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Pluralism and the Foundations of Mathematics

Contrary to the popular (mis)conception of mathematics as a cut-and-dried body of universally agreed-on truths and methods, as soon as one examines the foundations of mathematics, one encounters divergences of viewpoint and failures of communication that can easily remind one of religious, schismatic controversy. While there is indeed universal agreement on a substantial body of mathematical results, and while classical methods overwhelmingly dominate actual practice, as soon as one asks questions concerning fundamentals—such as "What is mathematics about?" "What makes mathematical truths true?" "What axioms can we accept as unproblematic?" and notoriously, even "What are the acceptable *logical rules* by which mathematical proofs can proceed?"—we find we have entered a minefield of contentiousness. Platonists treat mathematics as an objective study of abstract reality, no more created by human thought than the galaxies, and, accordingly, classical logic and a rich theory of the transfinite are entirely legitimate.[1] Radical constructivists (intuitionists) challenge even the meaningfulness of classical, objectivist thinking in connection with the infinite, and propose a reconstructed mathematics with restricted logic (e.g., no existence proofs by reductio ad absurdum) and different axioms (e.g., the least upper-bound principle is jettisoned). Classicists respond (if they respond at all, which is unusual) by accusing their critics of changing the subject. And between and beyond these camps there is a significant variety of positions or "schools," e.g., predicativism, or "semi-constructivism," which accepts classical logic but only those infinite sets we can actually describe in an acceptable way (which can be spelled out precisely); constructivism of the Bishop school, which, in contrast with intuitionism, adds no new, nonclassical mathematical axioms; constructivism of the Russian school, which lives with Church's thesis identifying constructive functions with the Turing-computable ones; strict finitism; and so on (see Beeson 1985). A plurality or multiplicity of approaches to central questions of *truth* and *proof* is simply an observable fact. What is the nature and significance of this multiplicity? Is it reasonable to think it can be transcended, or is it a permanent fact of life? What lessons, if any, does it hold for general questions concerning pluralism?

A second locus of pluralism in mathematics is ontology. Even within the classical framework, one may ask whether there is a single, all-embracing universe of discourse for mathematics, as set-theoretic reductionism on a customary reading would have it, that is to say, *the* cumulative hierarchy of sets, or should we think of a plurality of universes? Although ordinary mathematics—all that is required in typical graduate programs in the subject—can indeed be developed within set theory, specifically in the favored system known as ZF (Zermelo-Fraenkel), when one considers set theory itself (a branch of "extraordinary mathematics"), one in fact encounters a multiplicity of theories. Usually the Axiom of Choice is added (giving ZFC), but we know that its negation is a consistent option (relative to the consistency of ZF itself). We also know that we need not insist on well-foundedness (sets can be allowed to contain themselves; there can be infinitely descending membership chains). And then there is the whole subject of large cardinal extensions of ZFC, many of which are very natural from a set-theoretic standpoint but that cannot even be proved relatively consistent (a phenomenon known as "Gödel's curse"). Does it make sense to think of unique, determinate answers to all such questions, as talk of the "cumulative hierarchy" implies? Or should we rather think of "many worlds"? Furthermore, there is a different foundational approach with claims to universality, namely, *category theory,* more specifically *topos theory,* which generalizes on set theory in certain ways. Originating in algebraic geometry, toposes are categories in which certain key set-theoretic operations are generalized, notably, the formation of Cartesian products, function classes by exponentiation, and extensions of predicates. They have been proposed as universes of discourse for mathematics, introducing even more options. Thus topos relativity (unlike set-theoretic relativity with regard to large cardinals, for example) prima facie flies in the face of ordinary talk of "the real numbers," "the complex numbers," "the continuous functions of reals," and so on, where uniqueness is presupposed. This suggests a structuralist (re)interpretation of such talk, and even of set theory itself (better, set theories themselves), contrary to the single, fixed-universe view.

Let us elaborate on these two main topics in turn.

