A New Approach to Quantum Logic

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The idea of a 'logic of quantum mechanics' or *quantum logic* was originally suggested by Birkhoff and von Neumann in their pioneering paper [1936]. Since that time there has been much argument about whether, or in what sense, quantum 'logic' can be actually considered a true logic (see, *e.g.* Bell and Hallett [1982], Dummett [1976], Gardner [1971]) and, if so, how it is to be distinguished from classical logic. In this paper I put forward a simple and natural semantical framework for quantum logic which reveals its difference from classical logic in a strikingly intuitive way, *viz.* through the fact that quantum logic admits (suitably formulated versions of) the characteristic quantum-mechanical notions of *superposition* and *incompatibility* of attributes. That is, precisely the features that distinguish quantum from classical *physics* also serve, within this framework, to distinguish quantum from classical *logic*. Some light is shed on the question of whether quantum logic is a genuine logical system by introducing a natural *entailment* relation for quantum-logical formulas with the implication symbol. The novelty is that, although implication behaves as it should (*i.e.* the 'deduction theorem' holds), the *order of introduction of premises* is significant. The fact that a reasonable entailment relation can be formulated for quantum logic supports the view that it is a genuine logical system and not merely an algebraic formalism.

The paper is organised as follows. We begin with an account of the origins of quantum logic, based on Birkhoff and von Neumann [1936]. In §2 a common semantical framework for intuitionistic, classical and quantum logic is formulated, employing the notion of an *attribute* over a space with a distinguished lattice of subsets (this framework was first introduced in Bell [1983]). In §3 we define the central concept of *manifestation of attributes* and employ it to distinguish (intuitionistic and) classical logic from quantum logic. In §4 we introduce the logical operation of *implication* and show how the extension of the concept of manifestation to implication formulas leads both to general notions of *superposition* and *incompatibility* characteristic of quantum logic, and to the entailment relation mentioned above. Finally, in §5 we observe that the concept of superposition introduced here satisfies the conditions originally laid down by Dirac [1930] and that, interpreted within the 'orthodox' framework for quantum mechanics, it coincides with the usual notion of superposition of states.

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Readers not familiar with the mathematical formalism of quantum mechanics may omit sections 1 and 5 without substantial loss.

1 THE ORIGINS OF QUANTUM LOGIC

Let $\mathscr{S}$ be a classical physical system and let Σ be its phase space. We may regard an *observable* on $\mathscr{S}$ as being a function $f: \Sigma \to \Omega$ where the codomain Ω, the *observation space* of f, is the set of 'values' that f can assume. (Typically Ω will be a set of real numbers.) If $f_1, \ldots, f_n$ are observables on $\mathscr{S}$ with observation spaces $\Omega_1, \ldots, \Omega_n$, the observation space associated with the n-tuple of observables $(f_1, \ldots, f_n)$ is the Cartesian product $\Omega_1 \times, \ldots, \times \Omega_n$. Each subset X of $\Omega_1 \times \ldots \times \Omega_n$ is correlated with a *proposition* P_X concerning the state of x of $\mathscr{S}$, namely the assertion that the n-tuple of measured values of $f_1, \ldots, f_n$ lies in X when $\mathscr{S}$ is in state x. X has a *representative* $\hat{X}$ in Σ defined by

$$\hat{X} = \{x \in \Sigma : (f_1(x), \ldots, f_n(x)) \in X\}.$$

Thus $\hat{X}$ is the set of states x of $\mathscr{S}$ such that P_X is verified when $\mathscr{S}$ is in state x. Accordingly we may also call $\hat{X}$ the representative in Σ of the associated proposition P_X.

Notice that the relation of *entailment* between propositions corresponds to the relation of set-theoretic *inclusion* between their representatives; that the representative of the *negation* of a proposition is the set-theoretical complement (in Σ) of its representative; and that the representative of the *conjunction* of two propositions is the set-theoretical *intersection* of their representatives. It follows that the logic of propositions concerning a classical system $\mathscr{S}$ is isomorphic to a *Boolean algebra* of subsets of the phase space of $\mathscr{S}$.

Turning now to the case of *compatible* observables in a *quantum* system, we find that the situation is broadly similar. Thus let $\mathscr{Q}$ be a quantum system, H its phase space (Hilbert space) and $A_1, \ldots, A_n$ compatible observables on $\mathscr{Q}$, i.e. commuting self-adjoint linear operators on H. (For simplicity we shall assume that the eigenvalues of $A_1, \ldots, A_n$ are discrete and nondegenerate.) Since $A_1, \ldots, A_n$ commute, H has a basis $(b_0, b_1, \ldots)$ consisting of common eigenstates for the A_i. For each $i = 1, \ldots, n$; $j = 0, 1, 2, \ldots$ let λ_j^i be the eigenvalue of A_i corresponding to the eigenstate b_j. Then for each i, the set $\{\lambda_j^i : j = 0, 1, 2, \ldots\}$ lists all the possible values that the observable A_i can assume, and may accordingly be regarded as being the *observation space* of A_i.

What is the observation space for the n-tuple of observables $(f_1, \ldots, f_n)$? To determine this, let $(k_1, \ldots, k_n)$ be an n-tuple of natural numbers, and suppose that we are *certain* to get the result $(\lambda_{k_1}^1, \ldots, \lambda_{k_n}^n)$ by simultaneously measuring $A_1, \ldots, A_n$. The only state of $\mathscr{Q}$ in which we are *certain* to get the result $\lambda_{k_i}^i$ by measuring A_i is (up to a scalar factor) b_{k_i}. So we are only *certain* to get the result $(\lambda_{k_1}^1, \ldots, \lambda_{k_n}^n)$ by measuring $(A_1, \ldots, A_n)$ when all the b_{k_i} are *identical*, i.e. when $k_1 = k_2 = \cdots = k_n$. (To put it another way, if $\mathcal{Q}$ is in a state such that a measurement of one A_i is certain to give the result λ_j^i, then a measurement of *any* A_i is certain to give the result λ_j^i for the same j.) It follows that the *observation space* for $(A_1, \ldots, A_n)$ is the set of n-tuples $\{(\lambda_j^1, \ldots, \lambda_j^n): j = 0, 1, 2, \ldots\}$. Since the elements of this observation space are indexed by the set N of natural numbers, each subset X of N is correlated with a proposition P_X concerning the state x of $\mathcal{Q}$, viz., the assertion that the n-tuple of measured values of $A_1, \ldots, A_n$ is *certain* to be in the set $\{(\lambda_j^1, \ldots, \lambda_j^n): j \in X\}$ when $\mathcal{Q}$ is in state x. For each subset X of N we define the *representative* $\hat{X}$ of P_X (or X) to be the *closed subspace* of H generated by the set $\{b_n: n \in X\}$: this is natural since the elements of $\hat{X}$ are precisely those states x of $\mathcal{Q}$ such that the proposition P_X is verified when $\mathcal{Q}$ is in state x.

