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A geometric form of the axiom of choice

by

J. L. Bell (London) and **D. H. Fremlin** (Cambridge)

Consider the following well-known result from the theory of normed linear spaces ([2], p. 80, 4(b)):

(∗) the unit ball of the (continuous) dual of a normed linear space over the reals has an extreme point.

The standard proof of (∗) uses the axiom of choice (AC); thus the implication AC→(∗) can be proved in set theory. In this paper we show that this implication can be reversed, so that (∗) is actually *equivalent* to the axiom of choice. From this we derive various corollaries, for example: the conjunction of the Boolean prime ideal theorem and the Krein-Milman theorem implies the axiom of choice, and the Krein-Milman theorem is not derivable from the Boolean prime ideal theorem.

1. Preliminaries. Throughout this paper we shall assume that all linear spaces we consider have the real number field, $\boldsymbol{R}$, as their underlying field of scalars.

DEFINITION. Let L be a linear topological space. A subset A of L is said to be *quasicompact* if whenever $\mathcal{F}$ is a family of closed convex subsets of L such that $\{F \cap A\colon F \in \mathcal{F}\}$ has the finite intersection property, then $\bigcap \{F \cap A\colon F \in \mathcal{F}\} \neq \emptyset$. An element $a \in A$ is called an *extreme point* of A if $x, y \in A$ and $a = \frac{1}{2}(x+y)$ imply $a = x = y$.

Now consider the following propositions:

(BPI) Every Boolean algebra contains a prime ideal.

(HB) Let M be a linear subspace of a linear space L and let p be a sublinear functional on L (that is, $p(x+y) \leqslant p(x)+p(y)$ for all $x, y \in L$ and $p(\alpha x) = \alpha p(x)$ for all $0 \leqslant \alpha \in R$ and all $x \in L$). If f is a linear functional on M such that $f(x) \leqslant p(x)$ for all $x \in M$, then f can be extended to a linear functional g on L such that $g(x) \leqslant p(x)$ for all $x \in L$.

(KM) A compact convex subset of a locally convex Hausdorff linear topological space always has an extreme point.

(SKM) A quasicompact convex subset of a locally convex Hausdorff linear topological space always has an extreme point.

(AL) For any normed linear space L, the unit ball of the continuous dual L^* of L is quasicompact in the weak *-topology for L^*.

BPI is the well-known *Boolean prime ideal theorem*. HB *is the Hahn-Banach theorem*. KM and SKM are versions of the *Krein-Milman theorem*. Finally, AL is a weak version of *Alaoglu's theorem*, see [4].

For any propositions P and Q, we write $P \Rightarrow Q$ (resp. $P \not\Rightarrow Q$) for "the implication $P \to Q$ is provable (resp. is not provable) in Zermelo-Fraenkel set theory without the axiom of choice". We also write $P \Leftrightarrow Q$ for ($P \Rightarrow Q$ and $Q \Rightarrow P$).

THEOREM 1.1. [4] HB $\Leftrightarrow$ AL.

THEOREM 1.2. BPI & KM $\Rightarrow$ (*).

Proof. By [5], BPI is equivalent to the Tychonoff theorem for compact Hausdorff spaces. But this latter result implies in the usual way that the unit ball of the dual of a normed space is weak*-compact and KM implies that it has an extreme point. (*) follows.

THEOREM 1.3. AL & SKM $\Rightarrow$ (*).

Proof. By AL the unit ball of the dual of a normed space is quasi-compact; SKM then implies that it has an extreme point. Hence (*).

COROLLARY 1.4. HB & SKM $\Rightarrow$ (*).

Proof. By 1.1 and 1.3.

2. The main result and its consequences.

We now prove

THEOREM 2.1. (*) $\Rightarrow$ AC.