Constructivism versus (?) Classicism

The various forms of constructivism (apart from predicativism) have at their common core adherence to intuitionistic logic, usually described as "rejecting the law of excluded middle (LEM)," in the sense of not allowing it in proofs (*not* in the sense of affirming its negation—in fact, the double negation of LEM is a theorem of intuitionistic [propositional] logic). Similarly, allied principles are rejected, such as the law of double negation, proof of existence by reductio ad absurdum, and so on. Formally, intuitionistic logic appears simply to be a *proper part* of classical logic; if you restore LEM to the intuitionistic rules, you recover classical logic. So formally there is no inconsistency between the two.[2] But intuitionists are famous for holding that LEM and allied principles are "not correct"; it seems that we have a genuine disagreement over certain laws of logic! Is that really so? Can even pluralists tolerate such a disagreement? Is even propositional logic up for grabs? Whatever one thinks about the analytic/synthetic distinction in general, don't the (truth-functional) meanings of 'or' and 'not' guarantee that, within the intended domain of determinate propositions such as those of arithmetic, LEM has to be correct?

Indeed, if one looks more closely at intuitionistic usage—even as its proponents have explained it—it is abundantly clear that the key logical words are being used with very different meanings from the classical ones. The very idea of giving *truth-conditions* for logically complex statements is abandoned in favor of *proof-conditions* in which one explains when a (mathematical) *construction* counts as a *proof* of a complex statement. So, for example, intuitionistic 'or' is explained by a condition such as

c proves 'p or q' iff c proves p or c proves q.

(Here the 'or' on the right is supposed to be neutral or pretheoretic, somehow shared by all parties.) The conditional is explained by

c proves '$p \rightarrow q$' iff c is an operation on constructions tranforming any proof of p into a proof of q.

And intuitionistic *negation* is then explained via

c proves '$\neg p$' iff c proves '$p \rightarrow 0 = 1$'.

(Here "0 = 1" may be replaced by any other absurdity.) Given these meanings, no classicist would wish to affirm '*p or* $\neg p$' as a general logical principle, for, when spelled out, it asserts that every (mathematical) proposition is decidable! Similarly, considering that existential quantification is explained as a generalization of *'or,'* so that a proof of '$\exists x\varphi$' provides a method of *finding* an instance together with a (constructive) proof that it satisfies φ, no classicist would apply the method of *reductio* to establish such "existence." Clearly, to avoid confusion, all the connectives should carry subscripts indicating "intuitionistic" or "classical" readings. And then, we have not a single-law "LEM" but two radically distinct ones, the intuitionistic instance of which no one accepts, and similarly for the allied laws. So the controversy seems not to be over the correctness of any logical laws after all!

At this stage, constructivist positions split apart. The mere decision to eschew certain classical forms of proof can be made for a variety of reasons and does not by itself indicate any genuine disagreement with classical mathematics. Specifically, we must distinguish a *radical constructivist* view, which insists that mathematical reasoning must be intuitionistic and that classical reasoning is illegitimate or incoherent (views expressed in different ways by intuitionists from Brouwer to Dummett), from a *liberal* view, which, without challenging the meaningfulness or correctness of nonconstructive classical mathematics, prefers to pursue constructive mathematics for its own intrinsic interest and virtues. So here we have a stark contrast, within constructivism, between *hegemonists* and *pluralists*.

The hegemonist position, as Dummett (1977) has articulated it, rests on a verificationist view of meaning. Platonist or realist truth conditions pertaining to the infinite are in general incommunicable, as terminating procedures for testing them are not available. Rather than taking this (in addition to all the criticisms of verificationism developed by Quine, Sellars, Smart, Putnam, and others over many decades) as indicating a deficiency in the view of meaning, the hegemonist view leads to an extreme stance that Shapiro (1997, 6, passim) has dubbed "philosophy first," namely, that of rejecting mathematics itself for philosophical reasons. David Lewis's reaction (originally to a version of nominalism, but equally applicable here) is germane:

> I'm moved to laughter at the thought of how *presumptuous* it would be to reject mathematics for philosophical reasons. How would *you* like the job of telling the mathematicians that they must change their ways. . . . Can you tell them, with a straight face, to follow philosophical argument wherever it may lead? If they challenge your credentials, will you boast of philosophy's other great discoveries: that motion is impossible, that a Being than which no greater can be conceived cannot be conceived not to exist, that it is unthinkable that anything exists outside the mind, that time is unreal, that no theory has ever been made at all probable by evidence (but on the other hand that an empirically ideal theory cannot possibly be false), that it is a wide-open scientific question whether anyone has ever believed anything, and so on, and on, *ad nauseam?*
>
> Not me! (Lewis 1991, 59; italics in original)[3]

That Dummett's reasoning can also be invoked to challenge the determinateness of, for example, ordinary claims about the past (e.g., four years ago to this day, there were exactly twenty-seven paper clips on my desk) has not been a deterrent.

