

A NOTE ON GENERIC ULTRAFILTERS

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Let $\mathfrak{M} = \langle M, \epsilon \rangle$ be a transitive ϵ -model of ZFC (ZERMELO-FRAENKEL set theory with the axiom of choice) and let $B \in M$ be a complete BOOLEAN algebra in the sense of \mathfrak{M} . Thus, if $X \in P^{(\mathfrak{M})}(B) = P(B) \cap M$, then $\bigvee X$ and $\bigwedge X$, the join and meet of X , exist and are in M . (Here $P(B)$, $P^{(\mathfrak{M})}(B)$ are, respectively, the power set of B and the power set of B computed in \mathfrak{M} .) An ultrafilter U in B (U need not be in M !) is said to respect the join $\bigvee X$, for $X \in P^{(\mathfrak{M})}(B)$, if

$$\bigvee X \in U \Rightarrow X \cap U \neq \emptyset$$

(the reverse implication holding trivially). If $\mathcal{S} \subseteq P^{(\mathfrak{M})}(B)$, then U is said to be \mathcal{S} -complete if U respects the join $\bigvee X$ for every $X \in \mathcal{S}$. A $P^{(\mathfrak{M})}(B)$ -complete ultrafilter is said to be \mathfrak{M} -generic. In this note we show that several apparently weak conditions on an ultrafilter in B are actually sufficient for \mathfrak{M} -genericity.

We construct the B -extension $M^{(B)}$ of \mathfrak{M} in the usual way, following JECH [1]. We write $\|\sigma\|$ for the B -value of any sentence σ of the language of set theory (which may contain names for elements of $M^{(B)}$). Well-known is the fact that $\|\sigma\| = 1^1$ whenever σ is a theorem of ZFC. We also recall that there is a natural map $x \mapsto \hat{x}$ of \mathfrak{M} into $M^{(B)}$.

A subset $\{b_i : i \in I\}$ of B is called an \mathfrak{M} -partition of B if (1) $b_i \neq 0$ for all $i \in I$, (2) $\langle b_i : i \in I \rangle \in M$, (3) $\bigvee_{i \in I} b_i = 1$ and (4) $b_i \wedge b_j = 0$ whenever $i, j \in I$ and $i \neq j$. We

have the following well-known facts (the first of which is a special case of the so-called „Mixing Lemma“, see [1], Lemma 49):

Lemma 1. Let $\{b_i : i \in I\}$ be an \mathfrak{M} -partition of B and let $\{x_i : i \in I\} \in M$. Then there is $x \in M^{(B)}$ such that $b_i = \|x = \hat{x}_i\|$ for all $i \in I$.

Lemma 2. Let U be an ultrafilter in B . Then U is generic iff whenever $\{b_i : i \in I\}$ is an \mathfrak{M} -partition of B , there is $i \in I$ such that $b_i \in U$.

Now let U be an ultrafilter in B which we assume fixed once and for all. There is a natural way of factoring $M^{(B)}$ by U ; for $x, y \in M^{(B)}$ we put $x \sim_U y \Leftrightarrow \|x = y\| \in U$. It is easily verified that \sim_U is an equivalence relation on $M^{(B)}$. We write x^U for the \sim_U -class of x and define the relation \in_U on the set $M^{(B)}/U$ of all \sim_U -classes by

$$x^U \in_U y^U \Leftrightarrow \|x \in y\| \in U.$$

We define the quotient of $M^{(B)}$ by U to the structure $\mathfrak{M}^{(B)}/U = \langle M^{(B)}/U, \in_U \rangle$. We have the following well-known version of LOS' theorem for $\mathfrak{M}^{(B)}/U$:

Lemma 3. For any formula $\varphi(v_0, \dots, v_n)$ and any $x_0, \dots, x_n \in M^{(B)}$,

$$\mathfrak{M}^{(B)}/U \models \varphi[x_0^U, \dots, x_n^U] \Leftrightarrow \|\varphi(x_0, \dots, x_n)\| \in U.$$

It follows immediately from this lemma that $\mathfrak{M}^{(B)}/U$ is a model of ZFC.

Now let $\mathbf{Ord}(x)$ be the formula expressing that x is an ordinal. An element $a \in M^{(B)}/U$ is called an ordinal in $\mathfrak{M}^{(B)}/U$ if $\mathfrak{M}^{(B)}/U \models \mathbf{Ord}[a]$. Let $\mathbf{OR}^{(\mathfrak{M})}$ be the set of ordinals of \mathfrak{M} ; then for each $\alpha \in \mathbf{OR}^{(\mathfrak{M})}$, $\hat{\alpha}^U$ is an ordinal in $\mathfrak{M}^{(B)}/U$ called a standard ordinal in $\mathfrak{M}^{(B)}/U$.

