg\neg\beta$$

for sentences β is not valid. Such a topos can be extracted from [4]. In Exercise 5.4 on p. 162 of that volume, it is asserted that, if X is a topological space which is separable and zero-dimensional, then every subobject of 1 is projective in the topos $\mathsf{Shv}(X)$ of sheaves on X. In particular, this is the case for $\mathsf{Shv}(N^*)$, where N^* is the one-point compactification of the discrete space of integers. But it is well known that $(*)$ is valid in $\mathsf{Shv}(X)$ iff X is extremally disconnected, which N^* manifestly is *not*. So $(*)$ is not valid in $\mathsf{Shv}(N^*)$, as claimed.

Remark. Recall that a topos is said to be *Boolean* if its associated theory is classical. Now while it is easy enough to find Hilbertian toposes which are Boolean – for example, any topos in which the axiom of choice holds (cf. Section 7) – non-Boolean examples such as the one presented in I. above are harder to procure. A natural place to start looking is among the toposes of the form Set^P for a partially ordered set P. However, we are doomed to disappointment here in view of the following observation.

Suppose the object F in Set^P has the property that every arrow $U \to F$ with $U \rightarrowtail 1$ can be extended to an arrow $1 \to F$ (which would be the case if Set^P were Hilbertian and $F \twoheadrightarrow 1$). Then every transition map F_{pq} with $p \leq q$ in P is surjective. For choose any $q \in P$ and $a \in F(q)$ (if $F(q) = \emptyset$, then F is automatically surjective for any $p \leq q$). Define U in Set^P by $U(r) = \{F_{qr}(a)\}$ if $q \leq r$, $U(r) = \emptyset$ if $q \not\leq r$. Then U is a subobject of F and $U \rightarrowtail 1$ in Set^P. Suppose that there is $t : 1 \to F$ such that $U \subseteq \{t\}$ in Set^P. Given $p \leq q$, let $b = t_p(0) \in F(p)$. Then $\{a\} = U(q) \subseteq \{t\}(q) = \{t_q(0)\} = \{F_{pq}(b)\}$. So $a = F_{pq}(b)$ and F_{pq} is surjective as claimed.

It follows from this that if Set^P is Hilbertian, P is discrete and Set^P Boolean. (We note in passing that this leads to many more examples of non-Hilbertian toposes in which every subobject of 1 is projective: for instance, any topos Set^α with α an ordinal ≥ 2.) On the other hand, we might obtain non-Boolean Hilbertian toposes by turning our attention to toposes of *sheaves* (as opposed to *presheaves*). For example, is there a *topological space* whose category of sheaves meets these requirements? I have not been able to answer this question.

7. The ε-operator and the axiom of choice

We have only allowed the ε-operator to act on formulas with at most *one* free variable, thereby admitting only *closed* ε-terms. What happens if we relax this restriction? Suppose, for example, that in forming $\mathsf{L}_\varepsilon(S)$ and S_ε we drop the restriction that $\varepsilon_y\alpha$ can only be formed when α has at most y free and now allow α to contain an additional free variable x. Then we would have

$$\exists y\alpha(x,y) \vdash_{S_\varepsilon} \alpha(x, y/\varepsilon_y\alpha),$$

where the term $\varepsilon_y\alpha$ now has a free variable x. So if f is the map $x \mapsto \varepsilon_y\alpha$ we have

$$\vdash_{S_\varepsilon} \forall x[\exists y\alpha(x,y) \to \alpha(x, fx)].$$

Thus, if $\vdash_{S_\varepsilon} \forall x \exists y \alpha(x,y)$, then $\vdash_{S_\varepsilon} \forall x \alpha(x, fx)$. That is, under these conditions we have derived the *axiom of choice* in S_ε (cf. [2], 4.29).

Therefore the admission of ε-term with even *one* free variable to an intuitionistic type theory enables the axiom of choice to be derived. But it is well known (see 4.31(iv) of [2], for example) that *any intuitionistic type theory in which the axiom of choice can be derived is classical.* So the admission of ε-terms with free variables collapses an intuitionistic type theory to a classical theory. Since the ε-calculus has already been much investigated in a classical (albeit first order) context, the justice of confining our attention to closed ε-terms should now be evident.

In conclusion, we note that a type theory containing only closed ε-terms can be forced to become classical, albeit in a weakened sense, by adding as axioms the sequents asserting that the ε-terms satisfy an *extensionality principle*, viz.,

$$\forall x[\alpha(x) = \beta(x)] | \varepsilon_x\alpha = \varepsilon_x\beta.$$

For then, following the proof of the Theorem in Section 3 of [3], we will be able to derive $\vdash_T \gamma \vee \neg\gamma$ for all *sentences* γ. For completeness we briefly indicate how this is done. Let T be obtained by adding the above sequents to S. Let γ be any sentence. Then define $\alpha(x)$, $\beta(x)$ by

$$\alpha(x) \equiv x = \emptyset \vee \gamma, \quad \beta(x) \equiv x = 1 \vee \gamma$$

with $x : \mathbf{P1}$. It is now easily shown that $\vdash_T (\varepsilon_x\alpha \neq \varepsilon_x\beta) \vee \gamma$. But extensionality now gives $\gamma \vdash_T \varepsilon_x\alpha = \varepsilon_x\beta$, whence $\varepsilon_x\alpha \neq \varepsilon_x\beta \vdash_T \neg\gamma$. We conlude that $\vdash_T \neg\gamma \vee \gamma$ as claimed.

For more on extensionality in the first-order case, see [3].

Note added in proof (April 1993). ANDREAS BLASS (private communication) has shown that for any Hilbertian topos E, the algebra $\mathsf{E}(1,\Omega)$ of global E-elements of Ω is Boolean. It follows from this that any localic Hilbertian topos (in particular, any Hilbertian topos of sheaves) is Boolean, answering in the negative the question at the end of Section 6.

References

[1] ASSER, G., Theorie der logischen Auswahlfunktionen. Zeitschrift Math. Logik Grundlagen Math. **3** (1957), 30 – 68.

[2] BELL, J. L., Toposes and Local Set Theories: An Introduction. Oxford University Press, Oxford 1988.

[3] BELL, J. L., Hilbert's ε-operator and classical logic. J. Philosophical Logic **22** (1993), 1 – 18.

[4] JOHNSTONE, P. T., Topos Theory. Academic Press, New York 1977.

[5] LAMBEK, J., On the sheaf of possible worlds. In: Categorical Topology and its Relation to Analysis, Algebra and Combinatorics (J. ADAMEK and S. MacLane, eds.), World Scientific, Teaneck, N.J., 1989, pp. 36 – 53.

[6] LEISENRING, A. C., Mathematical Logic and Hilbert's ε-symbol. Gordon and Breach, New York 1969.

(Received: June 1, 1992)