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ON COMPACT CARDINALS

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Let $\varkappa$ be a cardinal and $\mathbf{L}$ a language. $\varkappa$ is said to be $\mathbf{L}$-*compact* if whenever Σ is a set of sentences of $\mathbf{L}$ such that any subset of Σ of power $< \varkappa$ has a model, so does Σ. If $\mathscr{L}$ is a class of languages, we say that $\varkappa$ is $\mathscr{L}$-*compact* if $\varkappa$ is $\mathbf{L}$-compact for each $\mathbf{L} \in \mathscr{L}$.

In Section 1 of this paper we investigate the properties and relative sizes of $\mathscr{L}$-compact cardinals for certain classes $\mathscr{L}$ of infinitary and second-order languages. In Section 2 we consider a natural extension of the notion of compactness to languages whose class of formulas is a proper class in the sense of MORSE-KELLEY set theory. Using a result of KUNEN [4], we show that there are no $\mathbf{L}$-compact cardinals for any *second order* language $\mathbf{L}$ whose class of individual constants is a proper class.

§ 1. Results on $\mathscr{L}$-compact cardinals

Notation. We write $\mathscr{L}^1$ for the class of all first order languages with equality, all of which are assumed to include a binary predicate symbol $\in$.

For cardinals $\varkappa, \lambda$, we write $\mathscr{L}_{\varkappa\lambda}$ for the class of all infinitary languages $\mathbf{L}_{\varkappa\lambda}$, where $\mathbf{L} \in \mathscr{L}^1$.

$\mathscr{L}^2$ is the class of all second-order languages $\mathbf{L}^2$ obtained from languages $\mathbf{L} \in \mathscr{L}^1$ by adding a countable sequence $X_0, X_1, \ldots$ of second-order variables, which are understood to range over all sets of individuals.

Definition. A cardinal $\varkappa$ is said to be

(a) *strongly compact* if it is $\mathscr{L}_{\varkappa\varkappa}$-compact;

(b) *supercompact* if for each cardinal $\lambda \geqq \varkappa$ there is a $\varkappa$-complete ultrafilter U over $\mathsf{P}_\varkappa(\lambda)$ (the family of all subsets of λ of power $< \varkappa$) such that, for all $\xi < \lambda$,

$$\{x \in \mathsf{P}_\varkappa(\lambda) : \xi \in x\} \in U$$

and, for all $f : \mathsf{P}_\varkappa(\lambda) \to \lambda$,

$$\{x \in \mathsf{P}_\varkappa(\lambda) : f(x) \in x\} \in U \to (\exists \xi < \lambda)\, \{x \in \mathsf{P}_\varkappa(\lambda) : f(x) = \xi\} \in U;$$

(c) *extendable* if for all $\alpha > \varkappa$ there is $\beta > \alpha$ and an elementary embedding j of $\langle R_\alpha, \in \rangle$ into $\langle R_\beta, \in \rangle$ such that $\varkappa$ is the first ordinal moved by j.

It is known [5] that every extendable cardinal is supercompact and that every supercompact cardinal is strongly compact. We first show that an extendable cardinal is a large supercompact cardinal.

Theorem 1. *Each extendable cardinal $\varkappa$ is a limit supercompact cardinal*, i.e. *there is a set of $\varkappa$ supercompact cardinals $< \varkappa$.*

Proof. Let $\mathrm{sc}(\xi)$ be a formula of the language of set theory expressing that ξ is a supercompact cardinal. By the reflection principle, we can find a limit ordinal $\alpha > \varkappa$ such that, for all $\xi < \alpha$,

$$\langle R_\alpha, \in \rangle \models \mathrm{sc}[\xi] \Leftrightarrow \mathrm{sc}(\xi).$$

Now, since $\varkappa$ is extendable, there is a $\beta > \alpha$ and an elementary embedding j of $\langle R_\alpha, \in\rangle$ into $\langle R_\beta, \in\rangle$ such that $\varkappa$ is the first ordinal moved by j. By [5], the set

$$U = \{x \subseteq \varkappa : \varkappa \in j(x)\}$$

is a normal ultrafilter over $\varkappa$. Now let $a = \{\xi < \varkappa : \mathrm{sc}(\xi)\}$. We claim that $a \in U$, which will prove the theorem since every member of U is evidently of power $\varkappa$. By the choice of α, we have

$$a = \{\xi < \varkappa : \langle R_\alpha, \in\rangle \models \mathrm{sc}[\xi]\}.$$

Hence, since j is elementary, we have

$$j(a) = \{\xi < j(\varkappa) : \langle R_\beta, \in\rangle \models \mathrm{sc}[\xi]\}.$$

Since α is a limit ordinal and $\langle R_\beta, \in\rangle$ is elementarily equivalent to $\langle R_\alpha, \in\rangle$, β is also a limit ordinal. Hence, since $\varkappa$ is supercompact we have $\langle R_\alpha, \in\rangle \models \mathrm{sc}[\varkappa]$. It follows that $\varkappa \in j(a)$, whence $a \in U$, as claimed. This completes the proof.