Thus far, we have seen that the dispute between radical constructivism and classicism is not really over logical laws per se; rather, it is over the meaningfulness of talk presupposing truth-determinate sentences or propositions of infinitistic mathematics. The classical logical connectives and quantifiers, however intelligible they may be in other contexts, are alleged to be unintelligible here in mathematics (except in its constructive part, on which the classical and intuitionistic theorems coincide), ironically the very domain in which the idealization of genuine bivalence built into classical logic has its clearest illustration, and for which it was originally developed!

Fortunately, not all constructivists are radicals. If you want to keep track of computational content in mathematics, requiring reasoning to obey intuitionistic logic makes eminent sense. It is an excellent bookkeeping device. So long as your starting points are constructively justifiable, your conclusions will also be. But if you try to recover standard mathematics along such lines, you will encounter many problems. As soon as you come to the real numbers (as convergent rational sequences), for example, you will realize that you cannot assert that they are totally linearly ordered. You will have to make do (and often can) with a weaker condition: if you know that $x < y$, then you will also be able to show, for any z, that either $x < z$ or $z < y$. You will also not be able to prove fundamental facts, such as the intermediate-value theorem (that every continuous function on [0,1] negative at 0 and positive at 1 has a 0 for some x, $0 < x < 1$), but you will be able to prove something very close to that by tinkering with the statement, strengthening the hypothesis of the theorem or weakening the conclusion (getting within ε of 0). Indeed, Errett Bishop (1967) took constructive analysis far beyond anything previously thought possible by the persistent and clever use of such methods, conquering even such apparently nonconstructive territory as measure theory. It is nontrivial to find genuine examples of scientifically applicable mathematics that cannot be recovered constructively in this sense, although there do appear to be some limitations.[4]

An important lesson we can learn from all this is that there are, indeed—as Carnap recognized through his principle of tolerance—*multiple logics*, legitimate for their own purposes. The notion of "*the* correct logic" is simply a mistake, one which fails to take account of the purpose-relativity and language-relativity of logic. Classical logic is designed for truth-preservation in an idealized setting in which we are dealing with bivalent propositions. The classical connectives (and quantifiers) are introduced as idealizations or simplifications of ordinary language expressions with simply statable, bivalent truth conditions, and classical logical principles and rules pertain to reasoning with *these* connectives. If a rule is *sound* (truth preserving), that is sufficient justification for it, regardless of (lack of) constructivity. But whether classical systems are applicable in a given domain or context is not a matter of logic, but a matter of usage and goals. *Logic does not proclaim its own applicability to particular situations.* Independently, however, it is clear that classical reasoning *is* especially useful in scientific, as well as purely mathematical, contexts in which we are interested in what holds or would hold in a certain situation or model, given certain assumptions, as an objective matter regardless of computability. Then there can be no objection to use of $\text{LEM}_{classical}$ or $\textit{reductio}_{classical}$, and indeed, forswearing their use would seem like tying a hand behind one's back. However, if computability or constructivity is our goal, then obviously it will not be achieved unless we modify our rules, and we may even introduce a new language (also rooted in ordinary language), as intuitionism does. For these new connectives, some but not all of the classical *forms* will be correct. Intuitionistic formal systems codify correct forms of reasoning from this standpoint, and no one can quarrel with that.[5] Classicists as well as constructivists can see all of this. Moreover, as both purposes—truth-preservation *simpliciter* and constructive interpretability—are worthy and important, we should certainly have peaceful coexistence and even cooperation.[6]

This brings us to a second main lesson. Mathematics as practiced is clearly very rich and diverse in its content and in the interests and purposes it supports. As just indicated, both classical and constructive purposes are encompassed; moreover, often they may be intertwined and not neatly separated by branch or subfield. The situation was well summed up by Feferman over twenty years ago:

> Since neither the realist nor constructivist point of view encompasses the other, there cannot be any present claim to a *universal foundation* for mathematics, unless one takes the line of rejecting all that lies outside the favored scheme. Indeed, *multiple foundations* in this sense may be necessary, in analogy to the use of both wave and particle conceptions in physics. Moreover, it is conceivable that still other kinds of theories [of operations and collections] will be developed as a result of further experience and reflection. (Feferman 1977, 151; italics in original)

This accords with the general hypothesis that the complexity and richness of scientific subject matter and practice may actually *require* a pluralistic approach, that any single one that we have contrived, or perhaps can contrive, will simply not do justice to an important aspect of the subject. The classicism-constructivism duality in mathematics is, we submit, an excellent illustration.

Many Worlds

At the end of a landmark paper, credited with the discovery of large cardinals in set theory, Zermelo wrote of "two polar opposite tendencies of the thinking mind, the idea of *creative progress* and that of *all-embracing completeness*" (italics in original). These, he continued:

> find their symbolic expression and resolution in the concept of the well-ordered transfinite number-series, whose unrestricted progress comes to no real conclusion, but only to relative stopping points, the "boundary numbers" [inaccessible cardinals] that divide the lower from the higher models. And so the "antinomies" of set theory, properly understood, lead not to a restriction and mutilation, but rather to a further, as yet unsurveyable, unfolding and enrichment of mathematical science. (Zermelo 1930, 47, trans. the author)

The central problem calling forth these "two polar opposite tendencies," in a nutshell, is this: over what totality do the unrestricted quantifiers of set theory range? We know that, on pain of contradiction, it cannot be taken to be a set, but if we take it as a collection of some higher type, we face the conundrum that we can apply set-like operations to it, leading to collections of higher and higher type, behaving just like sets, so that our effort to speak of absolutely all sets seems indistinguishable from speaking of all sets below a certain inaccessible level (one of Zermelo's "boundary numbers").[7] Indeed, whatever totality of collections we recognize—whatever we call it—can be properly extended, indeed, by the very operations that gave rise to set theory in the first place (forming singletons, power sets, etc.) The standard set-theoretic "way out" of remaining within a first-order language, officially recognizing *no* totality of all sets, while consistent and useful for mathematics in practice, does not really solve the problem, for the very *possibility* of considering new totalities and proper extensions is intrinsic to mathematics. As Mac Lane has put it:

> Understanding Mathematical operations leads repeatedly to the formation of totalities: the collection of all prime numbers, the set of all points on an ellipse . . . the set of all subsets of a set . . . , or the category of all topological spaces. There are no upper limits; it is useful to consider the "universe" of all sets (as a class) or the category *Cat* of all small categories as well as CAT, the category of all big categories. After each careful delimitation, bigger totalities appear. No set theory and no category theory can encompass them all—and they are needed to grasp what Mathematics does. (1986, 390; italics in original)

Still, we do seem to have unrestricted quantifiers in our language, allowing us to speak of *anything* and *everything.* But then we should be able to speak of *anything* and *everything mathematical,* among which would be all collections or set-like objects, totalities which would violate the general principle of *extendability* articulated by Mac Lane (generalizing Zermelo's own versions, independently arrived at also by Putnam).

It counts as a strike against set-theoretic foundations that it seems to be incapable of resolving this problem. Zermelo's resolution of recognizing an unending, ascending series of models of set theory, each of greater and greater ordinal characteristic (strongly inaccessible cardinal), is a major advance over the fixed universe view, but, as already indicated, it is only a partial resolution, for we still seem capable of speaking without contradiction of "all inaccessible cardinals," or "all full models of ZFC (characterized by Zermelo)," and so on, leading right back to our puzzle. (Indeed, if we formalize Zermelo's logic, which would be a fragment of second-order logic, the standard comprehension scheme leads to classes of all sets, all ordinals, all inaccessibles, all models, etc., after all, conflicting with general extendability, as already described.)

