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ISOMORPHISM OF STRUCTURES IN S-TOPOSES

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It is a well-known fact that two structures are $\infty\omega$-equivalent if and only if they are isomorphic in some Boolean extension of the universe of sets (cf. [4]; an early allusion to this result appears in [8]). My principal object here is to show that arbitrary toposes defined over the category of sets may be used instead. Thus $\infty\omega$-equivalence means isomorphism in the extremely general context of some universe of "variable" sets in which not only is much of the usual set-theoretic machinery unavailable but the underlying logic is not even classical. This provides further support for the view that $\infty\omega$-equivalence is a relation between structures of fundamental importance.

§1. We recall that a *topos* is a category E satisfying the following conditions (for notation, see [6] or [7]):

- E has a terminal object 1.
- E has finite products.
- There is an object Ω in E, the *subobject classifier* or *truth-value object*, and an arrow $1 \xrightarrow{\text{true}} \Omega$ such that for each monic $A \overset{m}{\rightarrowtail} B$ there is a unique $B \xrightarrow{\chi_m} \Omega$, the *characteristic arrow* of m (or A) such that

$$\begin{array}{ccc} A & \longrightarrow & \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \text{true}} \\ B & \xrightarrow{\chi_m} & \Omega \end{array}$$

is a pullback, and conversely every arrow to Ω arises in this way. Where there is no possibility of confusion, we write χ_A for χ_m.

- For each object A there is an object Ω^A, the *exponential* of A, and an arrow $A \times \Omega^A \xrightarrow{\text{ev}} \Omega$ such that, for each arrow $A \times B \xrightarrow{f} \Omega$ there is a unique arrow $B \xrightarrow{g} \Omega^A$ such that

$$\begin{array}{ccc} A \times B & & \\ {\scriptstyle \text{id}_A \times g}\downarrow & \searrow^{f} & \\ A \times \Omega^A & \xrightarrow{\text{ev}} & \Omega \end{array}$$

commutes. (Here id_A is the identity arrow on A.)

It is well known that any topos has an initial object 0, arbitrary finite coproducts,

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and pullbacks. We shall always assume that a topos has small hom-sets, and that it is nondegenerate, i.e. $0 \not\cong 1$.

Let S be the category of sets. A topos E is said to be *defined over* S or to be simply an *S-topos* if it has arbitrary set-indexed copowers of its objects. It is easy to see that for this to be the case it is enough to be able to form arbitrary set-indexed copowers of 1 in E. Moreover, it is well known (see [6, Chapter 4]) that in any S-topos one can form arbitrary coproducts of subobjects of any given fixed object, in particular, of subobjects of 1. If $\{X_i : i \in I\}$ is any family of objects, we write $\coprod_{i\in I} X_i$ or $\coprod X_i$ for the coproduct, assuming it exists.

We shall continually use the standard fact that, in a topos, if for each $i \in I$ the diagram

$$\begin{array}{ccc} X_i & \longrightarrow & Y_i \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

is a pullback, then so is the diagram

$$\begin{array}{ccc} \coprod X_i & \longrightarrow & \coprod Y_i \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

with the natural arrows.

Given an S-topos E, there is a natural functor $\hat{\ }: S \to E$ (left adjoint to the hom-functor $E(1, \cdot): E \to S$) defined as follows: for each set A we put $\hat{A} = \coprod_A 1$ and for $f: A \to B$ we define $\hat{f}: \hat{A} \to \hat{B}$ to be the unique arrow in E making the diagrams

$$\begin{array}{ccc} 1 & \xrightarrow{\hat{a}} & \hat{A} \\ & \searrow^{f(a)\hat{}} & \downarrow \hat{f} \\ & & \hat{B} \end{array}$$

commute for all $a \in A$, where $\hat{a}$, $\hat{b}$ are the canonical injections of 1 into $\hat{A}$, $\hat{B}$ respectively. It is easy to verify that $\hat{}$ is faithful and preserves products (in fact, all finite limits).

If $B \subseteq A$ and $a \in A$, it is not hard to see that[1]

$$a \in B \Leftrightarrow \chi_{\hat{B}}\hat{a} = \text{true};$$

$$a \notin B \Leftrightarrow \chi_{\hat{B}}\hat{a} = \text{false},$$

where $1 \xrightarrow{\text{false}} \Omega$ is the characteristic arrow of $0 \rightarrowtail 1$.