Corollary. *The least $\mathscr{L}^2$-compact cardinal is a limit supercompact, hence also a limit strongly compact, cardinal.*

Proof. By a result of [2], the least $\mathscr{L}^2$-compact cardinal is extendable, so the result follows immediately from Theorem 1.

We now consider $\mathscr{L}_{\omega_1\omega}$ and $\mathscr{L}_{\omega_1\omega_1}$-compact cardinals. Our next result is related to the main theorem of [1] and is proved in a similar way. We leave the reader to supply the details of the proof, using the techniques of [1].

Theorem 2. *Let $\varkappa$ be a cardinal. Then the following conditions are equivalent.*

(i) *$\varkappa$ is $\mathscr{L}_{\omega_1\omega}$-compact;*

(ii) *$\varkappa$ is $\mathscr{L}_{\omega_1\omega_1}$-compact;*

(iii) *for any transitive set M and any subset X of M with the property that any subset of X of power $<\varkappa$ has non-empty intersection, there is a transitive set N and an elementary embedding j of $\langle M, \in\rangle$ into $\langle N, \in\rangle$ such that $\bigcap j``X \neq \emptyset$.*

(iv) *for any cardinal λ, each $\varkappa$-complete filter over λ can be extended to an ω_1-complete ultrafilter over λ.*

The following result is an immediate consequence of Corollary 2.2 of [3].

Lemma. *Let μ_0 be the least (uncountable) measurable cardinal. Then, for any cardinal λ, each ω_1-complete ultrafilter over λ is μ_0-complete.*

Notice that this lemma together with Theorem 2 yields the conclusion that any $\mathscr{L}_{\omega_1\omega}$-compact cardinal is $\mathscr{L}_{\mu_0\mu_0}$-compact. For, if $\varkappa$ is $\mathscr{L}_{\omega_1\omega}$-compact, then, by (iv) of Theorem 2 and the lemma, any $\varkappa$-complete ultrafilter over a cardinal λ can be extended to a μ_0-complete ultrafilter, and the required conclusion can then be obtained by a straightforward ultrapower argument.

Now let $\varkappa_0$ be the least $\mathscr{L}_{\omega_1\omega}$-compact cardinal and λ_0 the least strongly compact cardinal. It is clear from (iv) of Theorem 2 that $\mu_0 \leqq \varkappa_0$ and trivially $\varkappa_0 \leqq \lambda_0$. It is still unknown whether $\mu_0 < \lambda_0$ is provable in set theory.

Theorem 3. $\mu_0 < \lambda_0 \Rightarrow \mu_0 < \varkappa_0$.

Proof. Suppose $\mu_0 = \varkappa_0$. Then by (iv) of Theorem 2 and the lemma, each μ_0-complete ultrafilter over a cardinal λ can be extended to a μ_0-complete ultrafilter, and it is well known that this condition is equivalent to the strong compactness of μ_0. Hence $\mu_0 = \lambda_0$, and the theorem follows.

In [6], VOPĚNKA and HRBÁČEK show that, if there is a cardinal $\varkappa$ such that, for any cardinal λ, each $\varkappa$-complete filter over λ can be extended to a $\varkappa$-complete ultrafilter (i.e. if $\varkappa$ is strongly compact), then the universe is not constructible from any set. Now it is quite easy to verify that the VOPĚNKA-HRBÁČEK proof only uses the fact that each $\varkappa$-complete filter over λ can be extended to an ω_1-*complete* ultrafilter over λ. But this is precisely condition (iv) of Theorem 2. Accordingly we have

Theorem 4. *If there is an $\mathscr{L}_{\omega_1\omega}$-compact cardinal, then the universe is not constructible from any set.*

Corollary. *Let* ZFM *be the theory* ZF + "*there exists a measurable cardinal*". *Then if* ZFM *is consistent, the statement* "*μ_0 is $\mathscr{L}_{\omega_1\omega}$-compact*" *cannot be proved in* ZFM.

