OPPOSITIONS AND PARADOXES IN MATHEMATICS AND PHILOSOPHY

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Abstract.

In this paper a number of oppositions which have haunted mathematics and philosophy are described and analyzed. These include the *Continuous and the Discrete, the One and the Many, the Finite and the Infinite, the Whole and the Part,* and *the Constant and the Variable.*

Underlying the evolution of mathematics and philosophy has been the attempt to reconcile a number of interlocking oppositions, oppositions which have on occasion crystallized into paradox and which continue to haunt mathematics to this day. These include the *Continuous and the Discrete, the One and the Many, the Finite and the Infinite, the Whole and the Part,* and *the Constant and the Variable.*

Let me begin with the first of these oppositions—that between *continuity* and *discreteness.* Continuous entities possess the property of being *indefinitely divisible* without alteration of their essential nature. So, for instance, the water in a bucket may be continually halved and yet remain water[i]. Discrete entities, on the other hand, typically cannot be divided without effecting a change in their nature: half a wheel is plainly no longer a wheel. Thus we have two contrasting properties: on the one hand, the property of being indivisible, separate or discrete, and, on the other, the property of being indefinitely divisible and continuous although not actually divided into parts.

Now one and the same object can, in a sense, possess both of these properties. For example, if the wheel is regarded simply as a piece of matter, it remains so on being divided in half. In other words, the wheel regarded as a wheel is discrete, but regarded as a piece of matter, it is continuous. From examples such as these we see that continuity and discreteness are complementary attributes originating through the mind's ability to perform acts of abstraction, the one obtained by abstracting an object's divisibility and the other its self-identity.

In mathematics the concept of *whole* number, of *integer,* provides an embodiment of the vision of pure discreteness, that is, of the idea of a collection of separate individual objects, all of whose properties—apart from their distinctness—have been refined away. The basic mathematical representation of the idea of continuity, on the other hand, is the geometric figure, and more particularly the straight line. By their very nature geometric figures are continuous; discreteness is injected into geometry, the realm of the continuous, through the concept of a point, that is, a discrete entity marking the boundary of a line.

Continuity and discreteness are united in the process of *measurement,* in which the continuous is expressed in terms of separate units, that is, numbers. But these separate units are unequal to the task of measuring in general, making necessary the introduction of fractional parts of the individual unit. In this way the humble fraction issues from the interaction between the continuous and the discrete.

A most striking example of this interaction—amounting, one might say, to a collision—was the Pythagorean discovery of *incommensurable magnitudes.*

The remarkable advances in mathematics made by the Pythagoreans had led them to the belief that mathematics (they are said to have coined the term from a root meaning "learning" or "knowledge") and more especially *number*, lies at the heart of all existence—the first "mathematical" philosophy. For the Pythagoreans the structure of mathematics took the form of a bifurcating scheme of oppositions:

Mathematics

splitting into

The discrete	**The continuous**

in turn splitting into

The absolute	**The relative**	**The static**	**The moving**

corresponding to

Arithmetic	**Music**	**Geometry**	**Astronomy**

This last is, of course, the so-called *quadrivium*, which served as the basis for Western pedagogy up to the Middle Ages. It was a fundamental principle of Pythagoreanism that all is explicable in terms of properties of, and relations between, whole numbers—that number, indeed, forms the very essence of the real. So it must have come as a great shock to the Pythagoreans to find, as they did, that this principle *cannot be upheld within geometry itself*. This followed upon the shattering discovery, probably made by the later Pythagoreans before 410 B.C., that ratios of whole numbers do not suffice to enable the diagonal of a square or a pentagon to be compared in length with its side. They showed that these line segments are *incommensurable*, that is, the impossibility of choosing a unit of length sufficiently small so as to enable both the diagonal of a square (or a pentagon) and its side to be measured by an integral number of the chosen units. Pythagorean geometry, depending as it did on the assumption that all line segments are *commensurable* (i.e., measurable by sufficiently small units), and with it the whole Pythagorean philosophy, was dealt a devastating blow by this discovery. The Pythagorean catastrophe shows how risky it can be to base a world outlook on a literal interpretation of mathematics: the exactness of mathematics may be used to undermine it!

