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Logical Reflections On The Kochen-Specker Theorem

IN THEIR WELL-KNOWN PAPER, Kochen and Specker (1967) introduce the concept of *partial Boolean algebra* (pBa) and show that certain (finitely generated) partial Boolean algebras arising in quantum theory fail to possess morphisms to any Boolean algebra (we call such pBa's *intractable* in the sequel). In this note we begin by discussing partial Boolean algebras within a *category-theoretic* framework[1]; our analysis will result in what appear to be some new formulations of intractability in purely *logical* terms, and an open problem.

Partial Boolean algebras arise naturally in connection with *ortholattices* (a concept we take to be familiar: see, e.g. Birkhoff (1984)). Let $L = (L, \vee, \perp, 1)$ be an ortholattice and let $\approx$ be the binary relation ("compatibility") on L defined by: $x \approx y \Leftrightarrow \{x, y\}$ generates a Boolean subalgebra (= distributive subortholattice) of L. Clearly $\approx$ is reflexive and symmetric. Now consider the partial algebra $L^{\approx}$ obtained from L by restricting the domain of $\vee$ to those pairs (x, y) of L for which $x \approx y$. The concept of partial Boolean algebra employed here is obtained by abstracting from the properties of $L^{\approx}$.

Thus a *partial Boolean algebra* (pBa) is a structure $B = (B, \vee, \perp, 1, \approx)$ in which B is a nonempty set, $1 \in B$, $\approx$ is a reflexive, symmetric relation (the *compatibility* relation) on B, and $\perp, \vee$ are maps to B from B and $\{(x, y): x \approx y\}$ respectively, satisfying the conditions:

$$1^{\perp} \neq 1 ;$$

for all $x, y \in B$: $1 \approx x$;

$$x \approx y \Rightarrow x \approx y^{\perp} \ \& \ (x \vee y) \approx x ;$$

for any $x \approx y$ in B: the substructure of B generated by $\{x, y\}$ is a Boolean algebra.

We note that the partial Boolean algebras of Kochen and Specker satisfy the additional condition that $\{x_1, \ldots, x_n\}$ generates a Boolean subalgebra whenever $x_i \approx x_j$ for any i, j: this condition is satisfied by pBa's induced by *orthomodular* ortholattices. For our purposes here, there is no need to impose this additional constraint.

We can now define the category of partial Boolean algebras. We de-

[1] An earlier investigation of the category of partial Boolean algebras, in which results different from those included here are obtained, appears in Kamber (1964).

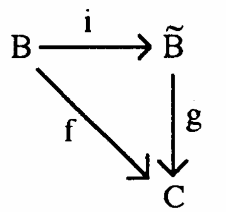
R. Clifton (ed.), Perspectives on Quantum Reality, 227–235.

fine a *morphism* of pBa's B, B′ to be a map $h: B \to B'$ such that:

$$\text{for any } x \approx y \text{ in B:} \quad \begin{array}{l} h(x) \approx h(y)\ ; \\ h(x \vee y) = h(x) \vee h(y)\ ; \\ h(x^{\perp}) = h(x)^{\perp}\ . \end{array}$$

The category $\mathcal{PBA}$ of partial Boolean algebras then has as objects all partial Boolean algebras and as arrows all morphisms between them. Clearly the category $\mathcal{Bool}$ of Boolean algebras and (Boolean) homomorphisms is a subcategory of $\mathcal{PBA}$ which is *full* in the sense that if B, B′ are objects of $\mathcal{Bool}$, then any morphism of B to B′ in $\mathcal{PBA}$ is also in $\mathcal{Bool}$ (i.e. is a Boolean homomorphism in the usual sense).

Although it is obvious that not every pBa is a Boolean algebra, it is natural to ask whether, nonetheless, any pBa B has a best "Boolean approximation" in the following sense: there is a pair $(\tilde{B}, i)$ consisting of a Boolean algebra $\tilde{B}$ and a morphism $i: B \to \tilde{B}$ such that, for any morphism $f: B \to C$ to a Boolean algebra C, there is a (unique) morphism $g: \tilde{B} \to C$ such that the diagram



commutes: we call this the *universal condition* on $(\tilde{B}, i)$. We shall call a pBa B *tractable* if a pair $(\tilde{B}, i)$ satisfying the universal condition exists.