In this case, too, we find that the relation of entailment between propositions corresponds to set-theoretical inclusion of their representatives and that the representative of the conjunction of two propositions is the set-theoretical intersection of their representatives. However, the representative of the *negation* of a proposition is *no longer* the set-theoretical complement, but rather the *orthogonal complement* of its representative. Nevertheless, it still follows that the logic of propositions involving *compatible* observables on a quantum system $\mathcal{Q}$ is isomorphic to a *Boolean algebra* of (closed) subspaces of the phase space of $\mathcal{Q}$.

So far so good. The difficulty arises when we try to extend the analysis to *incompatible* (*i.e.* non-commuting) observables: since non-commuting operators have no common eigenbasis, the whole procedure collapses. Thus, for example, given two incompatible observables A, B, we can perfectly well form the observation spaces of A and B separately and then consider the representatives in H of propositions involving *only* A and propositions involving *only* B. But we have no way of representing propositions involving *both* A and B, *e.g.* the conjunction of propositions of the above sort. In their original paper [1936], Birkhoff and von Neumann propose to remove this obstruction by *postulating* that the intersection of the representatives of *any* pair of propositions—even those involving incompatible observables—is still the representative of *some* proposition, namely the 'conjunction' of the pair. (Of course, this is already the case for propositions involving only *compatible* observables.) As they point out, the simplest (if Procrustean) way of ensuring that this postulate holds is to assume that all self-adjoint operators on H are (or correspond to) observables. In this event, every closed subspace of H is the representative of a proposition and the (ortho)lattice of closed subspaces may then be regarded as the mathematical embodiment of a 'logic' of propositions, the so-called *quantum logic*.

The problem with this approach is that, while the *mathematical* meaning of the operations of intersection and orthogonal complementation on the subspaces of H is perfectly clear, the *logical* meaning of the corresponding operations of 'conjunction' and 'negation' on the associated propositions is *not*. Thus arises the fundamental *problem of meaning* of quantum logic.

My attempt ro resolve this problem will hinge on two things: the replacement of Hilbert spaces by more perspicuous structures, the so-called *proximity spaces*, and the analysis of propositional quantum logic in terms of the concept of an *attribute* defined over such a space. We turn to this now.

2 PROPOSITIONAL LOGIC AS A LOGIC OF ATTRIBUTES

Let us think of *attributes* or *qualities* like 'blackness', 'hardness', 'having positive charge', etc. as being *possessed by* or *manifested over* parts of a *space* (sometimes called a manifold or field). For instance, if the space is my sensory field, part of it manifests blackness and part manifests hardness and, e.g., a blackboard manifests both attributes. Each attribute is correlated with a *proposition* (more precisely, a propositional function) of the form: '____ has the attribute in question.'

We shall use symbols A, B, C, to denote attributes. We assume that we are initially provided with a supply of *atomic* or *primitive* attributes, i.e. attributes not decomposable into simpler ones. For each such attribute A and each space S we also consider as given the total part of S which manifests A; this will be called the *A-part* of S and denoted by $[\![A]\!]_S$. (Thus, for instance, if S is my sensory field and A is the attribute 'red', then $[\![A]\!]_S$ is the total part of S that is coloured red: the red part of S.)

Attributes may be *combined* by means of the logical operators $\wedge$ (and), $\vee$ (or), $\neg$ (not) to form *compound* or *molecular* attributes.[1] The term 'attribute' will accordingly be extended to include compound attributes as well as primitive ones. It follows that (symbols for) attributes may be regarded as the formulas of a propositional language $\mathscr{L}$—the *language of attributes*—and we shall use the terms 'attribute' and 'formula' synonymously.

In order to be able to correlate parts of any given space S with *compound* attributes, i.e. to be able to define the A-part of S for compound A, we need to assume the presence of operations $\boldsymbol{\wedge}$, $\boldsymbol{\vee}$, $*$ (corresponding to $\wedge$, $\vee$, $\neg$) on the parts of S. For then we will be able to define the A-part $[\![A]\!]_S$ of the space for arbitrary attributes A by recursion on the number of logical operators in A according to the following scheme:

$$\left.\begin{aligned} [\![A \wedge B]\!]_S &= [\![A]\!]_S \boldsymbol{\wedge} [\![B]\!]_S \\ [\![A \vee B]\!]_S &= [\![A]\!]_S \boldsymbol{\vee} [\![B]\!]_S \\ [\![\neg A]\!]_S &= ([\![A]\!]_S)^* \end{aligned}\right\} \qquad (2.1)$$

($[\![A]\!]_S$ is also called the *value* of A in S.) Once this is done, we can then define the basic relation $\models_S$ of *entailment* or *inclusion* between attributes over S:

$$A \models_S B \quad \text{iff} \quad [\![A]\!]_S \subseteq [\![B]\!]_S.$$

Now the conventional meaning of '$\wedge$' dictates that, for any attributes A

[1] Note that '$\rightarrow$' (implication) is for the moment omitted. We make up for this deficiency in §4.

and B, we should have $A \wedge B \models_{\mathbf{S}} A$, $A \wedge B \models_{\mathbf{S}} B$ and, for any C, if $C \models_{\mathbf{S}} A$ and $C \models_{\mathbf{S}} B$ then $C \models_{\mathbf{S}} A \wedge B$. In other words, $[\![A \wedge B]\!]_{\mathbf{S}}$ should be taken to be the largest part (with respect to set-theoretic inclusion) of $\mathbf{S}$ included in both $[\![A]\!]_{\mathbf{S}}$ and $[\![B]\!]_{\mathbf{S}}$. By the first equation of (2.1), the same must then be true of $[\![A]\!]_{\mathbf{S}} \wedge [\![B]\!]_{\mathbf{S}}$. Consequently, for any parts U, V of $\mathbf{S}$, $U \wedge V$ should be the largest part of $\mathbf{S}$ included in both U and V.