The opposition between the continuous and the discrete may also be identified in the ancient Greek physicists' account of the nature of matter, or substance. This opposition first appeared in the third century B.C. with the emergence of two rival physical theories, each of which became the basis of a fully elaborated physical doctrine. One is the *atomic theory,* due to *Leucippus* and *Democritus.* The other—the *continuum theory*—is the creation of the *Stoic* school of philosophy and is associated with the names of *Zeno of Cition* and *Chrysippus.*

The continuum of Stoic philosophy is an infinitely divisible continuous substance which was presumed to furnish the ultimate foundation for all natural phenomena. In particular the Stoics held that *space* is everywhere occupied by a continuous invisible substance which they called *pneuma* (Greek for "breath"). This pervasive substance—which was regarded as a kind of synthesis of air and fire, two of the four basic elements, the others being earth and water—was conceived as being an elastic medium through which impulses are transmitted by wave motion. All physical occurrences were taken to be linked through tensile forces in the pneuma, and matter itself was held to derive its qualities from the “binding” properties of its indwelling pneuma.

The atomists, on the other hand, asserted that material form results from the arrangement of the atoms—the ultimate building blocks—to be found in all matter, that the sole form of motion is the motion of individual atoms, and that physical change can only occur through the mutual impact of atoms.

A major problem encountered by the Stoic philosophers was that of the nature of *mixture*, and, in particular, the problem of explaining how the pneuma mixes with material substances so as to “bind them together”. The atomists, with their granular conception of matter, did not encounter any difficulty in this regard because they could take a mixture of two substances to be a combination of their constituent atoms into a complex —a kind of lattice or mosaic. But the idea of a mixture posed difficulties for the Stoic conception of matter as continuous. For in order to mix fully two continuous substances, the substances would either have to interpenetrate in some mysterious way, or, failing that, they would each have to be subjected to an infinite division into infinitesimally small elements which would then have to be arranged, like finite atoms, into some kind of discrete pattern.

This controversy over the nature of mixture shows that the idea of continuity is inextricably entangled with the puzzles of infinite divisibility and of the infinitesimally small. The mixing of particles of finite size, no matter how small they may be, presents no difficulties. But this is no longer the case when dealing with a continuum, whose parts can be divided *ad infinitum.* The Stoic philosophers were confronted with what was, in the last analysis, a *mathematical* problem.

As is well-known, the problem of infinite divisibility had already been posed in a dramatic, yet subtle way more than a century before the rise of the Stoic school, by *Zeno of Elea*, a pupil of the philosopher *Parmenides*, who taught that the universe was a static unchanging unity. Zeno's arguments take the form of *paradoxes* which are collectively designed to discredit the belief in motion, and so in any notion of change. These paradoxes embody not only the opposition between the continuous and the discrete, but also that between the finite and the infinite. Let us review three of them in modern formulation. The first two, both of which rest on the assumption that space and time are continuous, purport to show that under these conditions continuous motion engenders, *per impossibile,* an actual infinity.

The first paradox, the *Dichotomy*, goes as follows. Before a moving body can reach a given point, it must first traverse half of the distance; before it can traverse half of the distance, it must traverse one quarter; and so on *ad infinitum.* So, for a body to pass from one point *A* to another, *B,* it must traverse an infinite number of divisions. But an infinite number of divisions cannot be traversed in a finite time, and so the goal cannot be reached.

The second paradox, *Achilles and the Tortoise*, is the best known. Achilles and a tortoise run a race, with the latter enjoying a head start. Zeno asserts that no matter how fleet of foot Achilles may be, he will never overtake the tortoise. For, while Achilles traverses the distance from his starting-point to that of the tortoise, the tortoise advances a certain distance, and while Achilles traverses this distance, the tortoise makes a further advance, and so on *ad infinitum.* Consequently Achilles will run forever without overtaking the tortoise.

This second paradox is formulated in terms of two bodies, but it has a variant involving, like the *Dichotomy,* just one. To reach a given point, a body in motion must first traverse half of the distance, then half of what remains, half of this latter, and so on *ad infinitum*, and again the goal can never be reached. This version of the *Achilles* exhibits a pleasing symmetry with the *Dichotomy*. For the former purports to show that a motion, once started, can never stop; the latter, that a motion, once stopped, can never have started.

