The Axiom of Choice

John L. Bell

© Individual author and College Publications 2009. All rights reserved.

ISBN 978-1-904987-54-3

College Publications
Scientific Director: Dov Gabbay
Managing Director: Jane Spurr
Department of Computer Science
King's College London, Strand, London WC2R 2LS, UK

http://www.collegepublications.co.uk

Original cover design by orchid creative www.orchidcreative.co.uk
Printed by Lightning Source, Milton Keynes, UK

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form, or by any means, electronic, mechanical, photocopying, recording or otherwise without prior permission, in writing, from the publisher.

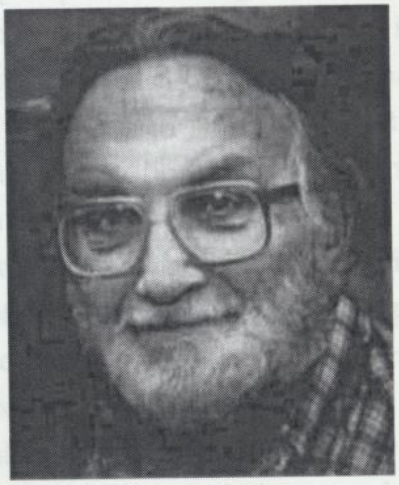
Dedicated to the Memory of

My Dear Wife Mimi

1947-2009

ERNST ZERMELO
1871 – 1953

FELIX HAUSDORFF
1868 – 1942



MAX ZORN
1906 – 1993

Preface

The Axiom of Choice has fascinated me from my student days. As a young aspiring mathematician, I was struck by the fact that through the areas of mathematics to which I was most attracted—abstract algebra, general topology, functional analysis —there ran a common thread: the Axiom of Choice and the "working mathematician's" surrogate for it known as Zorn's Lemma. Teasing out that thread is the purpose of the present book.

The book provides an overview of the development of the Axiom of Choice since its introduction by Zermelo at the beginning of the last century. In it the Axiom of Choice is surveyed from three perspectives. The first, or *mathematical* perspective, is that of the abovementioned "working mathematician". This perspective brings into view the manifold applications of the Axiom of Choice—usually in the guise of Zorn`s Lemma—in a great variety of areas of mathematics. The second, *foundational*, perspective, is that of the logician or constructive mathematician concerned with the foundational status of the Axiom of Choice. The third, *topos-theoretical*, perspective is that taken by the mathematician or logician investigating the role of the Axiom of Choice in topos theory.

In this book certain topics—for instance mathematical applications of the Axiom, and its relationship with logic—are discussed in considerable detail. Others—notably the consistency and independence of the Axiom of the usual systems of set theory—are given no more than summary treatment, the justification being that these topics have been given full expositions elsewhere. The book contains seven chapters and two appendices. Chapter I describes the origins of the Axiom of Choice and its status within set theory. Chapter II introduces

maximal principles, in particular, Zorn's Lemma, and discusses their development and their relationship to the Axiom of Choice. Chapter III provides a detailed account of the many applications of the Axiom of Choice and Zorn's Lemma within classical mathematics. Chapter IV offers a compressed account of how the Axiom of Choice is proved consistent with, and independent of, Zermelo-Fraenkel set theory. Chapter V describes the relationship between the Axiom of Choice and logic, chiefly focussing attention on intuitionistic logic. Chapter VI begins with a discussion of the Axiom of Choice in a categorical setting, and continues with what amounts to a crash course in topos theory and the associated local set theory, leading up to an account of the role played by the Axiom of Choice therein. Chapter VII is devoted to a discussion of the role of the Axiom of Choice in Constructive Type Theory. The book concludes with two Appendices, the first outlining intuitionistic logic and the second charting the basic concepts of category theory.

*

I wish to thank my old friends and fellow-logicians Peter Aczel and John Mayberry for their helpful comments on the first draft of the book.

Contents

Introduction **1**

Chapter I. The Axiom of Choice: Its Origins and Status within Set Theory **7**

THE ORIGINS OF **AC** *7*

THE INDEPENDENCE AND CONSISTENCY OF AC WITHIN SET THEORY 11

CHRONOLOGY OF **AC** *17*

Chapter II. Maximal Principles and Zorn's Lemma **19**

THE NATURE AND ORIGINS OF MAXIMAL PRINCIPLES 19

CHRONOLOGY OF MAXIMAL PRINCIPLES 30

Chapter III. Mathematical Applications of the Axiom of Choice **32**

AC-*PROPOSITIONS 33*

ZL-*PROPOSITIONS 38*

SOME CONSTRUCTIVE EQUIVALENTS AND CONSEQUENCES OF **ZL** *66*

DOING WITHOUT **AC**: *"POINTLESS" TOPOLOGY 71*

Chapter IV. Consistency and Independence of the Axiom of Choice **81**

ZERMELO-FRAENKEL SET THEORY 81

THE RELATIVE CONSISTENCY OF **AC** *83*

THE INDEPENDENCE OF **AC** *86*

Chapter V. The Axiom of Choice and Intuitionistic Logic **101**

AC *AND LOGIC* 101

CHOICE PRINCIPLES IN INTUITIONISTIC SET THEORY 108

AC *AND HILBERT'S* ε*-CALCULUS* 115

AC *AND THE LAW OF EXCLUDED MIDDLE IN WEAK INTUITIONISTIC SET THEORIES* 120

SOME WEAK FORMS OF **AC** *AND THEIR LOGICAL CONSEQUENCES* 134

Chapter VI. The Axiom of Choice in Category Theory, Topos Theory and Local Set Theory **144**

AC *IN CATEGORICAL CLOTHING* 144

LOCAL SET THEORIES 151

SET THEORY IN A LOCAL LANGUAGE 156

BASIC PROPERTIES OF TOPOSES 158

TOPOSES AS MODELS OF LOCAL SET THEORIES 161

THE STRUCTURE OF Ω *AND* **SUB**(A) 165

EXAMPLES OF TOPOSES 169

THE CHOICE RULE AND OTHER PRINCIPLES IN LOCAL SET THEORIES 174

THE FOREGOING PRINCIPLES INTERPRETED IN TOPOSES 186

CHARACTERIZATION OF $\mathcal{S}et$ 193

Chapter VII. The Axiom of Choice in Constructive Type Theory **197**

CONSTRUCTIVE TYPE THEORY 197

AC *IN CONSTRUCTIVE TYPE THEORY 203*

ZERMELO'S 1904 AND 1908 FORMULATIONS OF **AC** *CONTRASTED TYPE-THEORETICALLY 208*

INTENSIONAL AND EXTENSIONAL **AC** *COMPARED 212*

A LAST LOOK AT **AC** *AND THE "PROPOSITIONS AS TYPES" DOCTRINE 215*

Appendix I. Intuitionistic Logic **218**

Appendix II. Basic Concepts of Category Theory **220**

Bibliography **228**

Index **243**