Similarly, now using the conventional meaning of '$\vee$', we conclude that, for any parts U, V of $\mathbf{S}$, $U \vee V$ should be the smallest part of $\mathbf{S}$ which includes both U and V.

We suppose that '$\neg$' satisfies the law of *ex falso quodlibet*: thus if A is an attribute, then $A \wedge \neg A \models_{\mathbf{S}} B$ for any B. In other words $[\![A \wedge \neg A]\!]_{\mathbf{S}} \subseteq [\![B]\!]_{\mathbf{S}}$ or, using (2.1), $[\![A]\!]_{\mathbf{S}} \wedge [\![A]\!]_{\mathbf{S}}^{*} \subset [\![B]\!]_{\mathbf{S}}$ for any B. If we assume that there is a *vacuous* attribute B for which $[\![B]\!]_{\mathbf{S}} = \phi$, the empty part of $\mathbf{S}$, it follows that $[\![A]\!]_{\mathbf{S}} \wedge [\![A]\!]_{\mathbf{S}}^{*} = \phi$. Consequently, for any part U of $\mathbf{S}$ we should require that $U \wedge U^{*} = \phi$, *i.e.* that U and U^{*} be 'mutually exclusive'.

It follows from these considerations that we should take the parts of a space $\mathbf{S}$ to constitute a *lattice of subsets* of (the underlying set of) $\mathbf{S}$, on which is defined an additional operation $*$ ('complementation') corresponding to negation (or exclusion) satisfying the condition of mutual exclusiveness mentioned above. Formally, a *lattice of subsets* of a set S is a family L of subsets of S containing ϕ and S such that for any U, $V \in L$ there are elements $U \wedge V$, $U \vee V \in L$ such that $U \wedge V$ is the largest (with respect to $\subseteq$) element of L included in both U and V and $U \vee V$ is the smallest (with respect to $\subseteq$) element of L which includes both U and V. $U \wedge V$, $U \vee V$ are called the *meet* and *join*, respectively, of U and V. A lattice of subsets of S equipped with an operation $*: L \to L$ satisfying $U \wedge U^{*} = \phi$ for all $U \in L$ will be called a *$*$-lattice* of subsets of S.

We can now formally define a *space* to be a pair $\mathbf{S} = (S, L)$ consisting of a set S and a $*$-lattice L of subsets of S. Elements of L are called *parts* of $\mathbf{S}$, and L is called the *lattice of parts* of $\mathbf{S}$.

In practice we shall only need to consider the following sorts of space, so henceforth the term 'space' will connote one of the following 3 kinds:

(1) *Topological spaces*. In this case $\mathbf{S} = (S, L)$ is a set S equipped with a *topology* L. Here the meet and join operations in L are just set-theoretical intersection and union, and the $*$ operation is given by $U^{*} =$ interior of $S - U$, for $U \in L$.

(2) *Discrete spaces*. These are the special cases of (1) in which the topology L on S is the family PS of *all* subsets of S. The $*$-operation on L is then just ordinary set-theoretic complementation in S.

(3) *Proximity spaces*. *A proximity structure* is a set S equipped with a *proximity relation*, *i.e.* a symmetric reflexive binary relation $\approx$. (The reason for using the term 'proximity' is, as we shall see, that it is helpful to think of $x \approx y$ as meaning that x is *near* y. Caution: $\approx$ is *not* generally transitive!) For

each $x \in S$ we define the *quantum* at x, Q_x, by

$$Q_x = \{y \in S: x \approx y\}. \qquad (2.2)$$

Unions of families of quanta are called *parts* of S; thus a part of S is a subset of the form

$$\bigcup_{x \in A} Q_x$$

for $A \subseteq S$. It can be shown (see Bell [1983]) that the family Part(S) of parts of S forms a $*$-lattice[1] of subsets of S, in which the join operation is set-theoretical union, the meet of two parts of S is the union of all quanta included in their set-theoretical intersection, and, for $U \in \text{Part}(S)$,

$$U^* = \{y \in S: \exists x \notin U \cdot x \approx y\}. \qquad (2.3)$$

The pair $\mathbf{S} = (S, \text{Part}(S))$ is called a *proximity space*.

Observe that any discrete space is a proximity space in which $\approx$ is the equality relation. More generally, it is quite easily shown that a proximity space **S** is a topological space if and only if its proximity relation is transitive, and that in this case **S** is *almost* discrete in the sense that its lattice of parts is isomorphic to the lattice of parts of a discrete space.

Proximity structures (or spaces) S admit several interpretations which serve to reveal their significance.

(a) S may be viewed as a 'space' or field of perception, its points as *locations* in it, the relation $\approx$ as representing the *indiscernibility of locations*, the quantum at a given location as the *minimum perceptibilium* at that location, and the parts of S as the perceptibly specifiable subregions of S. This idea is best illustrated by assigning the set S a metric δ, choosing a fixed $\varepsilon > 0$ and then defining $x \approx y \leftrightarrow \delta(x, y) \leqslant \varepsilon$.

(b) S may be thought of as the set of *outcomes of an experiment* and $\approx$ as the relation of equality *up to the limits of experimental error*. The quantum at an outcome is then the 'outcome within a specified margin of error' of experimental practice.

(c) S may be taken to be the set of *states* of a quantum system and $s \approx t$ as the relation: 'a measurement of the system in state s has a non-zero probability of leaving the system in state t, or vice-versa.' More precisely, we take a Hilbert space H, put $S = H - \{0\}$, and define the proximity relation $\approx$ on S by $s \approx t \leftrightarrow \langle s, t\rangle \neq 0$ (s is not orthogonal to t). It is then readily shown that the $*$-lattice of parts of S is isomorphic to the $*$-(ortho)lattice of closed subspaces of H. Consequently, *$*$-lattices of parts of proximity spaces include the $*$-lattices of closed subspaces of Hilbert spaces—the lattices associated with Birkhoff and von Neumann's 'quantum logic'*. This observation will be employed later on.