The third paradox, the *Arrow*, rests on the assumption of the *discreteness of time*. Here we consider an arrow flying through the air. Since time has been assumed discrete we may "freeze" the arrow's motion at an indivisible instant of time. For it to move during this instant, time would have to pass, but this would mean that the instant contains still smaller units of time, contradicting the indivisibility of the instant. So at this instant of time the arrow is at rest; since the instant chosen was arbitrary, the arrow is at rest at any instant. In other words, it is always at rest, and so motion does not occur.

Let us review the current solutions to these paradoxes as provided by the mathematicians. In the case of the *Dichotomy*, the presentation may be simplified by assuming that the body is to traverse a unit spatial interval—a mile, say—in unit time—a minute, say. To accomplish this, the body must first traverse half the interval in half the time, before this one-quarter of the interval in one-quarter of the time, etc. In general, for every subinterval of length $1/2^n$ (n = 1, 2, 3,…), the body must first traverse half thereof, i.e. the subinterval of length $1/2^{n+1}$. In that case both the total distance traversed by the body and the time taken is given by a convergent series which sums to 1 as expected. So, *contra* Zeno, the infinite number of divisions is indeed traversed in a finite time.

More troubling, however, is the fact that these divisions, of lengths $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ (in reverse order) constitute an infinite *regression* which, like the negative integers, *has no first term.* Zeno seems to be inviting us to draw the conclusion that it cannot be supplied with one, so that the motion could never get started. However, from a strictly mathematical standpoint, there is nothing to prevent us from placing 0 before all the members of this sequence, just as it could be placed, in principle at least, before all the negative integers. Then the sequence of correlations $(\frac{1}{2}, \frac{1}{2}), (\frac{1}{4}, \frac{1}{4}), (\frac{1}{8}, \frac{1}{8}), \ldots$ (in reverse order) between the time and the body's position is simply preceded by the correlation (0, 0), where the motion begins. There is no contradiction here.

In the case of the *Achilles*, let us suppose that the tortoise has a start of 1000 feet and that Achilles runs ten times as quickly. Then Achilles must traverse an infinite number of distances—1000 feet, 100 feet, 10 feet, etc.—and the tortoise likewise must traverse an infinite number of distances—100 feet, 10 feet, 1 foot, etc.—before they reach the same point simultaneously. The distance of this point in feet from the starting points of the two contestants is given, in the case of Achilles, by a convergent series summing to $1111\frac{1}{9}$ feet. In the case of the tortoise the corresponding distance is a convergent series summing to $111\frac{1}{9}$ feet. And, assuming that Achilles runs 10 feet per second, the time taken for him to overtake the tortoise is given, in seconds, by a convergent series summing to $111\frac{1}{9}$ seconds. Thus, again *contra* Zeno, Achilles overtakes the tortoise in a finite time.

Although the use of convergent series does confirm what we take to be the evident fact that Achilles will, in the end, overtake the tortoise, a nagging issue remains (pointed out by Bertrand Russell in his entertaining essay *Mathematics and the Metaphysicians*[ii]). For consider the fact that, at each moment of the race, the tortoise is somewhere, and, equally, Achilles is somewhere, and neither is ever twice in the same place. This means that there is a biunique correspondence between the positions occupied by the tortoise and those occupied by Achilles, so that these must have the same number. But when Achilles catches up with the tortoise, the positions occupied by the latter are only *part* of those occupied by Achilles. This would be a contradiction if one were to insist, as did Euclid in his *Elements,* that the whole invariably has more terms than any of its parts. In fact, it is precisely this principle which, in the nineteenth century, came to be repudiated for infinite sets such as the ones encountered in Zeno's paradoxes. Once this principle is abandoned, no contradiction remains.

The paradox of the *Arrow* can be resolved by developing a theory of *velocity*, based on the differential calculus. By definition, (average) velocity is the ratio of distance travelled to time taken. It will be seen at once that in this definition *two* distinct points in space and *two* distinct points in time are required. Velocity at a point is then defined as the *limit* of the average velocity over smaller and smaller spatiotemporal intervals around the point. According to this definition, a body may have a nonzero "velocity" at each point, but at each instant of time will not "appear to be moving".

