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UNIVERSAL COMPLETE BOOLEAN ALGEBRAS AND CARDINAL COLLAPSING

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Let λ be an infinite cardinal and let A, B be Boolean algebras. A homomorphism $h: A \to B$ is said to be *λ-complete* if whenever X is a subset of A of cardinality $\leqq \lambda$ such that the join $\bigvee X$ of X exists in A, then $\bigvee h[X]$ exists in B and is equal to $h(\bigvee X)$. If $\varkappa$ is an infinite cardinal, B is said to be *$(\varkappa, \lambda)$-universal* if for each Boolean algebra A of cardinality $\leqq \varkappa$ there is a λ-complete monomorphism (i.e. one-one homomorphism) of A into B.

Our objective in this paper is to investigate, for *complete* Boolean algebras B, the relationship between $(\varkappa, \lambda)$-universality of B and the collapsing of cardinals to countable ordinals in the B-extension of the universe of sets. We assume familiarity with the technique of Boolean-valued models of set theory, as presented, e.g., in JECH [1]. For the theory of Boolean algebras, see SIKORSKI [2].

For each set X, we write $|X|$ for the cardinality of X. The symbols ξ, η will always denote ordinals, while $\alpha, \beta, \varkappa, \lambda$ will be used for cardinals. If A is a Boolean algebra, we write 0_A and 1_A for the least and greatest elements of A, respectively. If $x, y \in A$, then x^*, $x \vee y$, $x \wedge y$ denote the complement of x, and the join and meet of x and y, respectively. A subset X of A is called an *antichain* in A if $0_A \notin X$ and $x \wedge y = 0_A$ for any distinct $x, y \in X$.

Now let B be a *fixed infinite complete* Boolean algebra. As usual, we write $V^{(B)}$ for the B-extension of the universe V of sets and, for each sentence σ of the language $\mathrm{L}^{(B)}$ for $V^{(B)}$ we write $\|\sigma\|$ for the B-value of σ in $V^{(B)}$. We also write $V^{(B)} \models \sigma$ if $\|\sigma\| = 1_B$. We recall that there is a canonical mapping $x \mapsto \hat{x}$ of V into $V^{(B)}$. We shall also need the following two facts (JECH [1], Lemmas 49 and 50):

Lemma 1. *Let $\{x_\xi : \xi < \lambda\}$ be an antichain in B, and let $\{t_\xi : \xi < \lambda\} \subseteqq V^{(B)}$. Then there is $t \in V^{(B)}$ such that $x_\xi \leqq \|t = t_\xi\|$ for all $\xi < \lambda$.*

Lemma 2. *For each formula $\varphi(v)$ of $\mathrm{L}^{(B)}$ containing at most the variable v free, there is $t \in V^{(B)}$ such that $\|\exists v \varphi(v)\| = \|\varphi(t)\|$.*

Our first result gives a necessary and sufficient condition for a given cardinal to be collapsed to a countable ordinal in $V^{(B)}$.

Theorem 1. *Let $\varkappa \geqq \aleph_0$. Then the following conditions are equivalent:*

(i) $V^{(B)} \models$ *($\hat{\varkappa}$ is countable)*;

(ii) *there is a subset $\{b_{m\xi} : m \in \omega, \xi < \varkappa\} \subseteqq B$ such that $\bigvee_{m \in \omega} b_{m\xi} = 1_B$ for all $\xi < \varkappa$ and $\{b_{m\xi} : \xi < \varkappa\}$ is an antichain for each $m \in \omega$.*

Proof. (i) $\Rightarrow$ (ii). Suppose (i) is satisfied. Then we have

$$V^{(B)} \models \exists f[f \text{ is a map of } \hat{\omega} \text{ onto } \hat{\varkappa}].$$

Hence, by Lemma 2, there is $f \in V^{(B)}$ such that

(1) $$V^{(B)} \models (f \text{ is a map of } \hat{\omega} \text{ onto } \hat{\varkappa}).$$

Put $b_{m\xi} = \|f(\hat{m}) = \hat{\xi}\|$ for $m \in \omega$, $\xi < \varkappa$. Then if $\xi \neq \eta < \varkappa$,

$$b_{m\xi} \wedge b_{m\eta} = \|f(\hat{m}) = \hat{\xi} \wedge f(\hat{m}) = \hat{\eta}\| \leqq \|\hat{\xi} = \hat{\eta}\| = 0_B,$$

and, for $\xi < \varkappa$,

$$\bigvee_{m\in\omega} b_{m\xi} = \bigvee_{m\in\omega} \|f(\hat{m}) = \hat{\xi}\| = \|\exists x \in \hat{\omega}[f(x) = \hat{\xi}]\| = 1_B$$

by (1). Thus $\{b_{m\xi}: m \in \omega, \xi < \varkappa\}$ satisfies (ii).