Proof. Let $\{A_i : i \in I\}$ be a family of non-empty sets; we may assume without loss of generality that the A_i are *disjoint*. Let $A = \bigcup_{i \in I} A_i$, and define

$$K = \Big\{x \in \boldsymbol{R}^A : \sup_{i \in I} \sum_{t \in A_i} |x(t)| \leqslant 1\Big\},$$

$$L = \Big\{x \in \boldsymbol{R}^A : \sup_{i \in I} \sum_{t \in A_i} |x(t)| < \infty\Big\},$$

$$E = \{x \in \boldsymbol{R}^A : \forall \varepsilon > 0 [\{t \in A : |x(t)| > \varepsilon\} \text{ is finite}] \text{ and } \sum_{i \in I} \sup_{t \in A_i} |x(t)| < \infty\}.$$

Then L and E are normed linear spaces with norms defined by:

$$\|x\|_L = \sup_{i\in I} \sum_{t\in A_i} |x(t)| \quad \text{for} \quad x \in L\,,$$

$$\|y\|_E = \sum_{i\in I} \sup_{t\in A_i} |y(t)| \quad \text{for} \quad y \in E\,.$$

Also K is the unit ball of L. But L is isometrically isomorphic to the dual E^* of E (see, e.g., [2], p. 31, 11(b)), and therefore K may be regarded as the unit ball of E^*. By (*), K has an extreme point e. We claim that for each $i \in I$ there is a *unique* $t \in A_i$ for which $e(t) \neq 0$.

For suppose first that there is $i_0 \in I$ such that $e(t) = 0$ for all $t \in A_{i_0}$. Choose $v \in A_{i_0}$ and define $y, z \in K$ by

$$y(v) = 1\,, \quad z(v) = -1\,,$$
$$y(t) = z(t) = e(t) \quad \text{for all } t \in A\backslash\{v\}\,.$$

Then clearly $e = \frac{1}{2}(y+z)$ and $y \neq e \neq z$, contradicting the extremeness of e.

Now suppose that there is $i_0 \in I$ and two distinct members u, v of A_{i_0} such that $e(u) \neq 0$ and $e(v) \neq 0$. Define $y, z \in R^A$ by

$$\begin{aligned} y(u) &= e(u)(1+|e(v)|)\,, \\ y(v) &= e(v)(1-|e(u)|)\,, \\ z(u) &= e(u)(1-|e(v)|)\,, \\ z(v) &= e(v)(1+|e(u)|)\,, \\ z(t) &= y(t) = e(t) \quad \text{for all } t \in A\backslash\{u, v\}\,. \end{aligned}$$

It is easy to see that $y, z \in K$, $y \neq e \neq z$ and $e = \frac{1}{2}(y+z)$, again contradicting the assumption that e is an extreme point of K.

Thus the claim is proved. We can now define a choice function g for the family $\{A_i: i \in I\}$ by letting $g(i)$ be the unique $t \in A_i$ for which $e(t) \neq 0$. The axiom of choice follows.

COROLLARY 2.2. (*) $\Leftrightarrow$ AC.

COROLLARY 2.3. BPI & KM $\Rightarrow$ AC.

Proof. By 2.2 and 1.2.

COROLLARY 2.4. *If* ZF *is consistent, then* BPI $\not\Rightarrow$ KM.

Proof. This follows from 2.3 and the fact [3] that, if ZF is consistent, then BPI $\not\Rightarrow$ AC.

COROLLARY 2.5. HB & SKM $\Rightarrow$ AC.

Proof. By 2.2 and 1.4.

Corollary 2.4 improves a result of [1], where it was shown that HB & SKM $\Rightarrow$ BPI.

We conclude the paper with some open problems. Is it true that

HB ⇒ BPI?, or

KM ⇒ BPI?, or

SKM ⇔ KM?

Postscript (January 12, 1972). After this paper was submitted, we received a preprint of a review of [1] by W. A. J. Luxemburg in which the results of the present paper are arrived at independently. Corollary 2.3 has also been proved independently by Peter Renz. We have also been informed by Professor Luxemburg that D. Pincus has recently answered the first two of our open problems in the negative.

References

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THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE
CHURCHILL COLLEGE, Cambridge

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