Infinitesimals and the Continuum

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The belief that a continuum can be "composed of" or "synthesized from" points has been frequently challenged, as witness the following quotations:

Aristotle: ". . . no continuum can be made up out of indivisibles, as for instance a line out of points, granting that the line is continuous and the point indivisible." ([1], Book 6, Chap. 1)

Leibniz: "A point may not be a constituent part of a line." ([11], p. 109)

Kant: "Space and time are *quanta continua* . . . points and instants mere positions . . . and out of mere positions viewed as constituents capable of being given prior to space and time neither space nor time can be constructed." ([6], p. 204)

Weyl: "Exact time- or space-points are not the ultimate, underlying, atomic elements of the duration or extension given to us in experience." ([12], p. 94)

Brouwer: "The linear continuum is not exhaustible by the interposition of new units and can therefore never be thought of as a mere collection of units." ([4], p. 80)

René Thom: ". . . a true continuum has no points." ([5], p. 102)

These views are in striking contrast with the generally accepted set-theoretical account of mathematics according to which all mathematical entities are *discrete*: on this account there is, in particular, no "true" continuum.

Closely associated with the concept of continuum is that of *infinitesimal*, which is, roughly speaking, what remains after a continuum has been subjected to a mathematically or metaphysically exhaustive analysis. An infinitesimal may be regarded as a continuum "viewed in the small." On the set-theoretical or discrete account, infinitesimals are just points (or singletons); however, if continua are truly continuous and do not have points as parts, then an infinitesimal, as a part of the continuum from which it is extracted, *cannot be a point*. Let us call such infinitesimals *nonpunctual* or *continuous*.

(Nonpunctual) infinitesimals have a long and fascinating history. They first show up in the mathematics of the Greek mathematician-philosopher Democritus (himself an atomist!), only to be banished by Eudoxus (c. 350 B.C.) from what was to become official "Euclidean" mathematics. Taking the somewhat obscure form of "indivisibles," they reappear in the mathematics of the late middle ages and were systematically exploited during the 16th and 17th centuries by Kepler, Cavalieri, and others in determining areas and volumes of curvilinear figures. As "linelets" and "timelets" they play an essential role in Barrow's "method for finding tangents by calculation," which appears in his *Lectiones Geometricae* of 1670. As "evanescent quantities" they were instrumental in Newton's development of the calculus, and, as "inassignable quantities," in Leibniz's. De l'Hospital, the author of one of the first textbooks on the (infinitesimal!) calculus in 1696, invoked the concept in laying down the principle that "a curved line may be regarded as an infinite assemblage of infinitesimally small straight lines." Memorably derided by Berkeley as "ghosts of departed quantities" and condemned by Bertrand Russell as "unnecessary,

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erroneous, and self-contradictory," they were believed to be finally suppressed through the set-theorization of mathematics achieved in this century.

In fact, the suppression of infinitesimals within "respectable" mathematics did not eliminate them altogether but, instead, drove them underground. Physicists and engineers, for example, never abandoned their heuristic use for deriving (correct) results in the application of the calculus to physical problems. And even differential geometers as reputable as Lie and Cartan did not disdain to use them in formulating concepts which would later be put on a "rigorous" footing.

One of the greatest champions of the concept of (continuous) infinitesimal was Charles Sanders Peirce. He saw the concept of the continuous (as did Brouwer and Weyl and as does René Thom) as arising from the subjective grasp of the flow of time, and the subjective "now" as a continuous infinitesimal. Here are some quotations.