Kochen and Specker showed, in effect, that not every pBa is tractable: more on this later. For the moment, consider the subcategory $\mathcal{Trac}$ of $\mathcal{PBA}$ whose objects are all *tractable* pBa's: clearly $\mathcal{Bool}$ is a (full) subcategory of $\mathcal{Trac}$. Every object B of $\mathcal{Trac}$ has a "Boolean approximation" $(\tilde{B}, i)$ which is easily shown to be unique up to isomorphism in the evident sense. In that case, the map $B \mapsto \tilde{B}$: $\mathcal{Trac} \to \mathcal{Bool}$ defines what category-theorists call a *reflection*: for any object B of $\mathcal{Trac}$, $\tilde{B}$ is the "reflection" of B in the subcategory $\mathcal{Bool}$.

We are going to characterize the objects in $\mathcal{Trac}$, and provide an explicit description of the reflection $\tilde{B}$ of any tractable pBa.[2]

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[2]Kamber (1964) constructs the Boolean reflection of a pBa by a method different from that to be formulated here. Moreover, because he allows degenerate (= 1-element) pBa's - a possibility which has been explicitly excluded here - *every* (not necessarily tractable) pBa has a (possibly degenerate) reflection in his sense. However for that reason, the concept of tractability, which occupies centre stage here, plays no role in his discussion.

Given a partial Boolean algebra B, let $\mathcal{L}_B$ be the (classical) propositional language with propositional variables $\{p_b: b\in B\}$, and let Σ_B be the set of all sentences of the form:

$$p_a \vee p_b \leftrightarrow p_{a\vee b}\text{, for } a\text{﹏}b \text{ in B;}$$
$$\neg p_c \leftrightarrow p_{c^\perp}\text{, for } c\in B.$$

Theorem 1

Let B be a pBa. Then the following conditions are equivalent:

(i) B is tractable;

(ii) Σ_B is consistent;

(iii) there is a morphism from B to the two element Boolean algebra 2;

(iv) there is a morphism from B to some Boolean algebra.

Proof

(i)⇒(iv) is obvious, as is (iv)⇒(iii) since every Boolean algebra has a homomorphism to 2.

(iii)⇒(ii). Let $h: B\rightarrow 2$ be a morphism, and let $[\![\cdot]\!]$ be the valuation on $\mathcal{L}_B$ induced by h (i.e. such that $[\![p_b]\!]=h(b)$ for $b\in B$). It is easily verified that $[\![\phi]\!]=1$ for every $\phi\in\Sigma_B$, so that Σ_B is consistent.

(ii)⇒(i). Suppose that Σ_B is consistent. Then the Lindenbaum-Tarski algebra B of Σ_B (obtained by identifying formulas of $\mathcal{L}_B$ when their equivalence follows from Σ_B) is a nondegenerate Boolean algebra, and the canonical map $b\mapsto[p_b]$ (where $[\phi]$ is the image - i.e., equivalence class modulo provable equivalence from Σ_B - of a formula ϕ in $\tilde{B}$) is a morphism $i: B\rightarrow\tilde{B}$. We have to show that $(\tilde{B}, i)$ satisfies the universal condition.

To this end, suppose that f is a morphism of B to a Boolean algebra C. Let Form be the set of formulas of $\mathcal{L}_B$ and define the map $\underline{f}:\text{Form}\rightarrow C$ recursively by:

$$\underline{f}(p_b) = f(b) \text{ for } b\in B\ ,$$
$$\underline{f}(\phi\vee\psi) = \underline{f}(\phi)\vee\underline{f}(\psi)\ ,$$
$$\underline{f}(\neg\phi) = \underline{f}(\phi)^\perp .$$

Now define $g:\tilde{B}\rightarrow C$ by $g([\phi])=\underline{f}(\phi)$ for $\phi\in$Form. Clearly g will be a homomorphism provided we can show that it is well-defined. To do this, it suffices to show that, if $\phi,\psi\in$Form, then $[\phi]=[\psi] \Rightarrow \underline{f}(\phi)=\underline{f}(\psi)$. To this end, let h be any homomorphism $C\rightarrow 2$ and let $[\![\cdot]\!]$ be the truth valuation on Form induced by taking $[\![p_b]\!]= h(\underline{f}(p_b))$ for $b\in B$. It is easily shown by induction on formulas that $[\![\phi]\!]= h(\underline{f}(\phi))$ for any $\phi\in$Form. Clearly also $[\![\sigma]\!]=1$ for any $\sigma\in\Sigma_B$. Thus if $[\phi]=[\psi]$, then $\Sigma_B \vdash \phi\leftrightarrow\psi$, so $[\![\phi\leftrightarrow\psi]\!]=1$, whence $h(\underline{f}(\phi))= [\![\phi]\!] = [\![\psi]\!]=h(\underline{f}(\psi))$. Since this equality holds for arbitrary $h:C\to 2$, and homomorphisms to 2 distinguish the points of any Boolean algebra, it follows that $\underline{f}(\phi)=\underline{f}(\psi)$ as required.