§ 2. Compactness in languages whose class of formulas is a proper class

Suppose $\mathscr{L}$ is a proper class of languages with a common stock of logical symbols. Consider the language $\bigcup\mathscr{L}$ whose class of non-logical symbols is given by the union of the sets of non-logical symbols of all members of $\mathscr{L}$ and whose logical symbols are the same as those of each member of $\mathscr{L}$. Evidently the class of non-logical symbols, and hence of formulas, of $\bigcup\mathscr{L}$ is a proper class in the sense of MORSE-KELLEY set theory. It is natural to call a cardinal $\varkappa$ $\bigcup\mathscr{L}$-*compact for sets* if

(*) whenever Σ is a set of sentences of $\bigcup\mathscr{L}$ and each subset of Σ of power $< \varkappa$ has a model, then Σ has a model.

Clearly, then, if $\varkappa$ is $\bigcup\mathscr{L}$-compact, it must be $\mathscr{L}$-compact in the sense of § 1. Moreover, quite weak conditions on $\mathscr{L}$ are sufficient to ensure that the converse obtains, for example, if whenever $\mathscr{L}'$ is a subset of $\mathscr{L}$, we have $\bigcup\mathscr{L}' \in \mathscr{L}$. This condition is met by all the classes of languages considered in this paper.

We propose to investigate what happens if we extend the condition (*) to all classes of sentences Σ of $\bigcup\mathscr{L}$. Clearly, if $\mathscr{L}$-compactness was a stringent condition, then this new condition is in general still more stringent. We shall see that, in fact, when $\mathscr{L}$ is the class of second-order languages, this condition is so stringent as to be unsatisfiable.

Let us define a *class language* to be a language $\mathbf{L}$ with one binary predicate symbol $\in$ in addition to the equality symbol $=$, and a proper class C of constant symbols. An $\mathbf{L}$-*structure* is a triple[1]) $\mathfrak{A} = \langle A, E, F\rangle$ where A is a class, $E \subseteq A \times A$ and F is a function from C into A. If $c \in C$, we write $c^{\mathfrak{A}}$ for $F(c)$ as usual. The notion of satisfaction for formulas of $\mathbf{L}$ can now be defined in the customary way by interpreting $\in$ as E and c as $c^{\mathfrak{A}}$ for each $c \in C$. It is also easy to see that this definition of satisfaction can be formalized in MOESE-KELLEY set theory.

If $\varkappa$ is a cardinal and $\mathbf{L}$ is a class language, we say that is $\mathbf{L}$-*compact* if whenever Σ is a class of sentences of $\mathbf{L}$ such that each subset of Σ of power $< \varkappa$ has a model, then Σ has a model.

Observe that the usual HENKIN-style completeness proof for first-order logic can be extended in a straightforward way to show that any consistent class of first-

1) Here we use the customary notation in MORSE-KELLEY set theory for a finite sequence of

order sentences has a model. For the usual completeness proof to go through for a first order class language **L** it is evident that the following two conditions must be met:

(1) there is a proper class of new constants available for adding to **L**;

(2) each consistent class of sentences of **L** can be extended to a consistent complete class of sentences.

Now (1) is not true in general, because the class C of constants of **L** might exhaust the whole universe. But we can always replace C by a proper class C' equipollent with C and such that the complement of C' is a proper class. The new language **L**$'$ obtained in this way is equivalent to **L** and satisfies (1).

As for (2), the global axiom of choice yields an enumeration $\{\sigma_\alpha\colon \alpha \in \mathbf{ORD}\}$[1]) of all sentences of **L**. Starting with a consistent class of sentences Σ, we define by transfinite recursion a function f with domain **ORD** as follows:

$$f(\alpha) = \begin{cases} \sigma_\alpha & \text{if not } \Sigma \cup \{f(\beta)\colon \beta < \alpha\} \vdash \neg\sigma_\alpha \\ \neg\sigma_\alpha & \text{if } \Sigma \cup \{f(\beta)\colon \beta < \alpha\} \vdash \neg\sigma_\alpha \end{cases}$$

Then $\Sigma' = \Sigma \cup \{f(\alpha)\colon \alpha \in \mathbf{ORD}\}$ is clearly a consistent complete class of sentences containing Σ.

From this discussion we immediately infer

Theorem 5. *ω is* **L**-*compact for any first-order class language* **L**.