This naturally leads us to consider alternatives to set theory, and indeed *category theory* (CT) stands ready and waiting to step in. Its proponents have been maintaining for decades that it provides an autonomous, alternative foundational scheme that in fact is superior in a number of ways to set-theoretic foundations. Not only is it claimed to be more closely in contact with the actual content of advanced mathematics (e.g., algebraic topology and geometry problems, which it helps solve), it is also claimed to capture better certain key *structuralist* ideas, such as the interdependence of structures through various kinds of mappings and, in particular, the idea of a multiplicity of universes of discourse for mathematics in contrast with the fixed universe view of set theory. Unlike set theory, in which the content of a mathematical concept is fixed by referring it once and for all to a fixed absolute universe of sets, in "category theory" any mathematical concept acquires a *plural* reference through varying the category of discourse to which it is referred. This is well illustrated by the group concept. As a set-theoretical object, a group is a set equipped with a couple of operations satisfying certain elementary axioms expressed in terms of the elements of the set. By contrast, in category theory the group concept is given an "arrows only" formulation, in which it becomes a "group object" capable of living in virtually any category. In the category of topological spaces, for example, a group object is nothing other than a *topological group;* in the category of differentiable manifolds it is a *Lie group;* and in a category of sheaves it is a *sheaf of groups.*

Mac Lane, Bell, and others have proposed developing mathematics within suitable toposes, categories with a rich hierarchical structure generalizing certain key features of sets, roughly, those features that persist when sets are allowed to *vary* in some way. (As we have said, these features include the formation of Cartesian products, function classes by exponentiation, and extensions of predicates.) Any topos may be conceived as a possible universe of discourse in which mathematical arguments can be pursued and mathematical constructions carried out. A topos has its own internal language that describes it, and its own internal logic, which, in general, is not classical but intuitionistic. But classical logic emerges if certain further mathematical conditions, for example, the Axiom of Choice, are imposed. Thus, topos theory already accommodates both classical and constructive mathematics, allowing different universes for them built on a common core.

The plurality of reference already conferred on mathematical concepts by category theory is carried a stage further in topos theory. Take, for example, the concept *real-valued continuous function on a topological space X*. Any such function may be regarded as a real number, or quantity, *varying continuously* over X. Now consider the topos **Sh**(X) of sheaves on X. Here a sheaf on X may be conceived as a set undergoing continuous variation, in a suitable sense, over (the open subsets of) X. In that case, **Sh**(X) may be viewed as a universe in which everything is undergoing continuous variation over X, "co-moving," as it were, with the variation over X of any given varying real number. This causes the variation of the latter to be "unnoticed" in **Sh**(X); it is accordingly regarded there as being a *constant* real number. In other words, the concept *real number*, interpreted in **Sh**(X), corresponds to the concept *real-valued continuous function on X*. This shows that, from the standpoint of topos theory, a mathematical concept may be assigned a fixed *sense*, but may nevertheless have a plural *reference*. Indeed, we may take the sense of the concept *real number* as being fixed by a suitable definition in the common internal language of toposes, while its reference will depend on the topos of interpretation. In **Sh**(X), that reference will be, as we have seen, not the usual *real number* concept but *real-valued continuous function on X*. That is, reference is determined only *relative* to a topos of interpretation.

Another instance of the relativity of mathematical concepts, one familiar to all set-theorists, is the phenomenon of *cardinal collapse*. Here, given an uncountable set I, we can produce a "universe of sets"—actually a Boolean extension of the universe of sets—in which I is *countable*. This means that the cardinality of an infinite set is not an absolute or intrinsic feature of the set but is determined only in relation to the mathematical framework with respect to which that cardinality is "measured."

This shows that topos theory is *pluralistic*. But it is at the same time objective in that (certain) toposes may be seen as depicting, in an idealized way, objective aspects of the world, only no unique topos describes that world in its totality. For example, the *smooth topos* provides an idealized description of the geometric structure of the world, idealized through the assumption that all objects and maps are continuous and smooth. At the other extreme, the *topos of sets* presents the world as an entirely discrete structure in which objects are given purely in terms of their cardinality. Still another example is the *effective topos*, in which the world is viewed in terms of computability, and requires all functions to have algorithms. The evident pluralism we again see here arises not because we are dealing with competing theories, but because the alternatives are suited to different purposes. So it is not meaningful to ask whether it is "really" the case that all functions from the real line to itself are differentiable, or whether it is really "true" that all functions from the natural numbers to themselves are recursive, let alone whether any solid sphere is "really" decomposable into five pieces that can be fitted together to make two solid spheres of the same size. (This is an instance of the *Banach-Tarski paradox*, a consequence of the Axiom of Choice, which is generally assumed to hold in a topos of sets.) Instead, one recognizes such features as being tied to the relevant idealization, as being, if you like, "objective" features of that idealization, but not embodying any sort of claim about the (mathematical or physical) world *tout court*.