[1] Actually Part(S) has the structure of a *complete ortholattice* (see Bell [1983] or Birkhoff [1960]) for we have, for any $U, V \in \text{Part(S)}$, $U^{**} = U$, $U \cup U^* = S$, $U \wedge U^* = \phi$, $U \subseteq V \Rightarrow U^* \supseteq V^*$.

(d) S may be taken to be the set of *hyperreal numbers* in a model of Robinson's nonstandard analysis (see, *e.g.* Bell and Machover [1977]) and $\approx$ as the relation of infinitesimal nearness. In this case $\approx$ is *transitive*.

(e) S may be taken to be the *affine line* in a model of synthetic differential geometry (see Kock [1981]). In this case there exist many square-zero infinitesimals in S, *i.e.* elements $\varepsilon \neq 0$ such that $\varepsilon^2 = 0$, and we take $x \approx y$ to mean that the difference $x-y$ is such an infinitesimal, *i.e.* $(x-y)^2 = 0$. Unlike the situation in case (d), the relation $\approx$ here is *not* generally transitive.

Given a space $\mathbf{S} = (S, L)$ we define an *interpretation* of the language $\mathscr{L}$ of attributes to be an assignment, to each primitive attribute A (*i.e.* atomic formula of $\mathscr{L}$) of a part $[\![A]\!]_{\mathbf{S}}$ of $\mathbf{S}$. Then we can extend the assignment of parts of $\mathbf{S}$ to all attributes recursively as in (2.1).

Let us call a formula A $\mathbf{S}$-*valid* if $[\![A]\!]_{\mathbf{S}} = S$. If $\mathscr{M}$ is a class of spaces, we say that A is $\mathscr{M}$-*valid* if it is $\mathbf{S}$-valid for all $\mathbf{S} \in \mathscr{M}$. The purpose of introducing this concept of validity is that it enables us to characterise the tautological statements (truths) of various logical systems. Let $\mathscr{T}\!\mathit{op}$, $\mathscr{D}\!\mathit{is}$ and $\mathscr{P}\!\mathit{rox}$ be the classes of topological spaces, discrete spaces and proximity spaces, respectively. It is well known (*cf.* Rasiowa and Sikorski [1963], ch. IX, §3) that the *$\mathscr{T}\!op$-valid formulas of $\mathscr{L}$ coincide with the tautologies of intuitionistic logic in $\mathscr{L}$, and* (*ibid.*, *ch. VII*, §1) *the $\mathscr{D}\!is$-valid formulas with the tautologies of classical logic*. Now, as we have observed, the lattices of parts of proximity spaces include the lattices associated with Birkhoff and von Neumann's 'quantum logic'. So it is natural to *identify the $\mathscr{P}\!rox$-valid formulas* (*of $\mathscr{L}$*) *as the tautologies of* quantum logic (*in $\mathscr{L}$*).

Let us write I, K, Q for the sets of tautologies of intuitionistic, classical, and quantum logic, respectively. Clearly we have the relation

$$I \cup Q \subseteq K.$$

Moreover, we have

$$Q \underset{\neq}{\subset} I, \quad I \underset{\neq}{\subset} Q, \quad I \cup Q \neq K$$

since, for formulas A, B,

$$A \vee \neg A \in Q - I \tag{2.4}$$

$$\neg[A \wedge \neg(A \wedge B) \wedge \neg(A \wedge \neg B)] \in I - Q \tag{2.5}$$

$$\neg A \vee (A \wedge B) \vee (A \wedge \neg B) \in K - (I \cup Q). \tag{2.6}$$

To prove (2.4), we note that $A \vee \neg A \in Q$ is an immediate consequence of the evident fact that $U \cup U^* = S$ for any part U of a proximity space $\mathbf{S}$ (where U^* is defined in (2.3)). That $A \vee \neg A \notin I$ is, of course, well-known.

For (2.5), the formula C on the left-hand side is evidently a classical tautology and contains no connectives except $\wedge$ and $\neg$. So by a well-known result of Gödel (*ibid.*, ch. IX, §5) C is an intuitionistic tautology and hence

$C \in I$. To show that $C \notin Q$, it is enough to construct a proximity space S and an interpretation of primitive formulas A, B in S for which $[\![C]\!]_S \neq S$. To this end let S be the set $\{0, 1, 2, 3\}$; define the relation $\approx$ on S by

$$m \approx n \leftrightarrow |m-n| \neq 2.$$

Clearly $\approx$ is a proximity relation on S. Define an interpretation of A, B in the resulting proximity space S by $[\![A]\!]_S = Q_0$, $[\![B]\!]_S = Q_1$ (recalling the definition of Q_x given in (2.2)). It is then easily verified that

$$Q_0^* = Q_2, \quad Q_1^* = Q_3, \quad Q_0 \wedge Q_1 = Q_0 \wedge Q_1^* = \phi.$$

Consequently,

$$\begin{aligned}[\![C]\!]_S &= [\![\neg A]\!]_S \wedge [\![\neg(A \wedge B)]\!]_S \wedge [\![\neg(A \wedge \neg B)]\!]_S \\ &= Q_0^* \wedge S \wedge S = Q_2 \neq S.\end{aligned}$$

The result follows.

As for (2.6), the formula D on the left-hand side is evidently a classical tautology. It cannot, on the other hand, be an intuitionistic tautology since, if it were, by taking A to be itself an intuitionistic tautology, it would follow that $B \vee \neg B$ is an intuitionistic tautology, which as we know is not the case. To see, finally, that $D \notin Q$, one uses the proximity space S defined above and verifies that

$$[\![D]\!]_S = [\![C]\!]_S \neq S.$$

Thus quantum logic (as we have defined it) *may* be distinguished from classical (and intuitionistic) logic by the assertion that the formula displayed in (2.5)—a weak, if recherché, version of the *distributive law*—is a tautology of the latter systems but not of the former. But this, it seems to me, is a technical and somewhat opaque method of drawing the distinction: in the next section we show how to formulate it in a more striking and intuitively convincing way.