Let $\mathrm{Sub}(E)$ be the set of all subobjects of 1 in an S-topos E, where we agree to

[1]Composition of arrows is denoted, as usual, in the "set-theoretic" manner, i.e. by juxtaposition in the order opposite to that in which they appear in a diagram.

identify isomorphic subobjects. The relation $\leq$ on $\mathrm{Sub}(E)$ defined by $U \leq V$ iff there is a (necessarily unique) monic $U \rightarrowtail V$ is then a partial ordering on $\mathrm{Sub}(E)$ and, as is well known, $\langle \mathrm{Sub}(E), \leq \rangle$ is a complete Heyting algebra in which the meet $U \wedge V$ of two elements U, V coincides with their product $U \times V$ and, for any set $\{U_i : i \in I\}$, the join $\bigvee_{i \in I} U_i$ is the unique member of $\mathrm{Sub}(E)$ such that

$$\begin{array}{ccc} \coprod_{i \in I} U_i & \longrightarrow & 1 \\ & {\scriptstyle k}\searrow & \nearrow{\scriptstyle m} \\ & \bigvee_{i \in I} U_i & \end{array}$$

commutes with k epic and m monic. (We recall that in a topos any arrow has a "unique" epic-monic factorization.)

For each object $A \in E$ we define a (partial) *section* of A to be an arrow of the form $\mathrm{dom}(u) \overset{u}{\rightarrowtail} A$, where $\mathrm{dom}(u)$ is a subobject of 1. Let $\bar{A}$ be the set of sections of A, and let $u, v \in \bar{A}$. We define $[\![u = v]\!] \in \mathrm{Sub}(E)$ to be the subobject of 1 such that

$$\begin{array}{ccc} [\![u = v]\!] & \rightarrowtail & \mathrm{dom}(u) \\ \downarrow & & \downarrow \\ \mathrm{dom}(v) & \rightarrowtail & A \end{array}$$

is a pullback. Notice that we have, for $a, a' \in A$, $[\![\hat{a} = \hat{a}]\!] = 1$ and $[\![\hat{a} = \hat{a}']\!] = 0$ whenever $a \neq a'$.

Let H be a complete Heyting algebra. Following [5], we define an *H-set* to be a pair $\langle X, \delta \rangle$ consisting of a set X and a function $\delta \colon X \times X \to H$ satisfying

$$\delta(x, y) = \delta(y, x), \qquad \delta(x, y) \wedge \delta(y, z) \leq \delta(x, z).$$

(δ may be thought of as an "H-valued equality relation" on X.) The *category S_H of H-sets* is defined as follows. Its objects are all H-sets; an arrow between two objects $\langle X, \delta \rangle$ and $\langle Y, \varepsilon \rangle$ is a function $f \colon X \times Y \to H$ such that, for all $x, x' \in X$, $y, y' \in Y$,

$$\delta(x, x') \wedge f(x, y) \leq f(x', y),$$
$$f(x, y) \wedge \varepsilon(y, y') \leq f(x, y'),$$
$$f(x, y) \wedge f(x, y') \leq \varepsilon(y, y'),$$
$$\bigvee_y f(x, y) = \delta(x, x).$$

(Such a function may be thought of as an H-valued relation which is functional and defined on X.) Given two arrows $\langle X, \delta \rangle \xrightarrow{f} \langle Y, \varepsilon \rangle$ and $\langle Y, \varepsilon \rangle \xrightarrow{g} \langle Z, \eta \rangle$, the composition $\langle X, \delta \rangle \xrightarrow{gf} \langle Z, \eta \rangle$ is defined by $(gf)(x, z) = \bigvee_{y \in Y} f(x, y) \wedge g(y, z)$; the identity arrow on $\langle X, \delta \rangle$ is just δ. It is well known that S_H is an S-topos in which $\mathrm{Sub}(S_H) \cong H$ and, for any set A, $\hat{A}$ is the H-set $\langle A, \delta \rangle$ where δ is defined by $\delta(a, a) = 1$, $\delta(a, a') = 0$ for any $a, a' \in A$ such that $a \neq a'$.