All this has led Bell (1986), for example, to propose that mathematics should be seen as local, or relative to a choice of background topos. Theorems common to all the suitable toposes form the constructively provable common core. Beyond that, objectivity requires relativization to particular toposes, in analogy with relativistic physics. (Whether famous examples of undecidables of set theory, such as the Continuum Hypothesis, can be thought of as "objective" even in such a relative sense—i.e., in the case of CH, relative to a topos in which power objects are maximal—is a separate, debatable matter.)

This is an attractive view, as far as it goes. Category theory does provide a mathematically interesting generalization of set theory and does offer insights into "mathematical structure," revealing, for instance, how mathematical content is often only "up to isomorphism." However, it does not go far enough, or, better, it does not start early enough, or—more accurately still—it is not clear just where it starts. The problem can be brought out by attending to the term "category theory" itself. It is ambiguous, along with the term "axiom." On the one hand, there are first-order axioms defining what a category is, and various additions to these defining various types of topos (elementary, free, well-pointed, etc.). These are axioms only in the sense of *defining conditions,* telling us what these structures are, as in abstract algebra, where one has axioms for groups, rings, fields, and so on. As components of definitions, these so-called axioms assert nothing, and so are not proper axioms in the traditional, Fregean sense—evident truths, in an absolute sense, or at any rate *assertions* with a determinate truth-value, apart from being evident. (To be sure, this is compatible with such axioms decisively capturing a prior, well-motivated *conception* of a domain or type of object, as in the cases of the axioms for toposes mentioned above, for smoothness or discreteness or recursivity. Here the axioms are akin to the postulates of Euclidean geometry, if we read those as being true of our conception of space, rather than applying literally to actual physical space.) In fact, some category theorists have gone even further, explicitly reading their defining conditions in a Hilbertian, structuralist way: any objects whatever bearing a relation formally behaving like composition of functions (as spelled out in the CT axioms) constitute a category. In other words, the primitives of the language of CT are not even given a definite interpretation, but are treated as placeholders or variables. On the other hand, "category theory" as practiced by mathematicians involves substantive, even deep theorems, and surely these are assertory. But in what framework are these results proved? Not simply in the systems of definitions, as is clear from cases in which various categories or toposes are brought into functorial relations with one another. As Feferman (1977) pointed out, notions of *collection* and *operation* are presupposed just in saying what a category or a topos is as well as in relating them. And indeed, the typical text in the subject, which of course is presented as informal mathematics, makes reference early on to a given, background universe of sets, that is, category theory is not being presented as an autonomous foundational framework at all; rather, set theory is presupposed in the background as is standard in other branches of abstract mathematics (algebra, topology, etc.). As pure mathematics, this is fine; but clearly the CT foundationalist who would transcend the single-universe set-theoretic hierarchy must put on another hat and articulate an alternative framework. At a minimum, a background logic must be specified, including (asserted) axioms governing operations or relations and, presumably, governing the mathematical existence of categories and toposes.

Efforts to create such an alternative framework by explicitly axiomatizing the metacategory of all categories were in fact initiated by Lawvere (1966) and extended by Blanc and Donnadieu (1976). But there are difficulties with the claim that these axiomatizations could constitute an autonomous foundation for mathematics. Primitives such as "category" and "functor" must be taken as having definite, understood meanings, yet they are in practice treated algebraically or structurally, which leads one to consider interpretations of such axiom systems, that is, their semantics. But such semantics, as of first-order theories generally, rest on the set concept: a model of a first-order theory is, after all, a set. The foundational status of first-order axiomatizations of the metacategory of categories is thus still somewhat unclear.