3 THE MANIFESTATION OF ATTRIBUTES

Given a space S and an interpretation of the language of attributes $\mathscr{L}$ in S, an attribute A and a part U of S, it is natural to consider the relation $U \subseteq [\![A]\!]_S$ as meaning that the part U is *covered* by the attribute A. Now for topological (and discrete) spaces there is another way of obtaining the covering relation, which is reminiscent of the definition of set-theoretic forcing. Namely, we define the relation $U \Vdash_S A$, which shall be read *U manifests A* in S, by recursion on the number of logical symbols in A as follows:

$U \Vdash_S A \Leftrightarrow U \subseteq [\![A]\!]_S$ for primitive A

$U \Vdash_S A \wedge B \Leftrightarrow U \Vdash_S A \;\&\; U \Vdash_S B$

$U \Vdash_{\mathbf{S}} A \vee B \Leftrightarrow V \Vdash_{\mathbf{S}} A \;\&\; W \Vdash_{\mathbf{S}} B$ for some parts V, W of **S** such that $U = V \cup W$

$U \Vdash_{\mathbf{S}} \neg A \Leftrightarrow [V \Vdash_{\mathbf{S}} A \Rightarrow V \subseteq U^*]$ for all parts V of **S**.

Thus U manifests a disjunction $A \vee B$ provided there is 'covering' of U by two 'subparts' manifesting A and B respectively, and U manifests a negation $\neg A$ provided any part of **S** manifesting A is included in the 'complement' of U.

Now it is easily shown by induction on the number of logical symbols in formulas that for topological (and discrete) spaces **S**,

$$U \Vdash_{\mathbf{S}} A \Leftrightarrow U \subseteq [\![A]\!]_{\mathbf{S}}. \tag{3.1}$$

That is, for topological (and discrete) spaces, *the covering relation and the manifestation relation coincide*. However, as we shall see, *for proximity spaces this is no longer the case*. And, as we show presently, it is the *manifestation relation* which is of real interest in this situation.

The coincidence of the manifestation and covering relations for topological spaces has the following immediate consequence. Defining a space **S** to *support* an attribute (formula) A if $S \Vdash_{\mathbf{S}} A$ (which we shall abbreviate simply to $\Vdash_{\mathbf{S}} A$), then *the tautologies of intuitionistic* (*resp. classical*) *logic are those formulas which are supported by every topological* (*resp. discrete*) *space*. At the time of writing it is not known whether this result extends to quantum logic, *i.e.* whether the tautologies of quantum logic coincide with the formulas which are supported by every proximity space. (The claim in Bell [1983] that this is the case was based on a result (Theorem 2.4 of that paper) which has turned out to be false.) However, it can be shown that, for example, the quantum-logical tuatology $A \vee \neg A$ is supported by every proximity space (as are, additionally, all quantum-logical tautologies not containing '$\vee$').

Let us call an attribute A **S**-*persistent* (or persistent over **S**) if for all parts U, V of **S**

$$V \subseteq U \;\&\; U \Vdash_{\mathbf{S}} A \Rightarrow V \Vdash_{\mathbf{S}} A.$$

(Note that a *primitive* attribute is always **S**-persistent. More generally, it is not hard to show that the same is true for any attribute A not containing occurrences of the disjunction symbol $\vee$.) And let us call a space **S** *persistent* if every attribute is **S**-persistent (for any interpretation of $\mathscr{L}$ in **S**). By (3.1), every topological (or discrete) space is persistent, so in particular the tautologies of intuitionistic or classical logic are persistent over their associated spaces (topological or discrete, respectively). As we now show, in striking contrast, there are tautologies of quantum logic which are *not* persistent over their associated spaces, *viz.*, proximity spaces. This is revealed by the following simple example of a *non-persistent proximity space*.

Consider the real line R with the proximity relation: $x \approx y \leftrightarrow |x-y| \leqslant \frac{1}{2}$ and let **R** be the associated proximity space. The quantum at a point $x \in \mathsf{R}$ is then the closed interval of length l centred on x. Suppose now we are given

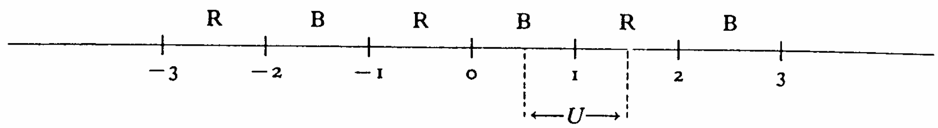

Figure 1.

two primitive attributes B ('black') and R ('red'). Define interpretations of B and R in $\mathbf{R}$ by

$$[\![B]\!]_{\mathbf{R}} = \cup\{[2n, 2n+1]: n \in Z\}$$
$$[\![R]\!]_{\mathbf{R}} = \cup\{[2n-1, 2n]: n \in Z\}.$$

(Here Z is the set of all positive and negative integers and $[a, b]$ is the closed interval with endpoints a, b.) To put it more vividly, we 'colour' successive unit segments of R alternately black and red. Clearly, then, $\mathbf{R}$ supports the disjunction $R \vee B$. But if U is the quantum $Q_1 = [\frac{1}{2}, \frac{3}{2}]$ then $R \vee B$ is *not* manifested over U, since U is evidently not covered by two subparts over which R and B are manifested, respectively (indeed, U has no proper subparts). Equally clearly, U does not manifest the quantum-logical tautology $R \vee \neg R$ (nor, of course, $B \vee \neg B$).

Thus arises the curious phenomenon that, although we can see, by surveying (a sufficiently large part of) the whole space $\mathbf{R}$, that the part U is covered by redness and blackness, nonetheless U—unlike R—does not split into a red part and a black part. In some sense redness and blackness are conjoined or *superposed* in U: it seems natural then to say that U manifests a *superposition* of these attributes rather than a *disjunction*.

This concept of superposition of attributes turns out to admit a very simple rigorous formulation. In the example we have just considered, the part U manifests a superposition of the attributes R and B just when there is a part V of the space which includes U and manifests $R \vee B$ (in this case, V may be taken to be the whole space R). Now this inevitably prompts the following definition. Given a proximity space $\mathbf{S}$, an interpretation of $\mathscr{L}$ in $\mathbf{S}$ and attributes A, B, we say that a part U of S *manifests a superposition* of A and B if there is a part V of S such that $U \subseteq V$ and $V \Vdash_{\mathbf{S}} A \vee B$. Now for any attribute C, it is readily shown that

$$\exists V \supseteq U \cdot V \Vdash_{\mathbf{S}} C \Leftrightarrow U \Vdash_{\mathbf{S}} \neg\neg C.$$

(Consequently, $\neg\neg C$ is persistent.) So the condition that U manifest a superposition of A and B is just

$$U \Vdash_{\mathbf{S}} \neg\neg(A \vee B).$$

It follows that ***superpositions are double negations of disjunctions***. We shall have more to say about superpositions in the sequel; in particular in the final section we shall see how this concept of superposition relates to the usual quantum-mechanical notion.