Therefore g is a homomorphism; clearly $g\circ i = f$, and it is easy to see that it is the unique such homomorphism. So $(\tilde{B}, i)$ satisfies the universal condition. ▮

As a consequence, we obtain the

Corollary

A pBa is tractable if and only if all of its finitely generated sub-pBa's are tractable.

Proof

If the pBa B is tractable, there is a morphism $h:B\to 2$ whose restriction to any sub-pBa B' of B is a morphism $B'\to 2$. Accordingly, B' is tractable.

Conversely, suppose B is intractable. Then Σ_B is inconsistent and so there are finite subsets $\{a_1,\dots,a_n\}$, $\{b_1,\dots,b_n\}$, $\{c_1,\dots,c_m\}$ of B for which $a_i \approx b_i$ $(i=1,\dots,n)$ and the set of sentences

$$\{p_{a_i} \vee p_{b_i} \leftrightarrow p_{a_i \vee b_i}: i=1,\dots,n\} \cup \{\neg p_{c_j} \leftrightarrow p_{c_j\perp}: j=1,\dots,m\}$$

is inconsistent. It follows that, if A is the sub-pBa of B generated by

$$\{a_1,\dots,a_n\} \cup \{b_1,\dots,b_n\} \cup \{c_1,\dots,c_m\}$$

then Σ_A is inconsistent, so that A is intractable. ▮

It is natural to single out those tractable pBa's B for which the canonical morphism $i:B\to\tilde{B}$ is *injective*. In this connection - following Kochen and Specker - we call a morphism $h:B\to B'$ in $\mathcal{PBA}$ an *embedding* (resp. *weak embedding*) if $a\neq b \Rightarrow h(a)\neq h(b)$ (resp. $a\neq b$ & $a\approx b \Rightarrow h(a)\neq h(b)$) for $a,b\in B$. It is straightforward to adapt the proofs of Theorem 1 to establish:

Theorem 2

The following conditions on a pBa B are equivalent:

(i) B is tractable and $i:B\to\tilde{B}$ is an embedding (resp. weak embedding);

(ii) $\Sigma_B \nvdash p_a \leftrightarrow p_b$ for any $a\neq b$ (resp. $a\neq b$ & $a\approx b$) in B;

(iii) for any $a\neq b$ (resp. $a\neq b$ & $a\approx b$) in B there is a morphism $h:B\to 2$ such that $h(a)\neq h(b)$;

(iv) there is an embedding (resp. weak embedding) of B into some Boolean algebra. ▮

Simple examples of pBa's which, while not Boolean algebras themselves, are embeddable therein, arise in the following way.

Let (A,f) be a structure consisting of a nonempty set A and a map $f: A \to A$ such that $f(x) \neq x$ and $f(f(x))=x$ for every $x \in A$. Let 0,1 be two distinct objects not in A, and on the set $\bar{A} = A \cup \{0,1\}$ define the relation $\approx$ by:

$$\approx \; = \{(x,x): x \in \bar{A}\} \cup \{(x,fx): x \in A\} \cup \{0,1\} \times \bar{A} \cup \bar{A} \times \{0,1\} .$$

Clearly $\approx$ is reflexive and symmetric. Define $\perp : \bar{A} \to \bar{A}$ by:

$$0^{\perp}=1 ;$$
$$1^{\perp}=0 ;$$
$$\text{for } x \in A: \quad x^{\perp}=fx ,$$

and $v: \{(x,y): x \approx y\} \to \bar{A}$ by:

$$0 \vee 0 = 0 ;$$
$$0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1 ;$$
$$\text{for every } x \approx y,\ x \neq y \text{ in A}: \quad x \vee y = 1 .$$

Then $(\bar{A}, \vee, \perp, 1, \approx)$ is a pBa which is evidently not a Boolean algebra if A has more than two elements.