> It is difficult to explain the fact of memory and our apparently perceiving the flow of time, unless we suppose immediate consciousness to extend beyond a single instant. Yet if we make such a supposition we fall into grave difficulties, unless we suppose the time of which we are immediately conscious to be strictly infinitesimal. ([9], p. 124)

> We are conscious only of the present time, which is an instant if there be any such thing as an instant. But in the present we are conscious of the flow of time. There is no flow in an instant. Hence the present is not an instant. ([9], p. 127)

> ...The fact that the continuity of space and time is a natural belief is perhaps evidence that it is true. Better evidence is that it explains the personal identity of consciousness in time, which is almost if not quite incomprehensible otherwise ([9], p. 62)

> This continuum does not consist of indivisibles, or points, or instants, and does not contain any except insofar as its continuity is ruptured. ([9], p. 925)

> It is singular that nobody objects to $\sqrt{-1}$ as involving any contradiction, nor, since Cantor, are infinitely great quantities objected to, but still the antique prejudice against infinitely small quantities remains. ([9], p. 123)

Recently, thanks to developments in category theory and mathematical logic, it has become possible to construct a consistent framework within which both "true" continua and continuous infinitesimals can be accommodated. This framework is the so-called *synthetic differential geometry* (SDG).[1] It is a theory of the *smoothly continuous* world: in it *all* functions or correlations between mathematical objects are smooth, thus realizing Leibniz's doctrine of continuity, *natura non facit saltus*. It is interesting to note that the idea of a framework of this kind was anticipated by Hermann Weyl in 1940:

> A natural way to take into account the nature of a continuum which, following Anaxagoras, defies "chopping off its parts from one another, as it were, with a hatchet", would be by limiting oneself to continuous functions. ([13], p. 294)

The pervasive nature of continuity within SDG forces a *change of logic*: from *classical* to *intuitionistic*—in which the law of excluded middle fails and so in which there must be *more than two "truth values"* (nonbivalence). How does this come about?

If excluded middle held in SDG, then each real number would be either $= 0$ or $\neq 0$, and so the correlation $0 \mapsto 0$, $x \mapsto 1$ for $x \neq 0$ would define a map from the real line to 2, which is clearly discontinuous. To see that logic cannot be bivalent in SDG, let Ω be its set of "truth values." Then as in ordinary set theory, for any object X, correlations $X \rightarrow \Omega$ are in bijective correspondence with parts of X. If X is a connected continuum (e.g., the real line), it presumably does have proper nonempty parts but certainly no continuous nonconstant maps to the two element set {true, false}. It follows that $\Omega \neq$ {true, false} in SDG.

In (many models of) SDG, any classical space X (e.g., $\mathbf{R}$, $\mathbf{R}^n$) has a counterpart X^* which is *indecomposable* whenever X is *connected*. (A space X is *indecomposable* if no proper nonempty part U is *detachable* in the sense that there is a part V such that $U \cup V = X$, $U \cap V = \emptyset$.) Thus the connected continua of SDG are *true* continua in something like the Anaxagoran sense.

Even more remarkably, perhaps, SDG embodies a fruitful concept of continuous infinitesimal—that of an *infinitesimal tangent vector*. A tangent vector to a curve C at a point p on it is a short (nondegenerate) straight line segment λ around p pointing along C. In SDG we may take λ to be a *part* of C: in SDG, therefore, curves are "composed" of infinitesimally small straight lines in something like de l'Hospital's sense. Since a curve is a continuous map f with domain a part of the real line, it turns out that we can take λ to be the image under f of the intersection D of a *circle* with a *tangent*: in particular, the intersection of the circle $x^2 + (y-1)^2 = 1$ with its tangent, the line $x = 0$. This choice makes D that part of the real line consisting of the points x for which $x^2 = 0$: the *square-zero infinitesimals*. Notice that, in SDG, D must be nondegenerate, that is, *not identical with* $\{0\}$!

In SDG, D is subject to the *principle of infinitesimal linearity*. This may be paraphrased by saying that D remains straight and unbroken under any map or that it is too small to bend or break (but larger than zero!); that is, D can be subjected solely to translations and rotations; it is, in other words, *a pure synthesis of location and direction*.[2]

These facts enable differential geometry [8] and the calculus [2] to be developed within SDG in a direct in-

[1] See [7] or [8]. The idea of basing such an approach to differential geometry on category theory is due originally to F. W. Lawvere.