Let \mathcal{S}_1 be the subfamily of $P^{(\mathfrak{M})}(B)$ consisting of all subsets of B of the form $\{\|x = \hat{\alpha}\| : \alpha \in \mathbf{OR}^{(\mathfrak{M})}\}$ for $x \in M^{(B)}$. It is well-known that, for any $x \in M^{(B)}$, we have

$$(1) \quad \|\mathbf{Ord}(x)\| = \bigvee_{\alpha \in \mathbf{OR}^{(\mathfrak{M})}} \|x = \hat{\alpha}\|.$$

¹⁾ $1, 0$ denote the unit and zero element of B , respectively.

We are now ready to state and prove our first result.

Theorem 1. *The following conditions are equivalent:*

- (i) *Each ordinal in $\mathfrak{M}^{(B)}/U$ is standard;*
- (ii) *U is \mathcal{S}_1 -complete;*
- (iii) *U is \mathfrak{M} -generic.*

Proof. (i) \Rightarrow (ii). Assume (i); then if $\bigvee_{\alpha \in \mathbf{OR}^{(\mathfrak{M})}} \|x = \alpha\| \in U$, we have $\|\mathbf{Ord}(x)\| \in U$ by (1), so $\mathfrak{M}^{(B)}/U \models \mathbf{Ord}[x^U]$ by Lemma 3. Hence (i) gives $x^U = \alpha^U$ for some $\alpha \in \mathbf{OR}^{(\mathfrak{M})}$, whence $\|x = \alpha\| \in U$, and (ii) follows.

(ii) \Rightarrow (iii). Assume (ii), let $\alpha \in \mathbf{OR}^{(\mathfrak{M})}$ and let $\{b_\xi : \xi < \alpha\}$ be an \mathfrak{M} -partition of B with $\alpha \in \mathbf{OR}^{(\mathfrak{M})}$. By Lemma 1 we can find $x \in M^{(B)}$ such that $b_\xi = \|x = \xi\|$ for all $\xi < \alpha$. We have

$$\bigvee_{\xi \in \mathbf{OR}^{(\mathfrak{M})}} \|x = \xi\| \geq \bigvee_{\xi < \alpha} \|x = \xi\| = 1 \in U,$$

so by (ii) there is $\xi_0 \in \mathbf{OR}^{(\mathfrak{M})}$ such that $\|x = \xi_0\| \in U$. If $\xi \geq \alpha$, $\xi \in \mathbf{OR}^{(\mathfrak{M})}$, it is clear that $\|x = \xi\| = 0$, so we must have $\xi_0 < \alpha$. Hence $b_{\xi_0} = \|x = \xi_0\| \in U$ and (iii) now follows from Lemma 2.

(iii) \Rightarrow (i). Assume (iii); then we have, using (1) and Lemma 3,

$$\begin{aligned} \mathfrak{M}^{(B)}/U \models \mathbf{Ord}[x^U] &\Leftrightarrow \|\mathbf{Ord}(x)\| \in U \\ &\Leftrightarrow \bigvee_{\alpha \in \mathbf{OR}^{(\mathfrak{M})}} \|x = \alpha\| \in U \\ &\Leftrightarrow \|x = \alpha\| \in U \text{ for some } \alpha \in \mathbf{OR}^{(\mathfrak{M})} \\ &\Leftrightarrow x^U = \alpha^U \text{ for some } \alpha \in \mathbf{OR}^{(\mathfrak{M})}, \end{aligned}$$

and (i) follows. \square

Corollary. *If U is \mathcal{S}_1 -complete, then \in_U is a well-founded relation.*

Proof. If U is \mathcal{S}_1 -complete, then Theorem 1 shows that the map $\alpha \mapsto \alpha^U$ sends the well-ordered set $\mathbf{OR}^{(\mathfrak{M})}$ onto the set of ordinals of $\mathfrak{M}^{(B)}/U$. This map is clearly order-preserving, and it follows that the ordinals of $\mathfrak{M}^{(B)}/U$ are well-ordered under \in_U . The usual rank argument now yields the well-foundedness of \in_U . \square

It is worth observing that, assuming the existence of a measurable cardinal, the converse of this corollary fails. This can be seen as follows. Take $M = V$, the universe of all sets, let κ be a measurable cardinal and let D be a non-principal κ -complete ultrafilter in the complete Boolean algebra $P\kappa$. Then by standard results $V^{(P\kappa)}/D$ is isomorphic to the ultrapower V^κ/D and is therefore well-founded by SCOTT [2]. On the other hand D cannot be \mathcal{S}_1 -complete, because if it were it would be V -generic by Theorem 1, and hence principal, contrary to assumption. (This argument is due to DANA SCOTT.)