To show that $\bar{A}$ is embeddable in a Boolean algebra, we argue as follows. Suppose given $a \neq b$ in A. If $b \neq fa$, let k be a function (whose existence is ensured by the axiom of choice for pairs) which selects an element from each pair $\{x, fx\}$ for $x \in A - \{a,b\}$. Now define $h: \bar{A} \to 2$ by:

$$h(a) = h(1) = 1 ;$$
$$h(b) = h(0) = 0 ;$$
$$\text{for } x \in A - \{a,b\}: \quad h(x) = 1, \text{ if } x \in \text{range}(k) ;$$
$$= 0, \text{ otherwise.}$$

It is easy to see that h is a morphism and $h(a) \neq h(b)$. When $b=fa$, or one or both of a or b is 0 or 1, the construction of h is similar. It follows from Theorem 2 that $\bar{A}$ is embeddable in a Boolean algebra.[3]

We now turn our attention to *intractable* pBa's. If B is intractable,

[3]The standard case arises of course when A is the set of rays in the plane E_2 and f the map assigning to each ray its orthogonal complement in E_2. In this case, we do not need to invoke the axiom of choice to prove the existence of two-valued morphisms, since a simple geometric argument suffices. However, in the general case we do need to use the axiom of choice, since it is not hard to show that the assertion that *every* partial Boolean algebra of the form $\bar{A}$ is embeddable in a Boolean algebra is *equivalent* to the axiom of choice for pairs.

then, as we have seen, Σ_B is inconsistent, which means that there exist finite subsets $\{a_1,\ldots,a_n\}$, $\{b_1,\ldots,b_n\}$, $\{c_1,\ldots,c_m\}$ of B such that $a_i \approx b_i$ $(i=1,\ldots,n)$ and the sentence

$$\bigwedge_{i=1}^{n} p_{a_i} \vee p_{b_i} \leftrightarrow p_{a_i \vee b_i} \;\wedge\; \bigwedge_{j=1}^{m} \neg p_{c_j} \leftrightarrow p_{c_j{}^\perp} \qquad (*)$$

is a classical contradiction. Now in (*) we may delete any conjunct in which a_i, b_i, or c_j is 0 or 1 and still ensure that the remaining conjunction - call it Φ - is a contradiction, since if Φ is satisfied by some truth valuation $[\![\cdot]\!]$, then (*) is satisfied by the valuation $[\![\cdot]\!]'$ which agrees with $[\![\cdot]\!]$ except that $[\![p_0]\!]'=0$, $[\![p_1]\!]'=1$. Next, in Φ delete each conjunct of the form $\neg p_{c_j} \leftrightarrow p_{c_j{}^\perp}$ and each conjunct $p_{a_i} \vee p_{b_i} \leftrightarrow p_{a_i \vee b_i}$ for which $a_i \vee b_i \neq c_j{}^\perp$ for any $j=1,\ldots,m$. In each remaining conjunct, replace the term $p_{a_i \vee b_i}$ by $\neg p_{c_j}$, where $a_i \vee b_i = c_j{}^\perp$. The result is a formula of the form

$$\bigwedge_{<i,j,k>\in R} q_i \vee q_j \leftrightarrow \neg q_k \qquad (**)$$

in which each q_i, $i=1,\ldots,N$ is a propositional variable and R is a set of triples of integers $1,\ldots,N$ such that $<i,j,k>\in R \Rightarrow i\neq j$. Since (*) was a contradiction, it is not hard to see that (**) is also. Moreover, for each $<i,j,k>\in R$, there exist (unique) $a,b,c\in B$ for which q_i is p_a, q_j is p_b, and q_k is p_c, where $a \approx b$ and $a \vee b = c^\perp$. It follows that, if we assign the values a, b, c in B to q_i, q_j, q_k respectively, the formula $q_i \vee q_j \leftrightarrow \neg q_k$ receives value 1. Doing this for each conjunct in (**) thus assigns value 1 to (**).

Now for each natural number $N \geq 1$ write $\underline{N}$ for the set $\{1,\ldots,N\}$. Let us define an N-*skeleton* to be a structure $\mathcal{S}$ of the form $(\underline{N},R)$ with $R \subseteq N^3$ satisfying $<i,j,k>\in R \Rightarrow i\neq j$. Given an N-skeleton $\mathcal{S}$, let $q_1,\ldots,q_N$ be propositional variables and denote the resulting formula (**) by $\phi_{\mathcal{S}}$: we shall call this the formula *associated* with $\mathcal{S}$. A *representation* of $\mathcal{S}$ in a pBa B is an injective map $b:\underline{N} \to B-\{0,1\}$ such that, writing b_i for $b(i)$,

$$\forall ijk \in \underline{N}\ [<i,j,k>\in R \Rightarrow b_i \approx b_j \ \&\ b_i \vee b_j = b_k{}^\perp]\ .$$