We observe that, for any S-topos E, and any object $A \in E$, the pair $\langle \hat{A}, [\![\cdot = \cdot]\!] \rangle$ is a Sub(E)-set.

If B is a complete Boolean algebra, the *Boolean extension* $V^{(B)}$ of the universe of sets may be regarded as a category in the following way. First, we identify elements of $V^{(B)}$ which are equal with probability 1. The objects of the category $V^{(B)}$ are then the (thus identified) elements of $V^{(B)}$ and the arrows are those elements of $V^{(B)}$ which are functions with probability 1. It is easy to see that the resulting category is a topos, and indeed an S-topos in which the natural functor $S \xrightarrow{\wedge} V^{(B)}$ defined above is just the usual injection of V into $V^{(B)}$ (see [1]). It is known [5] that $V^{(B)}$ and S_B are equivalent as categories.

Let E be a topos. A (binary) *E-structure*[2] is a pair $\mathfrak{A} = \langle A, R \rangle$ consisting of an object A of E and a subobject R of $A \times A$. If $\mathfrak{A} = \langle A, R \rangle$ and $\mathfrak{B} = \langle B, S \rangle$ are two E-structures, an arrow $f: A \to B$ is said to be *compatible* with R and S if

$$\chi_R = \chi_S(f \times f).$$

An *E-isomorphism* between A and B is an isomorphism f (in the usual categorical sense in E) which is compatible with R and S. Under these conditions we write $f: \mathfrak{A} \cong \mathfrak{B}$. If such an isomorphism exists in E we write $\mathfrak{A} \cong_E \mathfrak{B}$ and say that $\mathfrak{A}$ and $\mathfrak{B}$ are *E-isomorphic*.

If $\mathfrak{A} = \langle A, R \rangle$ is a (binary) structure in the usual sense, and E is an S-topos, we denote by $\hat{\mathfrak{A}}$ the E-structure[3] $\langle \hat{A}, \hat{R} \rangle$. If $\mathfrak{B}$ is a structure in the usual sense, we say that $\mathfrak{A}$ and $\mathfrak{B}$ are *E-equivalent*, written $\mathfrak{A} \equiv_E \mathfrak{B}$, if $\hat{\mathfrak{A}} \cong_E \hat{\mathfrak{B}}$. We shall call $\mathfrak{A}$ and $\mathfrak{B}$ *topos-equivalent*, and write $\mathfrak{A} \equiv_T \mathfrak{B}$, if $\mathfrak{A} \equiv_E \mathfrak{B}$ for some S-topos E.

Given two (usual) structures $\mathfrak{A} = \langle A, R \rangle$ and $\mathfrak{B} = \langle B, S \rangle$, a *partial isomorphism* between $\mathfrak{A}$ and $\mathfrak{B}$ is a nonempty family P of functions such that:

- for each $f \in P$, we have $\mathrm{dom}(f) \subseteq A$, $\mathrm{ran}(f) \subseteq B$, and f is an isomorphism of $\mathfrak{A} \upharpoonright \mathrm{dom}(f)$ to $\mathfrak{B} \upharpoonright \mathrm{ran}(f)$;
- if $f \in P$, $a \in A$, $b \in B$, then there exist g, $h \in P$ both extending f such that $a \in \mathrm{dom}(g)$, $b \in \mathrm{ran}(h)$ ("back and forth" property).

If these conditions are satisfied we write $P: \mathfrak{A} \cong_p \mathfrak{B}$; if there is such a P we write $\mathfrak{A} \cong_p \mathfrak{B}$ and say that $\mathfrak{A}$ and $\mathfrak{B}$ are *partially isomorphic*.

It is well known (see [3]) that *two structures are* $\infty\omega$*-equivalent if and only if they are partially isomophic*; it is this latter relation we shall employ in the sequel.

§2. We first want to formulate necessary and sufficient conditions on an S-topos E for two given structures to be E-equivalent. Since we are ultimately going to show that E-equivalent structures are partially isomorphic, our strategy will be to start with two partially isomorphic structures, use the partial isomorphism to construct an S-topos in which they turn out to be equivalent, and then 'read off' the appropriate conditions on the topos from the corresponding properties of the partial isomorphism.

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[2]There is really no necessity to confine attention to binary structures; we do so solely for the sake of notational simplicity. By complicating the symbolism the results given here go through for structures with any number of finitary relations.