(ii) $\Rightarrow$ (i). Assume (ii), and define $f \in V^{(B)}$ by $\mathrm{dom}(f) = \{\langle \hat{m}, \hat{\xi}\rangle^{(B)}: m \in \omega, \xi < \varkappa\}$ (where $\langle \cdot, \cdot \rangle^{(B)}$ is the "ordered pair in $V^{(B)}$") and, for $m \in \omega$, $\xi < \varkappa$,

$$f(\langle \hat{m}, \hat{\xi}\rangle^{(B)}) = b_{m\xi}.$$

Using the fact that $\{b_{m\xi}: \xi < \varkappa\}$ is an antichain, it follows easily that

$$V^{(B)} \models (f \text{ is a function such that Domain}(f) \subseteqq \hat{\omega} \text{ and Range}(f) \subseteqq \hat{\varkappa}).$$

Also, for each $\xi < \varkappa$ we have

$$\|\exists x \in \hat{\omega}[f(x) = \hat{\xi}]\| = \bigvee_{m\in\omega} \|f(\hat{m}) = \hat{\xi}\| = \bigvee_{m\in\omega} b_{m\xi} = 1_B.$$

It follows that $V^{(B)} \models (\hat{\varkappa} \subseteqq \mathrm{Range}(f))$ and so

$$V^{(B)} \models (\hat{\varkappa} \text{ is countable}).$$

This completes the proof.

Our next result relates $(\varkappa, \lambda)$-universality of B to the collapsing of $\varkappa$ to a countable ordinal in $V^{(B)}$. In the proof we employ a technique originally introduced by KRIPKE (cf. JECH [1], Theorem 40) in order to show that the collapsing $(\aleph_0, 2^{\varkappa})$-algebra is $(\varkappa, \varkappa)$-universal.

Theorem 2. *Let $\varkappa, \lambda \geqq \aleph_0$ satisfy $\varkappa^\lambda = \varkappa$ (thus $\lambda < \varkappa$). Then conditions* (i) *and* (ii) *of Theorem* 1 *are each equivalent to*

(iii) *B is $(\varkappa, \lambda)$-universal.*

Proof. (i) $\Rightarrow$ (iii). Assume (i); then, by Theorem 1, (ii) also holds. Let A be a Boolean algebra of cardinality $\alpha \leqq \varkappa$. If α is finite, it is well-known that A is isomorphic to a subalgebra of B, and the isomorphism is obviously a λ-complete monomorphism of A into B. Thus we may assume that α is infinite. Let S be the family of all subsets of A of cardinality $\leqq \lambda$. Then we have

$$|S| \leqq \sum_{\beta\leqq\lambda} \alpha^\beta \leqq \lambda \cdot \varkappa^\lambda = \varkappa.$$

Since $V^{(B)} \models (\hat{\varkappa} \text{ is countable})$, it follows that

$$V^{(B)} \models (\hat{S} \text{ is countable}).$$

Next, it follows immediately from (ii) that B contains an antichain $\{x_\xi: \xi < \alpha\}$ of cardinality α. By adjoining $(\bigvee x_\xi)^*$, if necessary, we may assume without loss of generality that $\bigvee_{\xi<\alpha} x_\xi = 1_B$. Let $\{a_\xi: \xi < \alpha\}$ enumerate the non-zero members of A. By Lemma 1, there is $b \in V^{(B)}$ such that $x_\xi \leqq \|b = \hat{a}_\xi\|$ for all $\xi < \alpha$. Hence

$$\|b \in \hat{A}\| \geqq \bigvee_{\xi<\alpha} \|b = \hat{a}_\xi\| \geqq \bigvee_{\xi<\lambda} x_\xi = 1_B.$$