Clearly any representation b of $\mathcal{S}$ in B assigns the (well-defined) value $b(\phi_{\mathcal{S}})=1$ to $\phi_{\mathcal{S}}$ in B. *If, in addition,* $\phi_{\mathcal{S}}$ *is a classical contradiction, then B must be intractable.* For if h were any morphism $B \to 2$, then $h(b(\phi_{\mathcal{S}}))$ is the value in 2 of $\phi_{\mathcal{S}}$ under the valuation obtained by assigning the value $h(b_i)$ to each q_i; this value must be 0 since $\phi_{\mathcal{S}}$ is a classical contradiction. On the other hand, since $b(\phi_{\mathcal{S}})=1$, we must also have $h(b(\phi_{\mathcal{S}}))=1$. This inconsistency shows that B has no two-valued morphisms, and so is intractable.

We may sum up our findings so far by the assertion:

> *A pBa B is intractable if and only if there is a skeleton which is representable in B and whose associated formula is a classical contradiction.*

If $\mathcal{S}$ is representable in B and $\phi_{\mathcal{S}}$ is a classical contradiction, it may be regarded as a *canonical* example of a formula which is true in B and at the same time a classical contradiction.

For any Hilbert space H, write B(H) for the pBa of closed subspaces of H, and E_n for n-dimensional Euclidean space. Kochen and Specker showed, in effect, that some finitely generated subalgebra of the pBa $B(E_3)$ is intractable. (We note that, by the Corollary above, it suffices for this purpose to show that $B(E_3)$ itself is intractable, which of course follows from Gleason's (1957) theorem.) For intractable subalgebras of $B(E_3)$, we can always find a skeletal representation of a particularly simple form which gives rise to a canonical contradiction. This is because each element of $B(E_3)$ is of the form $0, 1, a, a^{\perp}$ where a is an *atom* (= 1-dimensional subspace of E_3). Accordingly, corresponding to each conjunct $q_i \vee q_j \leftrightarrow \neg q_k$ in a canonical contradiction for an intractable subalgebra of $B(E_3)$ is an equality $b_i \vee b_j = b_k{}^{\perp}$ in which b_i, b_j, b_k are compatible, $b_i \neq b_j$, and each is either an atom or a complement of one. It is easy to see that there are then only three possibilities:

(i) b_i, b_j, b_k are mutually orthogonal atoms;
(ii) $b_i, b_j{}^{\perp}$ are atoms, $b_k = b_j{}^{\perp}$, and $b_i \leq b_j$;
(iii) $b_i{}^{\perp}, b_j$ are atoms, $b_k = b_i{}^{\perp}$, and $b_j \leq b_i$.

Now for each i define $e_i \in B(E_3)$ by $e_i = b_i$ if b_i is an atom, $e_i = b_i{}^{\perp}$ if $b_i{}^{\perp}$ is an atom. Then corresponding to each conjunct $q_i \vee q_j \leftrightarrow \neg q_k$ in a canonical contradiction is a triple (e_i, e_j, e_k) of atoms for which either (a) e_i, e_j, e_k are mutually orthogonal, or (b) e_i is orthogonal to e_j, and e_k is either e_i or e_j. (Similar, but more complicated, representations can be formulated for $B(E_n)$ with $n \geq 4$.)

It is interesting to note that the negation of the formula constructed by Kochen and Specker ((1967), Cor. to Thm. 4) to establish the intractability of a finitely generated subalgebra of $B(E_3)$ is canonical in the sense of this note (and in fact only involves conjuncts arising under clause (a) above). For the negation of their formula is a conjunction of formulas of the form

$$q_i + q_j + q_k + q_i \wedge q_j \wedge q_k \qquad (***)$$

(where "+" denotes exclusive disjunction) over a finite set of orthogonal triples of atoms (e_i, e_j, e_k) in $B(E_3)$. And it is easy to see that (***) is equivalent to the formula

$$(q_i \vee q_j \leftrightarrow \neg q_k) \wedge (q_k \vee q_i \leftrightarrow \neg q_j) \wedge (q_j \vee q_k \leftrightarrow \neg q_i)$$

since both assert that exactly one of q_i, q_j, q_k is true. Therefore, any conjunction of formulas of the form (***) is canonical.