While Zeno's paradoxes can be resolved from a strictly *mathematical* standpoint, they present difficulties for understanding the nature of *actual* motion which have persisted to the present day.

The opposition between the continuous and the discrete resurfaced with renewed vigour in the seventeenth century with the emergence of the *differential and integral calculus*. Here the controversy centred on the concept of *infinitesimal*[iii]. According to one school of thought, the infinitesimal was to be regarded as a real, infinitely small, indivisible element of a continuum, similar to the atoms of Democritus, except that now their number was considered to be infinite. Calculation of areas and volumes, i.e., integration, was thought of as summation of an infinite number of these infinitesimal elements. An area, for example, was taken to be the "sum of the lines of which it is formed". Thus the continuous was once again reduced to the discrete, but, with the intrusion of the concept of the infinite, in a subtler and more complex way than before.

Infinitesimals enjoyed a considerable vogue among seventeenth and eighteenth century mathematicians. As the charmingly named "linelets" and "timelets", they played an essential role in *Isaac Barrow's* (who was Newton's teacher) "method for finding tangents by calculation", which appears in his *Lectiones Geometricae* of 1670. As "evanescent quantities" they were instrumental (although later abandoned) in Newton's development of the calculus, and, as "inassignable quantities", in Leibniz's. The *Marquis de l'Hôpital*, who in 1696 published the first treatise on the differential calculus (entitled *Analyse des Infiniments Petits pour l'Intelligence des Lignes Courbes*), invokes the concept in postulating that "a curved line may be regarded as being made up of infinitely small straight line segments," and that "one can take as equal two quantities differing by an infinitely small quantity."

However, the conception of infinitesimals as actual entities was somewhat nebulous and even led, on occasion, to logical inconsistency. Memorably derided by Berkeley as "ghosts of departed quantities" and later condemned by Bertrand Russell as "unnecessary, erroneous, and self-contradictory", this conception of infinitesimal gave way to the idea—originally suggested by Newton—of the infinitesimal as a *continuous variable* which becomes arbitrarily small. By the start of the nineteenth century, when the rigorous theory of limits was in the process of being created, this revised conception of infinitesimal had been accepted by the majority of mathematicians. Now a line, for instance, was understood as consisting not of "points" or "indivisibles", but as the domain of values of a continuous variable, in which separate points are to be considered as locations. At this stage, then, the discrete had given way to the continuous.

But the development of mathematical analysis in the later part of the nineteenth century led mathematicians to demand still greater precision in the theory of continuous variables, and above all in fixing the concept of *real number* as the value of an arbitrary such variable. As a result, in the latter half of the 19th century a theory was fashioned in which a line is represented as a set of points, and the domain of values of a continuous variable by a set of real numbers. Within this scheme of things there was no place for the concept of infinitesimal, which accordingly left the scene for a time. Thus, yet again, the continuous was reduced to the discrete and the properties of a continuum derived from the structure of its underlying ensemble of points. This reduction, underpinned by the development of *set theory*, has met with almost universal acceptance by mathematicians.

The creators of set theory contrived to avoid the continuous/discrete opposition through the radical expedient of banishing the continuous altogether. But in so doing they left themselves open to the irruption of two other oppositions—that of the one and the many, and the finite and the infinite. As a result set theory was itself initially the breeding ground of paradox. Cantor, the visionary creator of set theory, held that there was no difference in principle between finite and infinite sets, as is revealed in his famous 1895 definition of the set concept:

> *By a* set *we understand every collection to a whole of definite, well-differentiated objects of our intuition or thought.*

The problem, of course, is to determine exactly which collections constitute legitimate "wholes". Traditionally, only *finite* collections were admitted to be "wholes" in this sense, since, it was held, infinite collections fail to accord with the time-honoured dictum (enunciated in Euclid's *Elements* as Axiom 5) that a whole must always be greater than any of its (proper) parts.

The curious way in which the principle that a whole must always exceed its parts is violated by infinite collections is strikingly conveyed by a fable attributed to the German mathematician *David Hilbert.* In Hilbert's tale, he finds himself the manager of a vast hotel, so vast, in fact, that it has an *infinite* number of rooms. Thus the hotel has a first, second,…,n^{th},… room, *ad infinitum.* At the height of the tourist season, Hilbert's hotel is full: each room is occupied.