It is easily verified that $V^{(B)} \models (\hat{A}$ is a Boolean algebra). Moreover, we have, for each $\xi < \varkappa$,

$$x_\xi \leqq \|b = \hat{a}_\xi\| = \|b = \hat{a}_\xi\| \wedge \|\hat{a}_\xi \neq \hat{0}_A\| \leqq \|b \neq \hat{0}_A\| = \|b \neq 0_{\hat{A}}\|$$

so that $1_B = \bigvee_{\xi<\varkappa} x_\xi \leqq \|b \neq 0_{\hat{A}}\|$. Since $V^{(B)} \models$ (Axiom of Choice), it follows that

$$V^{(B)} \models (\text{Rasiowa-Sikorski Lemma}),$$

and hence

$$V^{(B)} \models \exists U(U \text{ is an } \hat{S}\text{-complete ultrafilter in } \hat{A} \text{ and } b \in U).$$

Therefore, by Lemma 2, there is $U \in V^{(B)}$ such that

(1) $$V^{(B)} \models (U \text{ is an } \hat{S}\text{-complete ultrafilter in } \hat{A} \text{ and } b \in U).$$

Now we define $h: A \to B$ by

$$h(a) = \|\hat{a} \in U\|$$

for $a \in A$. It is easy to verify that h is a homomorphism of A into B. To see that h is λ-complete, observe that, if $X \in S$ and $a = \bigvee X$ in A, then $\|\hat{a} = \bigvee \hat{X}\| = 1_B$, so that, using (1),

$$h(a) = \|\hat{a} \in U\| = \|\bigvee \hat{X} \in U\| = \|\exists x \in \hat{X}(x \in U)\| = \bigvee_{x\in X} \|\hat{x} \in U\| = \bigvee_{x\in X} h(x).$$

And finally, h is one-one, because if $0_A \neq a \in A$, then $a = a_\xi$ for some $\xi < \varkappa$, so that

$$h(a) = \|\hat{a}_\xi \in U\| \geqq \|b \in U\| \wedge \|\hat{a}_\xi = b\| = \|\hat{a}_\xi = b\| \geqq x_\xi \neq 0_B.$$

This proves (iii).

(iii) $\Rightarrow$ (ii). Assume (iii), and let A be the collapsing $(\aleph_0, \varkappa)$-algebra, i.e. the regular open algebra of the product space $\varkappa^\omega$, where $\varkappa$ has been assigned the discrete topology. Let A' be the countably complete subalgebra of A generated by the elements $a_{m\xi} = \{g \in \varkappa^\omega : g(m) = \xi\}$ for $m \in \omega$, $\xi < \varkappa$. Then $\varkappa \leqq |A'| \leqq \varkappa^{\aleph_0} \leqq \varkappa^\lambda = \varkappa$. By (iii), there is a λ-complete monomorphism h of A' into B. It is easy to verify that, in A', $\bigvee_{m\in\omega} a_{m\xi} = 1_{A'}$ and $\{a_{m\xi}: \xi < \varkappa\}$ is an antichain in A' for each $m \in \omega$. Therefore, if we set $b_{m\xi} = h(a_{m\xi})$ for each $m \in \omega$, $\xi < \varkappa$, then $\{b_{m\xi}: m \in \omega, \xi < \varkappa\}$ is a subset of B satisfying (ii).

This completes the proof.

From Theorem 2 we immediately infer:

Corollary 1. *Let $\varkappa \geqq \aleph_0$ be such that $\varkappa^{\aleph_0} = \varkappa$. Then, if B is $(\varkappa, \aleph_0)$-universal, it is $(\varkappa, \lambda)$-universal for any λ such that $\varkappa^\lambda = \varkappa$. (In particular, if $\varkappa$ is inaccessible, and B is $(\varkappa, \aleph_0)$-universal, then B is $(\varkappa, \lambda)$-universal for all $\lambda < \varkappa$.)*

Assuming the generalized continuum hypothesis (GCH), Theorem 2 also yields the following characterization of the collapsing of a give successor cardinal to a countable ordinal.

Corollary 2 (GCH). *Let $\lambda \geqq \aleph_0$ and let $\varkappa = \lambda^+$. The the following conditions are equivalent:*

(i) $V^{(B)} \models (\hat{\varkappa}$ *is countable*);
(ii) *B is $(\varkappa, \aleph_0)$-universal.*

Proof. Assuming GCH, $\varkappa = 2^\lambda$ so that $\varkappa^{\aleph_0} = 2^{\lambda^{\aleph_0}} = 2^\lambda = \varkappa$ and Theorem 2 applies.

I do not know whether the GCH is essential for Corollary 2 to hold.

References

[1] JECH, T., Lecture notes on set theory. Springer Lecture Notes in Mathematics. Berlin 1971.
[2] SIKORSKI, R., Boolean Algebras. Berlin 1960.

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