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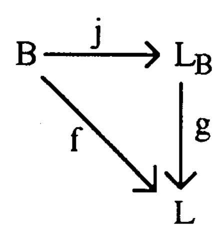
I conclude with two problems. The first of these is, to my knowledge, unresolved. The second was resolved some years ago, but this fact does not seem to be well-known.

In producing examples of intractable pBa's, Kochen and Specker in effect showed that there is no reflection from *PBA* into its subcategory *Bool*. Suppose, however, that we replace *Bool* by the category *Orth* of ortholattices and orthohomomorphisms. *Orth* may be identified as a (non-full) subcategory of *PBA* by identifying each ortholattice L with the corresponding pBa $L^{\sim}$ defined at the beginning of the paper. We may now ask whether there is a reflection *PBA*→*Orth*, that is, corresponding to each pBa B there is a pair (L_B, j) consisting of an *ortholattice* L_B and a morphism $j: B \to L_B$ such that, for any morphism f of B to an ortholattice L, there is a unique orthohomomorphism $g: L_B \to L$ such that the diagram



commutes. L_B, if it exists, would be the best "ortholattice approximation" to B. It is not hard to see that L_B exists if and only if there is a morphism of B to some ortholattice: since all the examples of intractable pBa's known to me arise as partial subalgebras of ortholattices, L_B certainly exists in these cases. I do not know whether L_B exists for arbitrary (intractable) B.

Finally, we note that the intractability of $B(E_3)$ implies the intractability of any $B(E_n)$ with $n>3$, despite the fact that $B(E_3)$ is not a partial subalgebra of $B(E_n)$ (if it were, the implication would be automatic). For if h were any two-valued morphism on $B(E_n)$ with $n>3$, then h must send some atom a to 1, so that the restriction of h to the partial

Boolean algebra of subspaces of any E_3 containing a is a two-valued homomorphism. Now consider any *infinite* dimensional Hilbert space H. Is B(H) intractable? Since no $B(E_n)$ is actually a partial subalgebra of B(H), and a two-valued morphism on B(H) need not send *any* finite-dimensional subspace of H to 1 (since it need not define a *countably* additive measure in the sense of Gleason (1957)), the intractability of the latter does *not* immediately follow.[4]

However, a straightforward argument, (essentially) due to Jost (1976)[5], shows that each $B(E_n)$ is *embeddable* in B(H), from which the latter's intractability follows easily. To show that $B(E_n)$ is embeddable in B(H), let $\{e_0, e_1, \ldots\}$, $\{a_0, \ldots, a_{n-1}\}$ be orthonormal bases for H and E_n respectively, and let H_n be the subspace of H generated by $\{e_{kn}: k=0,1,\ldots\}$. Then the map

$$e_{kn} \otimes a_i \mapsto e_{kn+i} \quad i=0,\ldots,n-1$$

is a bijection between the bases of $H_n \otimes E_n$ and H, which induces in a natural way an isomorphism between $H_n \otimes E_n$ and H, and hence an isomorphism j between $B(H_n \otimes E_n)$ and B(H). It follows that the map which sends each subspace S of E_n to $j(H_n \otimes S)$ is an embedding of $B(E_n)$ into B(H).

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REFERENCES

Birkhoff, G. (1984), *Lattice Theory* (3rd ed.). American Mathematical Society Colloquium Publications.

Cook, J. M. (1968), "Review of Kochen and Specker (1967)", *Mathematical Reviews* *36:* 2363.

Gleason, A. M. (1957), "Measures on the Closed Subspaces of a Hilbert Space", reprinted in Hooker (1975), pp. 123-134.

Hooker, C. A. (ed.) (1975), *The Logico-Algebraic Approach to Quantum Mechanics. Vol. I.* Dordrecht: Reidel.

Jost, R. (1976), "Measures on the Finite Dimensional Subspaces of a Hilbert Space: Remarks on a Theorem by A. M. Gleason", in E. H. Lieb, B. Simon, and S. Wightman (eds.), *Studies in Mathematical Physics: Essays in Honour of Valentine Bergmann*, Princeton: Princeton University Press, pp. 209-228.

Kamber, F. (1964), "The Structure of the Propositional Calculus of a Physical Theory", reprinted in Hooker (1975), pp. 221-245.

Kochen, S. and E. Specker. (1967), "The Problem of Hidden Variables in Quantum Mechanics", reprinted in Hooker (1975), pp. 293-328.

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[4]This observation seems first to have been made in Cook (1968). I thank William Demopoulos for bringing this review to my attention.

[5]I am grateful to Michael Kernaghan and William Demopoulos for bringing this paper to my attention.