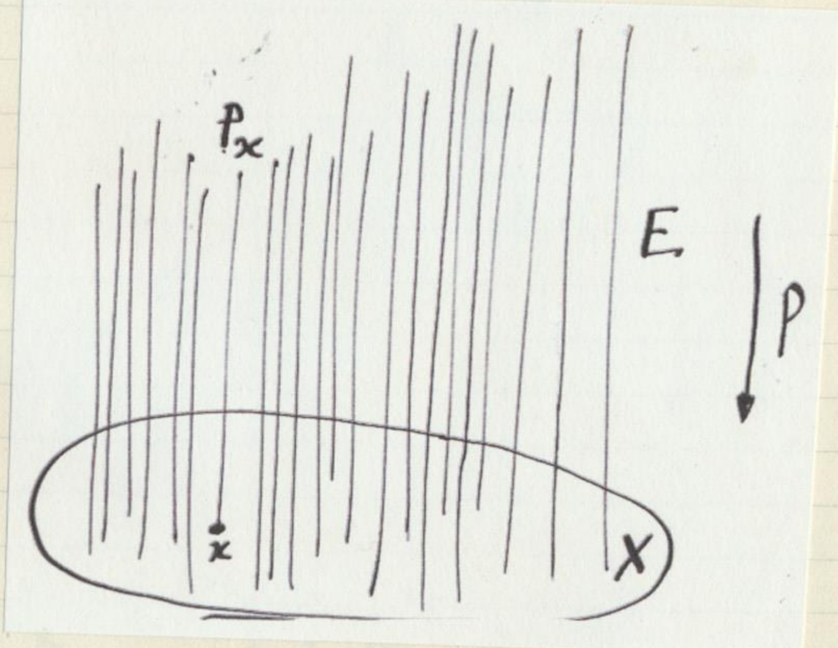
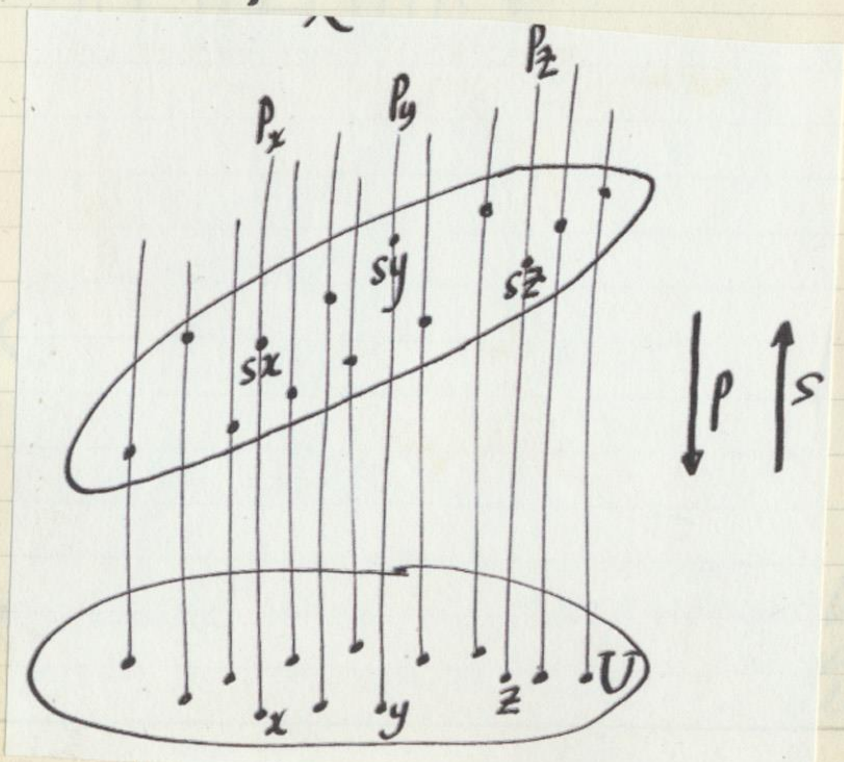
Bundles, Stacks & Sheaves

A *bundle* over a topological space X is just a continuous map $E \xrightarrow{p} X$ to X from some topological space E. (The domain of) A bundle over X may be viewed as the disjoint union of the fibres $P_x = p^{-1}(x)$ for $x \in X$:



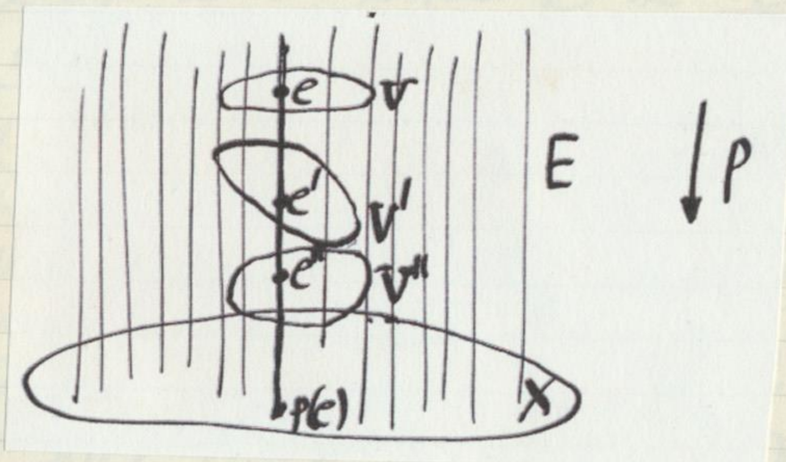
A Bundle

Let U be an open set in X. A (cross)-<u>section</u> of the bundle $E \xrightarrow{p} X$ <u>over</u> U is a map $s: U \to E$ for which $p \circ s$ is the inclusion map $U \hookrightarrow X$. If $x \in X$, a section of p <u>around</u> x is a section of p over some open neighbourhood of x.



A section over U

A *stack* over X is a bundle $E \xrightarrow{p} X$ for which p is a *local homeomorphism*, i.e. to each $e \in E$ there is an open neighbourhood V of E such that $p[V]$ is open in X and $p|V$ is a homeomorphism $V \to p[V]$:



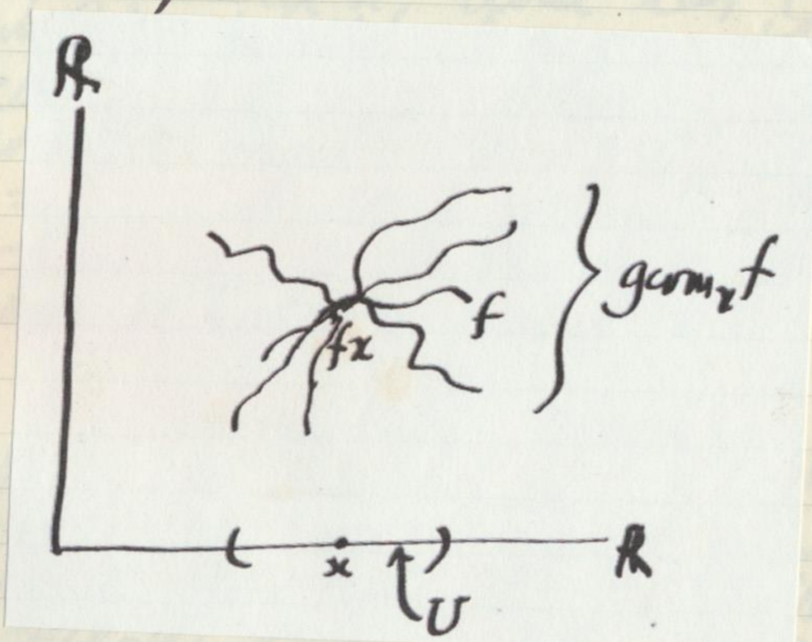
A Stack

If we call such a neighbourhood V of $e \in E$ a *stratum* at e, then the vicinity of each fibre in a stack can be pictured as a bunch of strata "stacked up".

Now consider topological spaces X and Y and a point $x \in X$. A continuous map to Y with domain an open neighbourhood of x is said to be *locally defined* at x.

Let f and g be locally defined around x. We say that f and g are *locally identical* around x, written $f \sim_x g$, if there is some open neighbourhood W of x such that $W \subseteq dom(f) \cap dom(g)$ and $f|W = g|W$. It is easy to see that $\sim_x$ is an equivalence relation; the equivalence class under this relation of a given locally defined f is called the <u>germ</u> of f at x and is written $\text{germ}_x f$.

When $X = Y = \mathbb{R}$ the germ of a map looks like this:



The germ of a map

Clearly $\mathrm{germ}_x f = \mathrm{germ}_x g$ $\Rightarrow fx = gx$, but not conversely.

Now let $E \xrightarrow{p} X$ be a stack over X and $x \in X$. Then points on the fibre P_x correspond bijectively to germs of sections of p around x. For suppose s is a section of p around x, and let $U = \mathrm{dom}(s)$. Write $\dot{s} = sx$; then clearly $\dot{s} \in P_x$. Also if $t \in \mathrm{germ}_x s$ then $sx = tx$, whence $\dot{s} = \dot{t}$. So each germ of a section of p around x determines a unique point of P_x.

Reciprocally, given a point $e \in P_x$, i.e. for which $p(e) = x$, there is an open neighbourhood U of e for which p/U is a homeomorphism $U \longrightarrow p[U]$, and the latter is an open neighbourhood of x. The map $s = p^{-1}/p[U]$ is then a section of p around x giving rise to its corresponding germ. Note that this germ is independent of the choice of U.