(We are of course assuming the existence of an infinite number of occupants.) Now a newcomer seeking accommodation shows up. "Alas," says Hilbert, "I have not a room to spare." But the newcomer is desperate, and at that point an idea occurs to Hilbert. He asks the occupant of each room to move to the next one; thus the occupant of room 1 is to move to room 2, that of room 2 to room 3, and so on up the line. This leaves the original occupants housed (one to a room, as before), only now the first room is vacant, and the relieved newcomer duly takes possession.

This is not the end of the fable. A vast mob of tourists desiring accommodation suddenly descends on the hotel. A quick tally by Hilbert shows that the mob is *infinite*, causing him some consternation. But now he has another idea. This time he requests each original occupant of his hotel to move to the room with *double* the number of the one presently occupied: thus the occupant of room 1 is to proceed to room 2, that of room 2 to room 4, etc. The result is again to leave all the original occupants housed, only now each member of the infinite set of rooms carrying *odd numbers* has become vacant. Thus each newcomer can be accommodated: the first in room 1, the second in room 3, the third in room 5, etc. It is clear that this procedure can be repeated indefinitely, enabling an infinite number of infinite assemblies of tourists to be put up.

Hilbert's fable shows that infinite sets have intriguingly counterintuitive properties, but not that they are *contradictory.* Indeed if, for example, the physical universe contains infinitely many stars—a proposition which Newton, for one, seems to have been perfectly happy to accept—then it can assume the role of "Hilbert's Hotel," with the stars (or orbiting planets) as "rooms".

While infinite sets are not contradictory in themselves, set theory itself, however, as originally formulated, *does* contain contradictions, which result not from admitting infinite totalities *per se,* but rather from countenancing totalities consisting of *all* entities of a certain abstract kind, "manies" which, on pain of contradiction, cannot be regarded as "ones". So it was in truth not the finite/infinite opposition, but rather the one/many opposition, which led set theory to inconsistency. This is well illustrated by the infamous *Russell paradox*, discovered in 1901.

Russell's paradox, it will be recalled, arises in the following way. It starts with the truism that any set is either a member of itself or not. For instance, the set of all cats is not a member of itself since it is not a cat, while the set of all non-cats is a member of itself since it is a non-cat. Now consider the set consisting precisely of all those sets which are *not* members of themselves: call this, the *Russell* set, *R*. Is *R* a member of itself or not? Suppose it is. Then it must satisfy the defining condition for inclusion in *R,* i.e. it must *not* be a member of itself. Conversely, suppose it is not a member of itself. Then it *fails* to satisfy the defining condition for inclusion in *R,* that is, it *must be* a member of itself. We have thus arrived at the unsettling, indeed contradictory, conclusion that *R* is a member of itself precisely when it is not. We note that whether *R* is finite or infinite is irrelevant; the argument depends solely on the defining property for membership in *R.*

The paradox also appears when we consider such bizarre entities as, for instance, the bibliography of all bibliographies that fail to list themselves: such a bibliography would, if it existed, list itself precisely when it does not. In this case, however, it may simply be inferred that the entity in question does not exist, a conclusion that cannot be drawn in the case of the Russell set without bringing into question the very basis on which sets have been introduced.

Russell's paradox has a purely *linguistic* counterpart known as the *Grelling-Nelson paradox*. Call an (English) adjective *autological* if it is true of itself and *heterological* if not. For instance, the adjectives “polysyllabic”, “English” are autological, and “palindromic”, “French” are heterological. Now consider the adjective “heterological”. Is it autological or not? A moment’s thought reveals that it *is* precisely when it *is not*.

Another principle of set theory whose enunciation in 1904 by *Ernst Zermelo* occasioned much dispute is the so-called *axiom of choice*[iv]. In its simplest form, the axiom asserts that, if we are given any collection *S* of sets, each of which has at least one member, there is a set *M* containing exactly one element from each set in *S*. No difficulty is encountered in assembling *M* when there are only finitely many sets in *S*, or if *S* is infinite, but we possess a *definite rule* for choosing a member from each set in *S*. The problem arises when *S* contains infinitely many sets, but we have *no rule* for selecting a member from each: in this situation, how can the procedure be justified of making infinitely, perhaps even nonenumerably, many arbitrary choices, and forming a set from the result?