The bearing of all this on the issue of pluralism should be evident: if indeed CT is dependent on a background universe of sets, then the plurality of universes of discourse for mathematics ultimately reduces to the plurality of universes of discourse for set theory. The plurality of toposes in which much mathematics can be developed may still be quite interesting in its own right, but all toposes would be seen as living inside models of set theory. On the level of theories, set theory would have to be seen as more fundamental, and CT's promise of an alternative, autonomous foundational approach would not be fulfilled. This is especially disappointing when we recall that if we ask, "What plurality of intended universes for set theory is there?" the standard answer is "None, there is just *the* cumulative hierarchy," although within this there may also be many (less than exhaustive) models. And it seems we are also stuck with set theory as a massive exception to a structuralist interpretation of mathematics.

It turns out, however, that there is a way out of this impasse, but at a price. If we introduce modality and tolerate talk of the possibility of large domains of discourse—essentially just large numbers of objects—then we have a natural way of recognizing a plurality of models of set theory and toposes, living side by side within these domains, of which there also can be many, but without ever allowing for any totality of *all* such domains. In this view, it does not even make sense to speak of collections or wholes of actual things combined with what merely *might* have existed! One makes sense of collecting, forming wholes, and so on, only within a world, so to speak, not across worlds. (Officially, worlds are not recognized; all this is spelled out with modal operators, ultimately with just one: "it is mathematically possible that . . .") In fact, surprisingly, second-order logical machinery is available to describe not only large domains, in the sense of having inaccessible cardinality, but also structures for set theory and category theory, without ever officially quantifying over classes or relations as objects. Clever combinations of mereology and plural quantification suffice. One must be able to speak of arbitrary wholes of enough pairwise nonoverlapping things (about whose nature we can remain neutral), and we must allow plural locutions, such as "Any things whatever that φ also ψ," as achieving the expressive power of quantification over arbitrary subcollections of the (given, hypothetical) domain of things.[8] For example, the second-order least upper-bound principle takes this form: "Any reals whatever which are all $\leq$ some real are all $\leq$ a least such." This, together with the usual axioms for a complete ordered field, characterizes the real number system up to isomorphism. Similar methods yield characterizations of other key mathematical structures such as the natural numbers, full models of set theory, and various toposes, and so on, again, without ever countenancing classes or relations as objects.

The upshot is that we do have at least one way of consistently combining set theory, category theory, and an open-ended plurality of universes of discourse for mathematics, in accordance with structuralist insights. The assertory axioms of the proposed framework are those of the background logic (essentially second-order logic with mereology) together with axioms asserting the possibility of large domains and guaranteeing extendability, that is, the possibility of ever larger ones. The axioms of set theories proper can then be interpreted structurally as defining conditions on certain kinds of structures. And category theory can be carried out relative to background domains without thereby becoming a (late-ish) chapter of set theory. (In effect, the Grothendieck method of universes has been recovered nominalistically.)

Unlike the first kind of pluralism discussed above, this pluralism in ontology seems distinctively attractive, even necessary, for mathematics, as compared with the natural sciences. After all, we live in a unique world, don't we? Pure mathematics is content to deal with mere conceptual possibilities, but the natural sciences aim to describe and explain reality.[9] Surely there is no analogue of the principle of extendability, articulating the "creative progress" that Zermelo found inherent in mathematics. Short of this, there may well be other multiplicities involving ontology. Of course, on the plane of metascience and perhaps in physics, there are multiple ways of conceiving even the material world, with or without properties, with or without space-time points as objects, with or without particles (e.g., with only quantum fields), and so on. Are these cases of genuine equivalence, and hence (?) only apparent choices, or undecidable questions? Or are we driven to a kind of ontological relativity favored by Carnap (1956, Suppl. A, 205–21)? If so, then in the natural sciences, as well as in mathematics, absolutist talk of "reality" or even the more humble sounding "everything," should really be given up.[10]