[3]In order for $\langle \hat{A}, \hat{R} \rangle$ to be an E-structure in the sense of the above definition we must *identify* the naturally isomorphic objects $(A \times A)^\wedge$ and $\hat{A} \times \hat{A}$. We shall continue to do this in the sequel, usually without comment.

So let $\mathfrak{A}$ and $\mathfrak{B}$ be two partially isomorphic structures and let $P: \mathfrak{A} \cong_p \mathfrak{B}$. We partially order the set P by inclusion and assign it the associated order topology in which the basic open sets are of the form $\{f \in P: p \subseteq f\}$ for $p \in P$. Now let C be the complete Boolean algebra of regular open subsets of P; for each $a \in A$, $b \in B$ let $V_{ab} = \{f \in P: f(a) = b\}$ and

$$U_{ab} = \{q \in P: \forall p \supseteq q \exists f \in V_{ab}[p \subseteq f]\}.$$

Then U_{ab} is the interior of the closure of V_{ab} in P, and so $U_{ab} \in C$. The fact that P is a partial isomorphism now readily yields the following facts about the U_{ab}: in C,

(a) $\bigvee_{b \in B} U_{ab} = 1$ for each $a \in A$;
(b) $\bigvee_{a \in A} U_{ab} = 1$ for each $b \in B$;
(c) $U_{ab} \wedge U_{a'b} = 0$ whenever $a \neq a'$;
(d) $U_{ab} \wedge U_{ab'} = 0$ whenever $b \neq b'$;
(e) if *either* $\langle a, a' \rangle \in R$ and $\langle b, b' \rangle \notin S$ *or* $\langle a, a' \rangle \notin R$ and $\langle b, b' \rangle \in S$, then $U_{ab} \wedge U_{a'b'} = 0$.

Properties (a) and (b) derive from the "back and forth" condition on P, (c) and (d) from the bijectiveness of the members of P, and (e) from the compatibility of each member of P with R and S.

Now consider the topos S_C of C-valued sets. Conditions (a) and (d) mean that the U_{ab} define an arrow $f: \hat{A} \to \hat{B}$ in S_C, and the remaining conditions ensure that this arrow is an isomorphism in S_C compatible with R and S (this will follow from our general result below but can in fact be easily verified in the special case at hand). Thus, in this case, the existence of a subfamily of $C \cong \mathrm{Sub}(S_C)$ satisfying conditions (a)–(e) above is *sufficient* for $\mathfrak{A}$ and $\mathfrak{B}$ to be S_C-equivalent. We are actually going to establish the much stronger result that, for *any* S-topos E, the existence of such a subset of $\mathrm{Sub}(E)$ is not only sufficient but also *necessary* for $\mathfrak{A}$ and $\mathfrak{B}$ to be E-equivalent. To do this, however, we require a

LEMMA. *Let E be an S-topos and let $f: A \to B$ be an arrow in E. Then*
(i) *f is monic iff $[\![f\hat{a} = f\hat{a}']\!] = 0$ whenever $a \neq a'$ in A;*
(ii) *f is epic iff $\bigvee_{a \in A} [\![f\hat{a} = \hat{b}]\!] = 1$ for all $b \in B$.*

PROOF. Suppose f is monic. For $a, a' \in A$, consider the pullback

$$\begin{array}{ccc} [\![f\hat{a} = f\hat{a}']\!] & \longrightarrow & 1 \\ \downarrow & & \downarrow f\hat{a} \\ 1 & \xrightarrow{f\hat{a}'} & \hat{B} \end{array}$$

Taking coproducts over a', we get the pullback

(1)
$$\begin{array}{ccc} \coprod_{a' \in A} [\![fa = fa']\!] & \longrightarrow & 1 \\ \downarrow & & \downarrow f\hat{a} \\ \hat{A} & \xrightarrow{f} & \hat{B} \end{array}$$

Since f is monic, so is the top line of (1). But clearly $[\![f\hat{a} = f\hat{a}]\!] = 1$, so $\coprod_{a' \neq a} [\![f\hat{a} = f\hat{a}']\!] = 0$ and hence $[\![f\hat{a} = f\hat{a}']\!] = 0$ when $a \neq a'$.