To conclude this section, we show how our space **R** can be enriched so as to furnish an interpretation of the quantum-mechanical notion of *incompatible* attributes. To this end, suppose that, in addition to the two primitive 'colour' attributes R and B, we are given two primitive 'charge' attributes $+$ and $-$. Write *Colour* for the disjunction $R \vee B$ and *Charge* for the disjunction $+ \vee -$. Interpret $+$ and $-$ in **R** by

$$\llbracket + \rrbracket_{\mathbf{R}} = \cup \left\{ \left[\frac{4n+1}{2}, \frac{4n+3}{2} \right] : n \in Z \right\}$$

$$\llbracket - \rrbracket_{\mathbf{R}} = \cup \left\{ \left[\frac{4n-1}{2}, \frac{4n+1}{2} \right] : n \in Z \right\}.$$

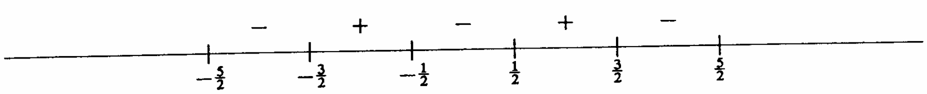

Figure 2.

Now clearly **R** supports *Colour* $\wedge$ *Charge*. But since, in **R**,

$$\llbracket + \rrbracket \wedge \llbracket R \rrbracket = \llbracket + \rrbracket \wedge \llbracket B \rrbracket = \llbracket - \rrbracket \wedge \llbracket R \rrbracket = \llbracket - \rrbracket \wedge \llbracket B \rrbracket = \phi,$$

we have

$$\Vdash_{\mathbf{R}} \neg(+ \wedge R) \wedge \neg(+ \wedge B) \wedge \neg(- \wedge R) \wedge \neg(- \wedge B).$$

In other words, *despite* the fact that the whole space R manifests both *Charge* and *Colour*, there is *no* non-empty part of the space which manifests both a *specific* charge and a *specific* colour. This situation is sufficiently similar to the familiar incompatibility of position and momentum measurements in quantum mechanics ('any particle has both a position and a momentum, but not a specific position and a specific momentum': *cf.* Putnam [1969]) to justify calling *Colour* and *Charge incompatible* attributes (over **R**). We shall have more to say about incompatibility once we have introduced the implication operation, a task we turn to in the next section.

4 INTRODUCING IMPLICATION

So far we have scrupulously avoided considering what is, in classical and intuitionistic logic, a logical operation of fundamental importance, *viz.*, *implication*. We shall now remedy this by expanding our language of attributes $\mathscr{L}$ to include the implication symbol $\rightarrow$.

When S is a *topological or discrete* space, its lattice L of parts has a naturally defined 'implication' operation → defined on it by

$$U \to V = \text{largest open set included in } V \cup (S - U)$$
$$(= V \cup (S - U) \text{ when } \mathbf{S} \text{ is discrete}).$$

So in either case $U \to V$ is the largest part of **S** whose intersection with U is included in V. We can extend the interpretation in **S** of formulas of $\mathscr{L}$ to include implication formulas $A \to B$ by the rule

$$[\![A \to B]\!]_{\mathbf{S}} = [\![A]\!]_{\mathbf{S}} \to [\![B]\!]_{\mathbf{S}}.$$

And it can then be readily shown that, if we extend the notion of validity to implication formulas in the obvious way, the $\mathscr{T}\!op$- (respectively $\mathscr{D}is$-) valid formulas (now also involving '→') continue to coincide with the intuitionistic (respectively, classical) tautologies.

In the case of proximity spaces, however, there is *no* entirely satisfactory way of defining the operation → on the lattice of parts, and so no evident way of interpreting '→'. (This, it may be said, is the source of the vexatious question of the meaning of '→' in quantum logic.) However, we can overcome this difficulty by extending the manifestation relation to implication formulas as follows. For any space **S**, we define

$$U \Vdash_{\mathbf{S}} A \to B \Leftrightarrow \forall V \subseteq U [V \Vdash_{\mathbf{S}} A \Rightarrow V \Vdash_{\mathbf{S}} B].$$

For topological and discrete spaces **S**, one can show that (3.1) continues to hold for any formulas, now including those involving '→', and so, again, the tautologies of intuitionistic (or classical logic) coincide with the formulas supported by every topological (or discrete) space. (Here the applicability of the term 'support' has been extended to include implication formulas.)

The introduction of → into $\mathscr{L}$ leads to simple and striking characterisations of the difference between classical and quantum logic. Let us identify the tautologies of what I shall term ***implicative quantum logic*** as those formulas, (now involving '→') supported by every proximity space. Now, one easily shows that for any space $\mathscr{S}$ an attribute A is $\mathscr{S}$-persistent if and only if, for any attribute B, $\mathscr{S}$ supports the formula $A \to (B \to A)$. Since, as we have seen, attributes are not generally persistent over proximity spaces, it follows that the formula $A \to (B \to A)$ is *not* a tautology of implicative quantum logic. This is consonant with the views of Mittelstaedt (*cf.*, *e.g.* Jammer [1974]) who regards the non-provability of $A \to (B \to A)$ as being *characteristic* of the difference between quantum and classical logic.