Suppose now that \in_U is a well-founded relation. Then $\mathfrak{M}^{(B)}/U$ can be collapsed uniquely to a transitive \in -structure $\mathfrak{M}[U] = \langle M[U], \in \rangle$ via the map h defined recursively on \in_U by

$$h(x^U) = \{h(y^U) : y^U \in x^U\} = \{h(y^U) : \|y \in x\| \in U\}.$$

Thus $h: \mathfrak{M}^{(B)}/U \rightarrow \mathfrak{M}[U]$ is a bijection satisfying $x^U \in_U y^U \Leftrightarrow h(x^U) \in h(y^U)$. Under these conditions we can define a surjection $i: M^{(B)} \rightarrow \mathfrak{M}[U]$ by putting $i(x) = h(x^U)$ for $x \in M^{(B)}$. Clearly we have, for $x \in M^{(B)}$,

$$(2) \quad i(x) = \{i(y) : \|y \in x\| \in U\}.$$

Our next lemma follows immediately from Lemma 3 and the fact that h is an isomorphism of $\mathfrak{M}^{(B)}/U$ onto $\mathfrak{M}[U]$.

Lemma 4. For any formula $\varphi(v_0, \dots, v_n)$ and any $x_0, \dots, x_n \in M^{(B)}$,

$$\mathfrak{M}[U] \models \varphi[i(x_0), \dots, i(x_n)] \Leftrightarrow \|\varphi(x_0, \dots, x_n)\| \in U.$$

Now define $j: \mathfrak{M} \rightarrow \mathfrak{M}[U]$ by putting $j(x) = i(\hat{x})$ for $x \in M$. Then j is an ϵ -monomorphism. Also, let \mathcal{S}_2 be the subfamily of $P^{(\mathfrak{M})}(B)$ consisting of all subsets of B of the form $\{\|x = \hat{z}\| : z \in y\}$ for $x \in M^{(B)}$, $y \in M$. We note the following well known fact

$$(3) \quad \|x \in \hat{y}\| = \bigvee_{z \in y} \|x = \hat{z}\|.$$

Theorem 2. The following conditions are equivalent:

- (i) U is \mathcal{S}_2 -complete;
- (ii) ϵ_U is well-founded and j is the identity on \mathfrak{M} ;
- (iii) ϵ_U is well-founded and $j[\mathfrak{M}]$ is transitive;
- (iv) U is \mathfrak{M} -generic.

Proof. (i) \Rightarrow (iv) Assume (i), and let $\{b_i : i \in I\} = y$ be an \mathfrak{M} -partition of B . Then by Lemma 1 we can find $x \in M^{(B)}$ such that $b_i = \|x = \hat{b}_i\|$ for all $i \in I$. Hence

$$\bigvee_{z \in y} \|x = \hat{z}\| = \bigvee_{i \in I} \|x = \hat{b}_i\| = \mathbf{1} \in U,$$

so by (i) there is $z \in y$ such that $\|x = \hat{z}\| \in U$. Since $z = b_i$ for some $i \in I$, it follows that $b_i = \|x = \hat{b}_i\| \in U$, and (iv) now follows from Lemma 2.

(iv) \Rightarrow (iii). Assume (iv). Then ϵ_U is well-founded by the Corollary to Theorem 1. Also, if $y \in M$ and $x \in j(y)$ then, since $\mathfrak{M}[U]$ is transitive, there is $x' \in M^{(B)}$ such that $x = i(x')$. Thus $i(x') \in j(y) = i(\hat{y})$, so that, by (2), $\|x' \in \hat{y}\| \in U$. It follows from (iv) and (3) that there is $z \in y$, hence $z \in M$ such that $\|x' = \hat{z}\| \in U$, whence $x = j(z) \in j[\mathfrak{M}]$. This proves (iii).

(iii) \Rightarrow (i). Assume (iii). If $x \in M^{(B)}$, $y \in M$, then

$$\begin{aligned} \bigvee_{z \in y} \|x = \hat{z}\| \in U &\Rightarrow \|x \in \hat{y}\| \in U && \text{(by (3))} \\ &\Rightarrow x^U \in_U \hat{y}^U \\ &\Rightarrow h(x^U) \in h(\hat{y}^U) \\ &\Rightarrow i(x) \in i(\hat{y}) = j(y) \\ &\Rightarrow i(x) = j(z) \text{ for some } z \in M && \text{(by (iii))} \\ &\Rightarrow z \in y. \end{aligned}$$

Hence $i(x) = j(z) = i(\hat{z})$, so, by Lemma 4, $\|x = \hat{z}\| \in U$. (i) follows.