Conversely. suppose that $[\![f\hat{a} = f\hat{a}']\!] = 0$ when $a \neq a'$. Then the pullback (1) becomes

$$\begin{array}{ccc} 1 = [\![f\hat{a} = f\hat{a}]\!] & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow {\scriptstyle f\hat{a}} \\ \hat{A} & \xrightarrow{\ f\ } & \hat{B} \end{array}$$

Taking coproducts over $a \in A$ yields the pullback

$$\begin{array}{ccc} \hat{A} & \xrightarrow{\ \mathrm{id}\ } & \hat{A} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow {\scriptstyle f} \\ \hat{A} & \xrightarrow{\ f\ } & \hat{B} \end{array}$$

from which it immediately follows that f is monic. This proves (i).

(ii) Suppose f is epic. For fixed $b \in B$ we get by taking coproducts a pullback

$$\begin{array}{ccc} \coprod_{a \in A} [\![f\hat{a} = \hat{b}]\!] & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow {\scriptstyle \hat{b}} \\ \hat{A} & \xrightarrow{\ f\ } & \hat{B} \end{array} \tag{2}$$

Since f is epic, so is the top line of (2), giving $\bigvee_{a \in A} [\![f\hat{a} = \hat{b}]\!] = 1$.

Conversely, suppose this last condition is satisfied. Then the top line in (2) is epic and so we get, by taking coproducts over $b \in B$, a pullback:

$$\begin{array}{ccc} \coprod_{b \in B} \coprod_{a \in A} [\![f\hat{a} = \hat{b}]\!] & \longrightarrow & \hat{B} \\ \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{id}} \\ \hat{A} & \xrightarrow{\ f\ } & \hat{B} \end{array}$$

But the top line of this diagram is clearly epic, and so therefore is f. This proves (ii). ■

We are now in a position to prove the promised

THEOREM. *Let E be an S-topos, and let $\mathfrak{A} = \langle A, R \rangle$, $\mathfrak{B} = \langle B, S \rangle$ be structures. Then the following are equivalent.*

(i) $\mathfrak{A} \equiv_E \mathfrak{B}$.

(ii) *There is a family* $\{U_{ab} : \langle a, b \rangle \in A \times B\} \subseteq \mathrm{Sub}(E)$ *such that conditions* (a)–(e) *above hold.*

PROOF. (i) ⇒ (ii) Assume (i), and let $f : \hat{\mathfrak{A}} \cong_E \hat{\mathfrak{B}}$. Then $f : \hat{A} \cong \hat{B}$ and

$$\chi_{\hat{R}} = \chi_{\hat{S}}(f \times f). \tag{1}$$

For each $\langle a, b \rangle \in A \times B$ put $U_{ab} = [\![f\hat{a} = \hat{b}]\!]$; then $U_{ab} \in \mathrm{Sub}(E)$.

(a) By taking coproducts over $b \in B$, we obtain for each $a \in A$ the pullback

$$\begin{array}{ccc} \coprod_{b \in B} U_{ab} & \longrightarrow & \hat{B} \\ \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{id}} \\ 1 & \xrightarrow{\ f\hat{a}\ } & \hat{B} \end{array}$$

But $\hat{B} \xrightarrow{\text{id}} \hat{B}$ is epic, and so therefore is $\coprod_{b \in B} U_{ab} \to 1$, giving (a).

(b) follows immediately from (ii) of the lemma.

(c) If $a \neq a' \in A$, then

$$U_{ab} \wedge U_{a'b} \leq [\![f\hat{a} = f\hat{a}']\!] = 0$$

by (i) of the lemma.

(d) If $b \neq b' \in B$ then

$$U_{ab} \wedge U_{ab'} \leq [\![\hat{b} = \hat{b}']\!] = 0.$$

(e) Before proving this we again remark that for any set X we have agreed to identify $(X \times X)\hat{}$ and $\hat{X} \times \hat{X}$. Thus, for any $\langle x, x' \rangle \in X \times X$ we shall also agree to identify $1 \xrightarrow{\langle \hat{x}, \hat{x}' \rangle} \hat{X} \times \hat{X}$ with $1 \xrightarrow{\langle x, x' \rangle\hat{}} (X \times X)\hat{}$.