It is natural at this point to introduce the relation of *entailment* among formulas. If $\mathscr{C}$ is any class of spaces, we say that a sequence $A_1, \ldots, A_n$ (with $n \geqslant 1$) of formulas *$\mathscr{C}$-entails* a formula B, and write

$$A_1, \ldots, A_n \models_{\mathscr{C}} B$$

if, for any $\mathbf{S} \in \mathscr{C}$ we have

$$\Vdash_{\mathbf{S}} A_1 \to (A_2 \to \cdots \to (A_n \to B) \ldots).$$

We extend this notion of entailment to the case of the empty sequence of formulas by agreeing that

$$\models_{\mathscr{C}} B \Leftrightarrow \Vdash_{\mathbf{S}} B \quad \text{for every } \mathbf{S} \in \mathscr{C}.$$

When $\mathscr{C}$ is $\mathscr{T}\!op$ or $\mathscr{D}\!is$ this definition of entailment is the familiar one:

$$\begin{cases} A_1, \ldots, A_n \models_{\mathscr{T}\!op} B \Leftrightarrow (A_1 \wedge \ldots \wedge A_n) \rightarrow B \\ \qquad\qquad\qquad\qquad \text{is an intuitionistic tautology} \\ A_1, \ldots, A_n \models_{\mathscr{D}\!is} B \Leftrightarrow (A_1 \wedge \cdots \wedge A_n) \rightarrow B \\ \qquad\qquad\qquad\qquad \text{is a classical tautology} \end{cases} \tag{4.1}$$

That is, $A_1, \ldots, A_n \models_{\mathscr{T}\!op} B$ if and only if $A_1, \ldots, A_n$ intuitionistically entails B and $A_1, \ldots, A_n \models_{\mathscr{D}\!is} B$ if and only if $A_1, \ldots, A_n$ classically entails B. Analogously, it is natural to say that $A_1, \ldots, A_n$ *implicative quantum logically* entails B and write

$$A_1, \ldots, A_n \models_Q B$$

when $A_1, \ldots, A_n \models_{\mathscr{P}\!rox} B$.

Implicative quantum-logical entailment has the curious feature, not shared by classical or intuitionistic entailment, that the *order* of the premises $A_1, \ldots, A_n$ must be taken into account. (Consequently, in particular, there is no analogy to (4.1) for $\models_Q$.) For instance, although it is evidently the case that

$$A, B \models_Q B,$$

it is *not* generally the case that

$$B, A \models_Q B.$$

(To see this, take A = *Colour* and B = *Charge* in the example at the end of §3.) It therefore seems appropriate to say, adapting a phrase of Saul Bellow's, that in quantum logic the postulates have a tendency to decay before the end of the argument!

Observe also that the rule of introduction of premises on the left—valid for classical and intuitionistic logic—*fails* for $\models_Q$. For instance it is certainly the case that

$$\models_Q B \vee \neg B,$$

but *not* generally that

$$A \models_Q B \vee \neg B.$$

Indeed, if A and B are *primitive* attributes, then it is *never* the case that $A \models_Q B \vee \neg B$. To establish this, return to the space **S** used to verify (2.5). It is easy to see that, with the interpretations of A and B given there, we have $Q_0 \nVdash_{\mathbf{S}} B \vee \neg B$, and hence that $\nVdash_{\mathbf{S}} A \rightarrow (B \vee \neg B)$, giving $A \not\models_Q B \vee \neg B$.

This leads once again to the idea of *(in)compatibility*. Let us say that two formulas (attributes) A, B are *compatible* if

$$A \models_Q B \vee \neg B \quad \text{and} \quad B \models_Q A \vee \neg A$$

and *coassertable* if

$$A, B \models_Q A \quad \text{and} \quad B, A \models_Q B.$$

Compatibility of A and B means that the introduction of the premise A does not affect the assertability of $B \vee \neg B$ (or *mutatis mutandis*). Coassertability of A and B means that, given the premise A, introducing the premise B does not affect the assertability of A (or *vice-versa*); in other words, A and B are simultaneously assertable.

It is easily shown that compatibility implies coassertability. However, the converse is false since all primitive formulas are evidently coassertable but, as we have shown above, incompatible. Note that it follows from this last fact that A and $B \vee \neg B$ are not coassertable for any primitive A, B.

We infer that implicative quantum logic is distinguished from *intuitionistic* (and, indeed, classical) logic by the presence of *non-coassertable* formulas, and from *classical* logic by the presence of *incompatible* formulas.

The concept of quantum-logical entailment also yields a precise formulation of a general notion of superposition of attributes. Given a space $\mathbf{S}$, let us say that an attribute A is a *superposition* of two attributes B, C *over* $\mathbf{S}$ provided that, for any part U of S, if U manifests A, then U manifests a superposition of B and C in the sense of §3. This condition is easily seen to be equivalent to:

$$\Vdash_{\mathbf{S}} A \rightarrow \neg\neg(B \vee C). \tag{4.2}$$

We say that A is a (quantum-logical) *superposition* of B and C if (4.2) holds for *every* proximity space $\mathbf{S}$, *i.e.* if

$$\models_Q A \rightarrow \neg\neg(B \vee C),$$

or in other words if

$$A \models_Q \neg\neg(B \vee C). \tag{4.3}$$

In the classical case, of course, we would be allowed to infer from (4.3) that $A \models B \vee C$; but in the implicative quantum-logical context we *cannot* do so. This follows from the evident fact that for any attributes A, B, A is a superposition of B and $\neg B$, but if they are both primitive, A is, as we have seen, incompatible with B. Thus implicative quantum logic is distinguished from classical logic by the presence of *superpositions which are not reducible to disjunctions*.

Despite the non-classical properties of $\models_Q$, we observe that the classically valid law

$$A, \neg A \vee B \models_Q B \tag{4.4}$$

still holds. And as an immediate consequence, the weaker 'orthomodular law' (*cf.* Goldblatt [1974])

$$A, \neg A \vee (A \wedge B) \models_Q B$$

also holds. (To establish (4.4) let $\mathbf{S}$ be any proximity space and let U, V be parts of $\mathbf{S}$ with $V \subseteq U$. Suppose that $U \Vdash_{\mathbf{S}} A$ and $V \Vdash_{\mathbf{S}} \neg A \vee B$. Then there are parts W, Z of $\mathbf{S}$ such that $W \cup Z = V$, $W \Vdash_{\mathbf{S}} \neg A$ and $Z \Vdash_{\mathbf{S}} B$. But since $W \subseteq U$, $U \Vdash_{\mathbf{S}} A$ and $W \Vdash_{\mathbf{S}} \neg A$ jointly imply $W = \phi$. Hence $V = Z$ and $V \Vdash_{\mathbf{S}} B$. This gives $\Vdash_{\mathbf{S}} A \rightarrow [(\neg A \vee B) \rightarrow B]$ and (4.4) follows.) That (4.4) holds is perhaps surprising since it is easily shown that the classical law (*modus ponens*) governing implication

$$A, A \rightarrow B \models B$$

fails for $\models_Q$. (To see this, take $A = B = \mathit{Colour}$ in the example at the end of §3.) However, we note that implication still satisfies the fundamental *deduction theorem* as a trivial consequence of the definition of $\models_Q$:

$$A_1, \ldots, A_n \models_Q B \Leftrightarrow A_1, \ldots, A_{n-1} \models_Q A_n \rightarrow B. \tag{4.5}$$