The difficulty here is well illustrated, as so often, by a Russellian anecdote. A millionaire possesses an infinite number of pairs of shoes, and an infinite number of pairs of socks. One day, in a fit of eccentricity, he summons his valet and bids him select one shoe from each pair. When the valet, accustomed to receiving precise instructions, asks for details as to how to perform the selection, the millionaire suggests that the left shoe be chosen from each pair. Next day the millionaire proposes to the valet that he select one *sock* from each pair. When asked as to how *this* operation is to be carried out, the millionaire is at a loss for a reply, since, in contrast with a pair of shoes, there is no intrinsic way of distinguishing one sock of a pair from the other. In other words, the selection of the socks must be truly arbitrary.

An odd consequence of the axiom of choice is the so-called *paradoxical decomposition of the sphere*[v], formulated in 1924 by the Polish mathematicians *Stefan Banach* and *Alfred Tarski*. An extreme form of the "paradox" is that, assuming the axiom, a solid sphere can be cut up into five pieces which can themselves be reassembled exactly to form *two* solid spheres, *each* of the same size as the original! Another version is that, given any two solid spheres, either one of them can be cut up into finitely many pieces which can be reassembled to form a solid sphere of the same size as the other. Thus, for example, a sphere the size of the sun can be cut up and reassembled so as to form a sphere the size of a pea! Of course, the phrase "can be cut up" here is to be taken in a metaphorical, not practical, sense; but this does not detract from the counterintuitiveness of these results. Strange as they are, however, unlike Russell's paradox, they do not constitute outright contradictions: sphere decompositions become possible in set theory only because continuous geometric objects have been analyzed into discrete sets of points which can then be rearranged in an arbitrary manner. This is really another instance of the opposition between the one and the many.

The finite/infinite and one/many oppositions play an important role in *Kant's* philosophy. His First Antinomy embodies the finite/infinite opposition: here the Thesis is the assertion that the world has a beginning in time, and is also limited as involves space; while the Antithesis asserts that the world has no beginning, and no limits in space—it is infinite as regards both time and space. And his second Antinomy embodies the one/many opposition: in this case the Thesis asserts that composite substances are made up of simple parts, while the Antithesis is that no composite thing is made up of simple parts.

Kant's First Antinomy has been the focus of a great deal of philosophical discussion. As far as the temporal aspect of the First Antinomy is concerned, Kant conceives of the world's past as a series, and he presents the following argument for the finiteness of that past:

> *The infinity of a series consists in the fact that it can never be completed through successive synthesis. It thus follows that it is impossible for an infinite world-series to have passed away.*

I think we can take "successive synthesis" here to mean "temporal succession". But in that case, what does it mean to say that the series *P* of past events is "completed" through temporal succession? The natural meaning would appear to be that *P* is "generated in the algebraic sense" that is, there is a class *E* of past events such that every past event is obtained by temporal succession from the events in *E*. Even granting this, however, one can infer that *P* is finite only if one assumes that *E is also finite.* In fact it is easy to see that these two assertions are actually *equivalent*, and, assuming either of them, one can take *E* to consist of a single "initial" event: the origin of time. But it is obviously consistent for *E,* and hence also *P,* to be infinite, and so Kant's argument, construed in this way, fails.

The situation boils down, it seems, to this. If the past *were* actually finite, it is of course generated by temporal succession from an initial event. But if it is infinite, although it *grows* by temporal succession, it is not generated or completed by temporal succession from any finite set of events. In this case the infinite past must be conceived at any instant as being *given as a whole*, and only *finite* parts of it as being "completed" by temporal succession.

In his thought-provoking paper, *The Age and Size of the World*[vi], Jonathan Bennett sums up the First Antinomy as follows:

> *Although Kant denies that the world can be infinitely old or large, he thinks that it cannot be finitely large or old either.*

In explicating this assertion, Bennett concludes that what Kant means by "the world is not finite in size" is "no finite amount of world includes all the world there is", or "every finite quantity of world excludes some world". Bennett submits that this last statement "seems to Kant to be a weaker statement than the statement that there is an infinite amount of world." More generally, Bennett suggests that

Kant is one of those who think that

Every finite set of F's excludes at least one F, (1)

though it contradicts the statement that there are only finitely many F's, is nevertheless weaker than

There is an infinite number of F's (2).