Finally, the equivalence of (ii) and (iii) follows immediately from the transitivity of \mathfrak{M} and the fact that j is an ϵ -isomorphism of \mathfrak{M} onto $j[\mathfrak{M}]$. \square

Remark. The equivalence of (i) and (iv) was suggested to me by B. BALCAR.

Now define $U_* \in M^{(B)}$ by $\text{domain}(U_*) = \{\hat{x} : x \in B\}$ and $U_*(\hat{x}) = x$ for $x \in B$. U_* is called the *canonical generic ultrafilter* in $\mathfrak{M}^{(B)}$, since, as is well-known, we have $\|U_*$ is a $P^{(\mathfrak{M})}(B)^\wedge$ -complete ultrafilter in $\hat{B}\| = \mathbf{1}$. We observe that, for $x \in M^{(B)}$,

$$(4) \quad \|x \in U_*\| = \bigvee_{y \in B} y \wedge \|x = \hat{y}\|.$$

Let \mathcal{S}_3 be the subfamily of $P^{(\mathfrak{M})}(B)$ consisting of all subsets of B of the form $\{y \wedge \|x = \hat{y}\| : y \in B\}$ for $x \in M^{(B)}$.

Our final result in this note is the following

Theorem 3. *The following conditions are equivalent:*

- (i) U is \mathcal{L}_3 -complete;
- (ii) \in_U is well-founded, j is the identity on U , and $i(U_*) = U$;
- (iii) U is \mathfrak{M} -generic.

Proof. (i) \Rightarrow (iii). Assume (i). Let $\{b_i: i \in I\}$ be an \mathfrak{M} -partition of B ; then by Lemma 1 we can find $x \in M^{(B)}$ such that $b_i = \|x = \hat{b}_i\|$ for $i \in I$. We have

$$\bigvee_{y \in B} y \wedge \|x = \hat{y}\| \geq \bigvee_{i \in I} b_i \wedge \|x = \hat{b}_i\| = \bigvee_{i \in I} b_i = \mathbf{1}.$$

Hence, by (i), there is $y \in B$ such that $y \wedge \|x = \hat{y}\| \in U$. If $y \notin \{b_i: i \in I\}$, then

$$\|x = \hat{y}\| = \|x = \hat{y}\| \wedge \mathbf{1} = \|x = \hat{y}\| \wedge \bigvee_{i \in I} \|x = \hat{b}_i\| \leq \bigvee_{i \in I} \|x = \hat{b}_i\| = \mathbf{0},$$

so that $\|x = \hat{y}\| \notin U$. It follows that $y = b_i$ for some $i \in I$, whence $b_i \wedge \|x = \hat{b}_i\| \in U$, and so $b_i \in U$. (iii) now follows from Lemma 2.

(iii) \Rightarrow (ii). Assume (iii). Then \in_U is well-founded and j is the identity on U by Theorem 2. We also claim that for $x \in M^{(B)}$, $\{i(x): \|x \in U_*\| \in U\} = \{i(\hat{y}): y \in U\}$. In fact this follows from the chain of equivalences:

$$\begin{aligned} \|x \in U_*\| \in U &\Leftrightarrow \bigvee_{y \in B} y \wedge \|x = \hat{y}\| \in U && \text{(by (4))} \\ &\Leftrightarrow \exists y \in B [y \wedge \|x = \hat{y}\| \in U] && \text{(by (iii))} \\ &\Leftrightarrow \exists y \in U [\|x = \hat{y}\| \in U]. \\ &\Leftrightarrow \exists y \in U [i(x) = i(\hat{y})]. \end{aligned}$$

Now we have $i(U_*) = \{i(x): \|x \in U_*\| \in U\} = \{i(\hat{y}): y \in U\} = \{j(y): y \in U\} = U$ and so (ii) follows.

(ii) \Rightarrow (i). Assuming (ii), we have

$$(5) \quad U = i(U_*) = \{i(x): \|x \in U_*\| \in U\}.$$

Hence

$$\begin{aligned} \bigvee_{y \in B} y \wedge \|x = \hat{y}\| \in U &\Rightarrow \|x \in U_*\| \in U && \text{(by (4))} \\ &\Rightarrow i(x) \in U && \text{(by (5))} \\ &\Rightarrow i(x) = y = j(y) \text{ for some } y \in U \text{ (since } j \text{ is the identity on } U) \\ &\Rightarrow i(x) = i(\hat{y}) \\ &\Rightarrow \|x = \hat{y}\| \in U \\ &\Rightarrow y \wedge \|x = \hat{y}\| \in U, \end{aligned}$$

and (i) follows. \square

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References

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