It is tempting to conjecture that the implicative quantum-logical entailment relation is *axiomatisable*. That is, one should be able to specify a 'quantum-logical provability relation' $\vdash_Q$ based on a set of formal axioms and rules of inference and then proceed to show that

$$A_1, \ldots, A_n \models_Q B \Leftrightarrow A_1, \ldots, A_n \vdash_Q B.$$

(As axioms and rules one would presumably include correct assertions such as (4.4) and (4.5).) The logical calculus based on $\vdash_Q$ would then be, in my view, a promising candidate for the role of *formal quantum logic*. So far, however, I have not succeeded in carrying this out and it remains an open problem. Nevertheless, the fact that the quantum-logical entailment relation is definable in a way similar to that for classical and intuitionistic logic, and satisfies the deduction theorem, suggests that, from a semantical standpoint at least, implicative quantum logic is a geniune logical system and not merely an algebraic formalism.

5 SUPERPOSITION OF STATES

In this final section we relate the concept of superposition of attributes to the quantum-mechanical notion of superposition of *states*.

We may regard a discrete space as being essentially the same as a classical phase space (*cf.* §1). In such a space $\mathbf{S}$, a *state* may be identified with a one-point subset of S, *i.e.* a minimal non-empty part of $\mathbf{S}$. If every such part is the value in $\mathbf{S}$ of a primitive attribute, then we may identify states of $\mathbf{S}$ with *minimal* primitive attributes over $\mathbf{S}$, *i.e.* primitive attributes A such that, for any part U of $\mathbf{S}$,

$$U \Vdash_{\mathbf{S}} A \Leftrightarrow U = \phi \quad \text{or} \quad U = [\![A]\!]_{\mathbf{S}}. \tag{5.1}$$

We shall retain this definition of state *when* **S** *is an arbitrary proximity space.*

Given a proximity space **S**, and states A, B, C of **S**, we recall that A is a superposition of B and C over **S** if

$$\Vdash_{\mathbf{S}} A \to \neg\neg(B \vee C).$$

If we agree to identify two states A and B whenever

$$\Vdash_{\mathbf{S}} (A \to B) \wedge (B \to A),$$

it is then readily shown that some of the most important of Dirac's rules governing superpositions ([1936], chapter 1) are satisfied, *e.g.*

- The result of superposing any state with itself is the same as the original state.
- For any states B, C, both are superpositions of B and C.
- Superposition is independent of order.
- Each pair of states has (in general) many different superpositions.

To complete the picture, consider finally the 'orthodox' quantum-mechanical framework based on a Hilbert space H. Here the associated proximity structure is $(H-\{o\}, \approx)$ where $\approx$ is the relation of non-orthogonality of vectors. For each $c \neq 0$ in H we introduce a primitive attribute A_x and interpret A_x in the resulting proximity space H by setting

$$[\![A_x]\!] = Q_x = \{y \neq 0 \colon x \approx y\}.$$

Then the A_x are the minimal primitive attributes over **H**. Moreover, we identify A_x and A_y precisely when $Q_x = Q_y$, which is easily seen to be equivalent to: x is in the one-dimensional subspace of H generated by y. In other words, the (identified) minimal attributes over **H**—the states of **H** in the above sense—correspond to the one-dimensional subspaces of H, *i.e.* to the states of H in the usual quantum-mechanical sense. And lastly, it is easy to show that A_x is a superposition of A_y and A_z in our sense, *i.e.*

$$\Vdash_{\mathbf{H}} A_x \to \neg\neg(A_y \vee A_z)$$

if and only if $Q_x \subseteq Q_y \cup Q_z$, which is in turn equivalent to 'x is in the subspace spanned by y and z', *i.e.* 'state x is a quantum-mechanical superposition of states y and z'.

We conclude that the concept of superposition of minimal attributes is the correct extension of the quantum-mechanical concept of superposition to our more general framework.

Concluding Remark. Here we have only dealt with *propositional* logic. But since all the lattices involved are *complete*, it is not difficult to extend the framework to accommodate *predicate* logic (*cf*. Bell [1983]). As far as I can determine, however, no fundamentally new features emerge.

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REFERENCES

BELL, J. L. [1983]: 'Orthologic, Forcing and the Manifestation of Attributes'. *Southeast Asian Conference on Logic. Studies in Logic* **111**. North Holland.

BELL, J. L. and HALLETT, M. F. [1982]: 'Logic, Quantum Logic and Empiricism', *Philosophy of Science* **49**, pp. 355–79.

BELL, J. L. and MACHOVER, M. [1977]: *A Course in Mathematical Logic*. North-Holland.

BIRKHOFF, G. [1960]: *Lattice Theory*, 3rd edition. American Math. Soc. Colloq. Publs., vol. XXV.

BIRKHOFF, G. and VON NEUMANN, J. [1936]: 'The Logic of Quantum Mechanics', *Annals of Mathematics* **37**, pp. 823–43.

DIRAC, P. A. M. [1930]: *Quantum Mechanics*. Clarendon Press.

DUMMETT, M. [1976]: 'Is Logic Empirical?' In H. D. LEWIS (*ed.*), *Contemporary British Philosophy*. Allen and Unwin.

GARDNER, M. [1971]: 'Is Quantum Logic Really Logic?' *Philosophy of Science* **38**, pp. 508–29.

GOLDBLATT, R. I. [1974]: 'Semantic Analysis of Orthologic', *Journal of Philosophical Logic* **3**, pp. 19–35.

JAMMER, M. [1974]: *The Philosophy of Quantum Mechanics*. Wiley.

KOCK, A. [1981]: *Synthetic Differential Geometry*. London Math. Soc. Lecture Notes 51. Cambridge University Press.

PUTNAM, H. [1969]: 'Is Logic Empirical?' In COHEN and WARTOFSKY (*eds.*), *Boston Studies in the Philosophy of Science*, vol. 5. Reidel.

RASIOWA, H. and SIKORSKI, R. [1963]: *The Mathematics of Metamathematics*. PWN, Warsaw.