By contrast, however, we have

Theorem 6. *Let* **L** *be a second-order class language. Then there are no* **L**-*compact cardinals.*

Proof. First of all, it is clear that, if $\varkappa$ is **L**-compact for one second-order class language **L**, it is **L**-compact for all second-order class languages **L**. This is so because there is – assuming the global axiom of choice – a bijection between the classes of constants of any pair of second-order class languages.

Thus, without loss of generality, we may assume that **L** is a second-order class language whose class C of constants does not exhaust the whole universe V. By the global axiom of choice, there is a bijection F from C onto V. Let $\mathfrak{B}$ be the **L**-structure $\langle V, \in F\rangle$. For each $a \in V$, we write $\bar{a}$ for $F^{-1}(a)$; thus $\bar{a}$ is the constant of **L** which denotes a in $\mathfrak{B}$.

Now suppose, if possible, that $\varkappa$ is an **L**-compact cardinal. Let λ be the least regular cardinal $\geqq \varkappa$. Then since any cardinal $\geqq \varkappa$ is **L**-compact, so is λ. Let c be a constant not in C and let **L**$'$ be the language obtained by adjoining the constant c to **L**. Then, by the above remarks, λ is **L**$'$-compact.

Let Σ be the class of sentences of **L**$'$ consisting of:

(1) all sentences of **L** holding in $\mathfrak{B}$;
(2) the sentences $\bar{\alpha} \in c$ for all $\alpha < \lambda$;
(3) the sentence $c \in \bar{\lambda}$.

We claim that each subset Σ' of Σ of power $< \lambda$ has a model. For let β be the supremum of all ordinals $\alpha < \lambda$ such that $\bar{\alpha}$ occurs in a sentence of type (2) in Σ'. Then, since

[1]) Here **ORD** is the class of all ordinals.

there are $< \lambda$ such α and λ is regular, we have $\beta < \lambda$. Accordingly, $\langle V, \in, F \cup \{\langle c, \beta \rangle\}\rangle$ is a model of Σ'.

Since λ is $\mathbf{L}'$-compact, Σ has a model whose universe may be taken to be transitive since Σ contains the second-order sentence which asserts that $\in$ is a well-founded relation. Thus let $\mathfrak{A} = \langle A, \in, G \rangle$ be a transitive class model of Σ. Clearly the map $j\colon V \to A$ defined by $j(a) = \bar{a}^{\mathfrak{A}}$ for $a \in V$ is an elementary embedding of $\langle V, \in \rangle$ into $\langle A, \in \rangle$. Thus j is order-preserving on the ordinals and it follows that, for each ordinal α, we have $\alpha \leqq j(\alpha)$. Accordingly A contains arbitrarily large ordinals, so, since it is transitive, it contains all the ordinals. Now put σ for the second-order sentence

$$\forall X_0 [\exists x \, \forall y [X_0(y) \to y \in x] \to \exists x \, \forall y [X_0(y) \leftrightarrow y \in x]].$$

σ says that every subset of an individual is (coextensive with) an individual. Now certainly σ holds in $\langle V, \in \rangle$, so it also holds in $\langle A, \in \rangle$. But it is well-known that the only transitive class model of $\mathbf{ZF} + \sigma$ containing all ordinals is V itself; it follows that $A = V$. Therefore j is an elementary embedding of $\langle V, \in \rangle$ into itself. Since $j(\lambda) = \bar{\lambda}^{\mathfrak{A}} > c^{\mathfrak{A}} > \bar{\alpha}^{\mathfrak{A}}$ for all $\alpha < \lambda$ it follows that $j(\lambda) > \lambda$, so that j is not the identity. But this contradicts a result in [4] which asserts that there are no non-trivial elementary embeddings of $\langle V, \in \rangle$ into itself.

This completes the proof.

Remark. Since the sentence characterizing well-foundedness and the sentence σ introduced in the proof of Theorem 6 are both Π_1^1, it is clear that this proof actually establishes the following ostensibly stronger result: *For each cardinal $\varkappa$ there is a class Σ of Π_1^1-sentences whose only non-logical constants are $\in$ and individual constants such that each subset of Σ of power $< \varkappa$ has a model but Σ itself has no model.*

Added in proof: In the proof of Theorem 6, to justify the claim that the model $\mathfrak{A}$ may be taken to be transitive, it is also necessary to show that Σ contains a sentence σ' which asserts that each initial segment under $\in$ is a set. We may take σ' to be the second-order sentence

$$\forall x \, \exists X_0 \, \forall y [X_0(y) \leftrightarrow y \in x];$$

it is now not hard to see that $\sigma' \in \Sigma$.

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