Now suppose $\langle a, a' \rangle \in R$ and $\langle b, b' \rangle \notin S$. It is easy to see that, from the definition of U_{ab}, we have the commutative diagram

(2)
$$\begin{array}{ccc} U_{ab} \times U_{a'b'} & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow{\scriptstyle \langle \hat{b}, \hat{b}' \rangle} \\ 1 & \xrightarrow{\langle f\hat{a}, f\hat{a}' \rangle} & \hat{B} \times \hat{B} \end{array}$$

Since $\langle b, b' \rangle \notin S$, we have $\chi_{\hat{S}} \langle \hat{b}, \hat{b}' \rangle = \text{false}$, and so, putting $V = U_{ab} \times U_{a'b'}$, we get, using (1) and (2)

$$\begin{aligned} & V \longrightarrow 1 \xrightarrow{\text{false}} \Omega \\ &= V \longrightarrow 1 \xrightarrow{\langle \hat{b}, \hat{b}' \rangle} \hat{B} \times \hat{B} \xrightarrow{\chi_{\hat{S}}} \Omega \\ &= V \longrightarrow 1 \xrightarrow{\langle f\hat{a}, f\hat{a}' \rangle} \hat{B} \times \hat{B} \xrightarrow{\chi_{\hat{S}}} \Omega \\ &= V \longrightarrow 1 \xrightarrow{\langle \hat{a}, \hat{a}' \rangle} \hat{A} \times \hat{A} \xrightarrow{f \times f} \hat{B} \times \hat{B} \xrightarrow{\chi_{\hat{S}}} \Omega \\ &= V \longrightarrow 1 \xrightarrow{\langle \hat{a}, \hat{a}' \rangle} \hat{A} \times \hat{A} \xrightarrow{\chi_{\hat{R}}} \Omega \\ &= V \longrightarrow 1 \xrightarrow{\text{true}} \Omega, \end{aligned}$$

since $\langle a, a' \rangle \in R$. But then clearly $V = 0$, whence $U_{ab} \wedge U_{a'b'} = 0$ as required. A similar argument shows that if $\langle a, a' \rangle \notin R$ and $\langle b, b' \rangle \in S$, then $U_{ab} \wedge U_{a'b'} = 0$. This proves (e).

(ii) $\Rightarrow$ (i) Let $\{U_{ab} : \langle a, b \rangle \in A \times B\}$ be a subset of $\text{Sub}(E)$ satisfying (a)–(e). We first claim that, for any $a \in A$,

(3)
$$\coprod_{b \in B} U_{ab} \cong 1.$$

Since by (a) we have $\coprod_{b \in B} U_{ab} \to 1$ epic, it suffices to show that this arrow is monic. We consider the pullback

$$\begin{array}{ccc} U_{ab} \times U_{ab'} & \longrightarrow & U_{ab'} \\ \big\downarrow & & \big\downarrow \\ U_{ab} & \longrightarrow & 1 \end{array}$$

Then

$$\begin{array}{ccc} \coprod_{b\in B} U_{ab}\times U_{ab'} & \longrightarrow & U_{ab'} \\ \big\downarrow & & \big\downarrow \\ \coprod_{b\in B} U_{ab} & \longrightarrow & 1 \end{array}$$

is a pullback. But by (d), $U_{ab}\times U_{ab'} = 0$ if $b \neq b'$, so the above pullback diagram becomes

$$\begin{array}{ccc} U_{ab'} \cong U_{ab'}\times U_{ab'} & \longrightarrow & U_{ab'} \\ \big\downarrow & & \big\downarrow \\ \coprod_{b\in B} U_{ab} & \longrightarrow & 1 \end{array}$$

Hence, taking coproducts over $b' \in B$, we get the pullback

$$\begin{array}{ccc} \coprod_{b'\in B} U_{ab'} & \xrightarrow{\text{id}} & \coprod_{b'\in B} U_{ab'} \\ \big\downarrow{\scriptstyle \text{id}} & & \big\downarrow \\ \coprod_{b\in B} U_{ab} & \longrightarrow & 1 \end{array}$$

It follows easily from this that the bottom line of this diagram is monic, and so we get (3).