For a scientific example of "many worlds" in a very different sense, there is, of course, the notorious "many worlds" interpretation of quantum mechanics (the de Witt version of the Everett interpretation, with actual splitting practically whenever "anything definite happens"), but the objections that have been raised against this seem to us decisive. More promising, perhaps, cosmologists now explore ideas about a *multiverse* instead of the universe, multiple real cosmoses arising from quantum mechanical processes, including inflation. Certain seemingly intractable questions are then blocked, for example, "Why does the actual cosmos satisfy the very special conditions of the constants of nature permitting the formation of galaxies, let alone life?" (Answer: Bad question, for there are many actual cosmoses in which galaxies never form. The reformulated question, "Why does *this* cosmos—the one *we* experience—satisfy those special conditions?" seems like a nonsense question, something like "Why am I me and not you?" which nevertheless kept us from getting to sleep sometimes as children.) And, of course, there is the whole issue of emergence (versus reduction), still not entirely resolved, even at the level of chemistry vis-à-vis quantum physics. Should we recognize multiple categories of *properties* and *relations (attributes),* corresponding to different, irreducible levels of scientific inquiry? But these questions cannot be addressed within the scope of this essay (which we hereby guarantee by stopping).

Notes

1. To be sure, classical practice itself does not imply endorsement of Platonism, as many mainstream mathematicians, if pressed, fall back on some kind of formalism or fictionalism. "Platonism" designates a reflective view, based on a literal, face-value reading of mathematical discourse, which would justify the practice. It may well not be the only, or the best, justification, however.

2. While it is true that no consistent intuitionistic *propositional* theory can be in formal contradiction with classical logic, this is far from being the case for intuitionistic first or higher-order theories. For example, the sentence $\neg\forall x\forall y\ (x = y \vee x \neq y)$ is consistent in intuitionistic, but not classical logic. Indeed, such striking "conflicts" with classical mathematics—famously Brouwer's continuity theorem—arise in intuitionistic analysis, where nonclassical axioms of continuity governing choice sequences are available. The Bishop framework abandons any such nonclassical axioms and so generates no such conflicts. However, even in the intuitionistic case, these formal conflicts are only apparent, not real, turning on ambiguity of the logical notation, as will be explained later. (For a fuller discussion bringing out certain expressive limitations of intuitionism, see Hellman 1989.)

3. For a sustained critique of Dummett's case, see Burgess 1984.

4. These arise in connection with quantum mechanics and general relativity, but need not concern us here. See, e.g., Hellman 1993 and 1998.

5. To be sure, one can raise questions concerning the "universe of constructions" to which constructive proof-conditions appeal. But at least on a rough-and-ready, ordinary understanding of those conditions, anyone can see that the intuitionistic rules are correct and why certain classical principles and rules must be dropped.

6. According to anecdote, even the intuitionist Heyting seems to have shared this perspective, as he liked to teach classical recursion theory. He said he found it interesting.

7. On the iterative conception, sets are arranged in a hierarchy of stages corresponding to (finite and transfinite) ordinals. These "go on and on" in virtue of two main operations, passing from a set to its power set (set of all subsets of the given set), and taking the limit of any ordinal sequence of sets (or taking as a set the range of any function on a given set or ordinal (the content of the Axiom of Replacement). A stage so large that it cannot be reached from below by either of these operations is called "(strongly) inaccessible."

As Zermelo (1930) proved, an inaccessible stage provides a model for the ZF axioms, and so, by Gödel's second incompleteness theorem, the existence of inaccessibles cannot be proved within ZF. Nevertheless, they are regarded as quite legitimate by set theorists.

8. A famous example of Geach, "Some critics admire only one another," illustrates that the logic of plurals goes well beyond first order. On the usual, "singularist" view, there is hidden quantification over *classes* (of critics, in this case), but Boolos (1985) proposed turning this on its head, taking plural quantification as already understood and interpreting class quantification through it. This idea has been applied by Burgess, Hazen, and Lewis (Appendix to Lewis 1991), to get the effect of ordered pairing of arbitrary individuals without any set-theoretic machinery. Lewis (ibid.) has also argued, persuasively in our view, that we do have an independent grasp of plural quantifiers. These ingredients have played an important role in recent developments of a modal-structuralist approach to mathematics (e.g., Hellman 1996, 2003). A systematic, more ambitious treatment of the logic of plurals is given by Yi (2005).

9. Notoriously, Nelson Goodman (1978) challenged this assumption, where the "real world" literally gives way to multiple world versions (even apart from mathematics). We think, along with Scheffler, however, that here Goodman goes too far. See, e.g., Scheffler 1980.

10. This is quite compatible with an open-ended, context-relative understanding of quantifier phrases, which, arguably is all that is needed for ordinary expression and reasoning.

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