A CHARACTERIZATION OF UNIVERSAL COMPLETE BOOLEAN ALGEBRAS

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Let B be an (infinite) Boolean algebra† and let κ be an infinite cardinal such that $\kappa \leqslant |B|$. (We write $|B|$ for the cardinality of B.) B is *κ-complete* if each subset X of B of cardinality $< \kappa$ has a join $\bigvee X$. Following Morley and Vaught [3], B is said to be *κ-universal* if for each Boolean algebra A of cardinality $< \kappa$ there is a monomorphism of A into B. A subset X of B is an *antichain* if $0 \notin X$ and, for any pair of distinct elements $x, y \in X$, we have $x \wedge y = 0$. For each cardinal λ, we write A_λ for the Boolean algebra of all finite and cofinite subsets of λ.

Our aim in this note is to prove the following

THEOREM. *Let κ be an infinite cardinal and let B be an infinite κ-complete Boolean algebra. Then the following conditions are equivalent:*

(i) *B is κ-universal;*

(ii) *for each $\lambda < \kappa$, there is a monomorphism of A_λ into B;*

(iii) *for each cardinal $\lambda < \kappa$, B contains an antichain of cardinality λ.*

Remark. My original proof of the implication (iii) $\Rightarrow$ (i) used the technique of Boolean-valued models of set theory. The present elementary proof was discovered later.

Proof of the theorem

(i) $\Rightarrow$ (ii). Suppose that B is κ-universal and $\lambda < \kappa$. Then $|A_\lambda| < \kappa$ and (ii) follows.

(ii) $\Rightarrow$ (iii). Assume (ii), and let $\lambda < \kappa$. Obviously A_λ contains an antichain X of cardinality λ, so if h is a monomorphism of A into B, $\{h(x) : x \in X\}$ is an antichain in B of cardinality λ.

(iii) $\Rightarrow$ (i). Suppose that (iii) holds, and let A be a Boolean algebra of cardinality $\lambda < \kappa$. If λ is finite, it is easy to show that there is a monomorphism of A into B. (For example, construct a continuous mapping of the Stone space of B onto that of A.) Thus we may assume that λ is infinite. Let $\{a_\xi : \xi < \lambda\}$ be an enumeration of the non-zero elements of A, and for each $\xi < \lambda$ let U_ξ be an ultrafilter in A containing a_ξ. By assumption, B contains an antichain $\{b_\xi : \xi < \lambda\} = X$ of cardinality λ. By adjoining the complement of $\bigvee X$ to X, if necessary, we may assume without loss of generality that $\bigvee_{\xi<\lambda} b_\xi = 1$. Now define $h : A \to B$ by $h(x) = \bigvee\{b_\xi : x \in U_\xi\}$ for each $x \in A$.

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† If B is a Boolean algebra, we write 0, 1 respectively for the least and greatest elements of B. For $x, y \in B$ we write $x \vee y$, $x \wedge y$ and x^* for the join and meet of x and y, and the complement of x, respectively.

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We claim that h is a monomorphism of A into B. First, for $x, y \in A$ we have:

$$\begin{aligned} h(x \vee y) &= \bigvee\{b_\xi : x \vee y \in U_\xi\} = \bigvee\{b_\xi : x \in U_\xi \text{ or } y \in U_\xi\} \\ &= \bigvee\{b_\xi : x \in U_\xi\} \vee \bigvee\{b_\xi : y \in U_\xi\} \\ &= h(x) \vee h(y). \end{aligned}$$

Also,

$$h(x) \vee h(x^*) = h(x \vee x^*) = h(1) = \bigvee_{\xi < \lambda} b_\xi = 1,$$

and

$$\begin{aligned} h(x) \wedge h(x^*) &= \bigvee\{b_\xi : x \in U_\xi\} \wedge \bigvee\{b_\eta : x^* \in U_\eta\} \\ &= \bigvee_\xi \bigvee_\eta \{b_\xi \wedge b_\eta : x \in U_\xi \ \& \ x^* \in U_\eta\} \\ &= 0. \end{aligned}$$

Accordingly $h(x^*) = h(x)^*$, and it follows that h is a homomorphism.

Finally, h is one-one since, if $0 \neq x \in A$, then $x = a_\xi$ for some $\xi < \lambda$, so that $x \in U_\xi$ and $h(x) \geqslant b_\xi \neq 0$. This completes the proof.

COROLLARY 1. *Every infinite $\aleph_1$-complete Boolean algebra is $\aleph_1$-universal.*

Proof. It is well known (*cf.* Dwinger [**1**; Thm. 4.8]) that every infinite Boolean algebra contains an infinite antichain, so the result is an immediate consequence of the theorem.

Let $P\kappa$ be the complete Boolean algebra of all subsets of κ. Since $P\kappa$ clearly contains an antichain of cardinality κ, the theorem implies

COROLLARY 2. *If $\kappa \geqslant \aleph_0$, $P\kappa$ is κ^+-universal.*

Remarks

1. The assumption that B is κ-complete cannot be dropped in the statement of the theorem. For example, take $\kappa = \aleph_1$ and $B = A_{\aleph_0}$. Then B obviously satisfies condition (ii) of the theorem. On the other hand, B cannot be $\aleph_1$-universal. For it is easy to see that every subalgebra of B contains atoms, so since the free Boolean algebra A on countably many generators is atomless, there can be monomorphism of A into B. I do not know a necessary and sufficient condition for an arbitrary Boolean algebra to be κ-universal.

2. Let us call a κ-complete Boolean algebra B *strongly* κ-universal if for any Boolean algebra A of cardinality $< \kappa$ there is a *complete* monomorphism of A into B, i.e. a monomorphism which preserves any infinite joins which exist in A. If B is the collapsing $(\aleph_0, \kappa)$-algebra, the regular open algebra of the product space $\kappa^{\aleph_0}$ with the product topology—where κ is assigned the discrete topology—then it follows from work of Kripke [**2**] that B is strongly λ^+-universal for any λ satisfying $2^\lambda \leqslant \kappa$. (Since B obviously contains an antichain of cardinality κ, it follows from the present theorem that it is κ^+-universal.) Again, I do not know a necessary and sufficient condition for a κ-complete Boolean algebra to be strongly κ-universal.

References

1. P. Dwinger, *Introduction to Boolean algebras* (Würzburg, 1961).
2. S. Kripke, "An extension of a theorem of Gaifman–Hales–Solovay", *Fund. Math.*, 61 (1967), 29–32.
3. M. Morley and R. L. Vaught, "Homogeneous universal models", *Math. Scand.*, 11 (1962), 37–57.

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