Bennett implies that Kant is simply mistaken here, that in fact (1) and (2) are equivalent. But is this right? Let us investigate the matter a little.

Call a set *A finite* if for some natural number *n*, all the members of *A* can be enumerated as a list a_0, …, a_n; *potentially infinite* if it is not finite, that is, if, for any natural number *n*, and any list of *n* members of *A*, there is always a member of *A* outside the list; *actually infinite* if there is a list a_0, …, a_n, …(one for each natural number *n*) of *distinct* members of; and *Kantian* if it is potentially, but not actually infinite, that is, if it is neither finite nor actually infinite. Now *it is possible for a set to be Kantian*, just as Kant (according to Bennett) thought the world was.

For suppose that we are given a potentially infinite set *A*, and we attempt to show that it is actually infinite by arguing as follows. We start by picking a member a_0 of *A;* since *A* is potentially infinite, there must be a member of *A* different from a_0; pick such a member and call it a_1. Now again by the fact that *A* is potentially infinite, there is a member of *A* different from a_0, a_1—pick such and call it a_2. In this way we generate a list a_0, a_1, a_2, … of distinct members of *A;* so, we are tempted to conclude, *A* is actually infinite. But clearly the cogency of this argument hinges on our presumed ability to "pick", for each *n*, an element of *A* distinct from a_0, …, a_n—an ability enshrined in the axiom of choice. Now the axiom of choice is, as Gödel showed in 1938, a perfectly consistent mathematical assumption. But, as Paul Cohen showed in 1964, it is also consistent to deny it. In fact, it can be denied in such a way as to prevent the argument just presented from going through, that is, to allow the presence of potentially infinite sets which are not at the same time actually infinite. That is, the existence of Kantian sets is consistent with the axioms of set theory *as long as the axiom of choice is not assumed.*

Another, more direct way of obtaining a Kantian set is to allow our "sets" to undergo *variation*, to wit, variation over discrete time (that is, over the natural numbers). For consider the following universe of discourse[1] $\mathscr{U}$. Its objects are all sequences of maps between sets:

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots A_n \xrightarrow{f_n} A_{n+1} \longrightarrow \cdots$$

Such an object may be thought of as a set $\boldsymbol{A}$ "varying over (discrete) time": A_n is its "value" at time n. Now consider the temporally varying set

$$\boldsymbol{K} = (\{0\} \longrightarrow \{0,1\} \longrightarrow \{0,1,2\} \cdots \longrightarrow \{0,1,\ldots,n\} \longrightarrow \cdots)$$

in which all the arrows are identity maps. In $\mathscr{U}$ $\boldsymbol{K}$ "grows" indefinitely and so is certainly potentially infinite. On the other hand at each time it is finite and so is not actually infinite. In short, in the universe of "sets through time", $\boldsymbol{K}$ is a Kantian set.

Thus, I contend, Kant's vision of the universe as being "potentially infinite without being actually infinite" is, it seems to me, coherent after all.

I conclude with a few words on the last of our oppositions: the constant and the variable.

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[1] Those "in the know" will recognize $\mathscr{U}$ as the topos of sets varying over the natural numbers.

The world as we perceive it is in a perpetual state of flux. But the objects of mathematics are usually held to be eternal and unchanging. How, in that case, is the phenomenon of variation given mathematical expression? Consider, for example, a fundamental and familiar form of variation: *change of position,* or *motion,* a form of variation so basic that the mechanical materialist philosophers of the 18th and 19th centuries held that it subsumes all forms of physical variation. Now motion is itself reducible to a still more fundamental form of variation—*temporal* variation. (It may be noted here that according to Whitehead even this is not the ultimate reduction: cf. his notion of "passage of nature".) But this reduction can only be effected once the idea of *functional dependence* of spatial locations on temporal instants has been grasped. Lacking an adequate formulation of this idea, the mathematicians of Greek antiquity were unable to produce a satisfactory analysis of motion, or more general forms of variation, although they grappled mightily with the problem. The problem of analyzing motion was of course compounded by Zeno's paradoxes, which, as we know, were designed to show that motion was impossible.