[2] D may also be regarded as a geometric representation of the *specious present* (as opposed to the pointlike instant).

tuitive manner. For example, a *tangent vector* at a point x of a space ("manifold") X is an infinitesimal straight path on X passing through x, that is, a map $\rho : D \to X$ with $\rho(0) = x$. So the *tangent bundle* of X is just X^D, the space of all maps[3] $D \to X$, and a *vector field* on X any map $X \to X^D$ whose composite with the base point map $\rho \mapsto \rho(0) : X^D \to X$ is the identity on X. This is just the beginning of the remarkable conceptual simplification of differential geometry made possible by SDG.

Since, as we have observed, the law of excluded middle fails in models of SDG, it follows that an assertion is, in general, no longer implied by its double negative. This fact is exploited [10] in a remarkable way in certain models of SDG, where infinitesimal real numbers can be equivalently identified either as *nilpotent* elements or as elements *not unequal to* or *indistiguishable from* 0. In this situation, we can say that two points are *infinitesimally close* if they are not unequal, or indistinguishable; the assertion that all maps are continuous then reduces to the purely logical fact that any map automatically preserves the relation of indistinguishability. Here we have a remarkable example of a reduction of *topology* to *logic*.

The infinitesimals of SDG are to be contrasted with those of Abraham Robinson's *nonstandard analysis* (NSA) (see, e.g., [3], Chap. 11). In Robinson's approach, the field **R** of (standard) real numbers is "enlarged" to a field $\mathbf{R}^*$ containing "infinitely large" elements in such a way as to preserve the usual algebraic properties of the real number field. In particular, every nonzero element of $\mathbf{R}^*$ is (multiplicatively) invertible; the infinitesimals of NSA are obtained as the inverses of the infinitely large elements of $\mathbf{R}^*$. By contrast, the nilpotent infinitesimals of SDG do not (of course) possess multiplicative inverses and so cannot be obtained in this way.[4] The infinitesimals and infinities of NSA may be regarded as "ideal" elements playing much the same role with respect to the standard real number system as the "ideal" points and lines "at infinity" of classical projective geometry play with respect to the standard Euclidean plane. In neither case does the use of these ideal elements add to the (classically) provable facts about the standard elements. NSA is, accordingly, *conservative*, in that $\mathbf{R}^*$ possesses no mathematical features not already possessed by **R**. This is to be contrasted with SDG whose version of the real number system differs essentially from its standard counterpart, not being a field.

The enlarged system $\mathbf{R}^*$ of NSA may be regarded as just the standard system **R** viewed through new mathematical "spectacles" whose resolving power is sufficient to reveal the presence of ideal elements among the standard ones. ($\mathbf{R}^*$ is, in particular, still a discrete structure composed of "distinguishable" elements.) In view of the conservative nature of the enlargement, these ideal elements are infinitesimal or infinite not in an *absolute* sense, but only *in relation* to the standard elements; that is, speaking metaphorically, an "observer" situated within a model of NSA would be unable to detect the presence of infinitesimals or infinities in $\mathbf{R}^*$: this is because, within any such model, $\mathbf{R}^*$ satisfies the usual axioms for the real number field which of course *excludes* infinitesimals (Archimedean property). By contrast, the nilpotency of the infinitesimals of SDG is an *absolute* property which is perfectly "detectable" within a model of SDG.

I conclude with a final quotation from Peirce which reveals that, even before Brouwer, he was aware that a faithful account of the truly continuous will involve jettisoning the law of excluded middle:

> Now if we are to accept the common idea of continuity . . . we must either say that a continuous line contains no points or we must say that the principle of excluded middle does not hold of these points. The principle of excluded middle applies only to an individual . . . but places being mere possibilities without actual existence are not individuals. ([9], p. xvi: the quotation is from a note written in 1903.)

A remarkable insight, indeed!

[3] This implies that, in SDG, the tangent space to any manifold M at any point on it may be identified with a part of M. In other words, in SDG, just as every curve is "infinitesimally linear," so every manifold is "infinitesimally flat."

[4] It should be noted, however, that models of SDG incorporating "Robinsonian" infinitesimals *in addition* to nonpunctual ones have been constructed: see [8].

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