These correspondences are readily shown to be mutually inverse.

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A __presheaf__ F on a topological space X is an assignment, for each open set U in X, of a set FU and for each pair U, V of open sets for which $U \subseteq V$, of a map $F_{UV} : FV \to FU$ satisfying

- $F_{UU} = 1_U$
- $F_{UW} = F_{UV} \circ F_{VW}$ for $U \subseteq V \subseteq W$

If $v \in FV$ and $U \subseteq V$ we write $v|_U$ for $F_{UV}(v)$ and call it the <u>restriction</u> of v to U.

A presheaf F on X is a <u>sheaf</u> if it satisfies the following condition:

- Given any open covering $\{U_i : i \in I\}$ of an open set U in X and any family $\{u_i : i \in I\}$ with $u_i \in FU_i$ such that for any pair (i,j) with $u_i|_{U_i \cap U_j} = u_j|_{U_i \cap U_j}$, there exists a <u>unique</u> $u \in FU$ for which $u_i = u|_{U_i}$ for all $i \in I$.

Each bundle $E \xrightarrow{p} X$ over X determines a sheaf $\tilde{p}$ on X called the <u>sheaf of sections</u> of p. Here, if U is open in X,

$\tilde{p}U$ is the set of sections of p over U

and for $U \subseteq V$,

$$\tilde{p}_{UV} : \tilde{p}V \to \tilde{p}U : s \mapsto s|U.$$

It is easy to check that $\tilde{p}$ is a sheaf.

Conversely, each sheaf on X can be regarded as the sheaf of sections of a suitable bundle — in fact a <u>stack</u> — over X

The bijective correspondence — noted above — between points of (the domain of) a stack and germs of sections provides the "germ" of the construction of the stack whose sheaf of sections coincides with a given sheaf. The idea is to extend the concept of germ to an arbitrary presheaf.

Suppose then that F is a presheaf on X, $x \in X$, and U, V are open neighbourhoods of x. If $s \in F\bar{U}$, $t \in F\bar{V}$, we say that s and t are <u>coincident</u> around x, and write $s \sim_x t$, provided there exists an open neighbourhood $\bar{W}$ of x such that $W \subseteq U \cap V$ and $s|_W = t|_W$.

It is not hard to show that coincidence around x is an equivalence relation; for $s \in F\bar{U}$, the equivalence class containing s is called the <u>germ</u> of s at x and denoted by $\mathrm{germ}_x\, s$.

Now define

$$P_x = \{\text{germ}_x s : s \in FU,\ U \text{ open nbd of } x\}^{*}$$

to be the set of all germs at x. We amalgamate the various sets P_x of germs in the disjoint union (over $x \in X$), so obtaining the set

$$\Sigma_P = \coprod_{x \in X} P_x,$$

and define $p = p_F : \Sigma_P \to X$ as the map sending each $\text{germ}_x s$ to the point $x \in X$ where it is taken.

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* P_x is called the <u>stalk</u> of F at x.

To turn p into a stack over X we have to topologize Σ_p. This is done as follows: Each $s \in F\bar{U}$ determines a map $s^*: U \longrightarrow \Sigma_p$ given by

$$s^* x = \mathrm{germ}_x s \quad \text{for } x \in U.$$

Clearly s^* is a section of p. (Notice that in this way each element s of the given presheaf is replaced by an actual function s^* to the set Σ_p of germs.) Now we topologize Σ_p by taking as a base of open sets all the image sets $s^*[U] \subseteq \Sigma_p$; thus an open set in Σ_p is a union of images of the sections s^*. This topology makes p and every s^* continuous, and it is not hard to show that $s^*: U \to s[U]$ is a homeomorphism.

So $p: \Sigma_p \to X$ is a bundle; it is in fact a stack. For each point germ s in Σ_p has the open neighborhood $s^*[U]$, and $p|s^*[U]$ has $s^*: U \to s^*[U]$ as a two-sided inverse, making it a homeomorphism.

ΔT can then be known

that, if F is a <u>sheaf</u>, then F is isomorphic in the appropriate sense to the sheaf of sections of the stack Σ_F. In other words, each sheaf is the sheaf of sections of a <u>stack</u>.

Finally, let $Sh(X)$ be the category of sheaves on X with natural transformations as arrows, and let $St(X)$ be the category of stacks over X in which an arrow between stacks

$E \xrightarrow{p} X$, $E' \xrightarrow{p'} X$ is a continuous map $f: E \to E'$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & p \searrow \quad \swarrow p' & \\ & X & \end{array}$$

commutes. It can then be shown that the correspondence $F \longmapsto p_F$ between sheaves and stacks extends to an equivalence between $Sh(X)$ and $St(X)$. This is the precise sense in which sheaves and stacks are "equivalent".