It was not in fact until the 17th century that motion came to be conceived as a functional relation between space and time, as the manifestation of a dependence of variable spatial position on variable time. This enabled the manifold forms of spatial variation to be reduced to the one simple fundamental notion of temporal change, and the concept of motion to be identified as the *spatial representation* of temporal change. (The "static" version of this idea is that space curves are the "spatial representations" of straight lines.)

Now this account of motion (and its central idea, functional dependence) in no way compels one to conceive of either space or time as being further analyzable into static indivisible atoms, or *points.* All that is required is the presence of two domains of variation—in this case, space and time— correlated by a functional relation. True, in order to be able to *establish* the correlation one needs to be able to *localize* within the domains of variation, (e.g. a body is in place x_i at "time" t_i, $i = 0, 1, 2, \ldots$) and it could be held that these domains of variation are just the "ensemble" of all conceivable such "localizations". But even this does not necessitate that the localizations themselves be atomic points—in this connection there comes to mind Whitehead's method of "extensive abstraction" and, latterly, the rise of the amusingly named "pointless topology".

The incorporation of variation into mathematics in the 17th century led, as is well known, to the triumphs of the calculus, mathematical physics, and the mathematization of our understanding of nature. But difficulties surfaced in the attempt to define the instantaneous rate of change of a varying quantity—the fundamental concept of the differential calculus. Like the ancient Pythagorean effort to reduce the continuous to the discrete, the endeavour of the mathematicians of the 17th century to reduce the varying to the static through the use of infinitesimals led to outright *contradictions.*

The response of their successors was, effectively, to replace variation by completed infinity. Cantor in particular abandoned the concept of a varying quantity in favour of a completed, static *domain of variation,* itself to be regarded as an ensemble of atomic individuals—thus, like the Pythagoreans, at the same time replacing the continuous by the discrete. He also banished infinitesimals and the idea of geometric objects as being generated by points or lines in motion.

Mathematicians, such as Brouwer and Weyl, and philosophers, such as Brentano and Peirce, raised objections to the idea of “discretizing” or “arithmetizing” the linear continuum. Peirce, for example, rejected the idea that a true continuum can be completely analyzed into a collection of discrete points, no matter how many of them there might be.

It was with Brouwer at the start of the 20th century that logic itself enters the argument. Rejecting the Cantorian account of the continuum as discrete, Brouwer identified points on the line as entities “in the process of becoming”, that is, as embodying a certain kind of variation. Brouwer saw that the variable nature of these entities would cause them to violate certain laws of logic which had been affirmed since Aristotle, in particular, the *law of excluded middle* or *bivalence*—“each statement is either true or false”. This revolutionary insight led to a new form of logic, *intuitionistic logic*—a “non-Aristotelian” logic in which the law of excluded middle is no longer affirmed. It was later shown that the intuitionistic logic of Brouwer embraces a very general concept of variation, embracing all forms of continuous variation, and which is in particular compatible with the use of infinitesimals.

Certain philosophers—notably Hegel and Marx—believed that achieving a true understanding of the phenomenon of change would require the fashioning of a dialectical logic or “logic of contradiction”, in which the *law of noncontradiction*— “no statement can be both true and false”—is repudiated. It is a striking fact that, so far at least, more light has been thrown on the problem of variation by challenging the law of excluded middle rather than questioning the law of non-contradiction.

Notes.

[i] For the purposes of argument we are here ignoring the atomic nature of matter that has been established by modern physics.

[ii] In *Mysticism and Logic*, London: Pelican Books, 1954.

[iii] See J .L. Bell, *A Primer of Infinitesimal Analysis*, Cambridge University Press, 1998.

[iv] See G. H. Moore, *Zermelo's Axiom of Choice*: *Its Origins, Development and Influence*, Studies in the History of Mathematics and Physical Sciences 8, Springer-Verlag, 1982.

[v] See S. Wagon, *The Banach-Tarski Paradox,* Cambridge University Press, 1985.

[vi] *Synthese* **23** (1971), 127-146