Given $a \in A$, we let $\coprod_{b\in B} U_{ab} \xrightarrow{j_a} \hat{B}$ be the unique arrow making the diagrams

$$\text{(4)} \qquad \begin{array}{ccc} U_{ab} & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow{\scriptstyle \hat{b}} \\ \coprod_{b\in B} U_{ab} & \xrightarrow{j_a} & \hat{B} \end{array}$$

commute for each $b \in B$. Using (3), we let i_a be the composition of the arrows $1 \cong \coprod_{b\in B} U_{ab} \xrightarrow{j_a} \hat{B}$. We now define $\hat{A} \xrightarrow{f} \hat{B}$ to be the unique arrow making the diagrams

$$\begin{array}{ccc} 1 & \xrightarrow{\hat{a}} & \hat{A} \\ & {\scriptstyle i_a}\searrow & \big\downarrow{\scriptstyle f} \\ & & \hat{B} \end{array}$$

commute for all $a \in A$.

We claim that $f\colon \hat{\mathfrak{A}} \cong_E \hat{\mathfrak{B}}$. To establish this we first observe that

$$\text{(5)} \qquad U_{ab} = [\![f\hat{a} = \hat{b}]\!]$$

for $a \in A$, $b \in B$. For it follows immediately from (4) and the definition of i_a that the diagrams

$$\begin{array}{ccc} U_{ab} & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow{\scriptstyle \hat{b}} \\ 1 & \xrightarrow{f\hat{a}} & \hat{B} \end{array}$$

commute for all $a \in A$, $b \in B$. Hence $U_{ab} \leq [\![f\hat{a} = \hat{b}]\!] = V_{ab}$, say. Clearly we have $V_{ab} \wedge V_{ab'} = 0$ whenever $b \neq b'$, so $V_{ab} \wedge U_{ab'} = 0$ when $b \neq b'$. Now, by (a), we have

$$V_{ab} = V_{ab} \wedge 1 = V_{ab} \wedge \bigvee_{b' \in B} U_{ab'}$$

$$= \bigvee_{b' \in B} V_{ab} \wedge U_{ab'} = V_{ab} \wedge U_{ab}.$$

So $V_{ab} \leq U_{ab}$ and (5) follows.

Now we can show:

f is monic. This follows from (5), (c) and (i) of the lemma.

f is epic. This follows from (5), (b) and (ii) of the lemma.

($*$) $\chi_{\hat{R}} = \chi_{\hat{S}}(f \times f)$. Using (a), it is easy to show that

$$\bigvee_{b \in B} \bigvee_{b' \in B} U_{ab} \wedge U_{a'b'} = 1,$$

so that the arrow

$$\coprod_{b \in B} \coprod_{b' \in B} U_{ab} \times U_{a'b'} \longrightarrow 1$$

is epic. In order to prove ($*$) it suffices to show that, for any $a, a' \in A$,

(6) $$\chi_{\hat{R}}\langle \hat{a}, \hat{a}' \rangle = \chi_{\hat{S}}(f \times f)\langle \hat{a}, \hat{a}' \rangle = \chi_{\hat{S}}\langle f\hat{a}, f\hat{a}' \rangle.$$

To establish (6) we treat separately the cases $\langle a, a' \rangle \in R$ and $\langle a, a' \rangle \notin R$.

Suppose then that $\langle a, a' \rangle \in R$. Using (5) we get a commutative diagram

$$\begin{array}{ccccc} U_{ab} \times U_{a'b'} & \longrightarrow & 1 & & \\ \big\downarrow & & \big\downarrow{\scriptstyle \langle \hat{b}, \hat{b}' \rangle} & & \\ 1 & \xrightarrow{\langle f\hat{a}, f\hat{a}' \rangle} & \hat{B} \times \hat{B} & \xrightarrow{\chi_{\hat{S}}} & \Omega \end{array}$$

If $\langle b, b' \rangle \in S$ then $\chi_{\hat{S}}\langle \hat{b}, \hat{b}' \rangle = \text{true}$, while if $\langle b, b' \rangle \notin S$ then $U_{ab} \times U_{a'b'} = 0$ by (e). Hence for any $b, b' \in B$ we have

$$U_{ab} \times U_{a'b'} \longrightarrow 1 \xrightarrow{\text{true}} \Omega$$

$$= U_{ab} \times U_{a'b'} \longrightarrow 1 \xrightarrow{\langle f\hat{a}, f\hat{a}' \rangle} \hat{B} \times \hat{B} \xrightarrow{\chi_{\hat{S}}} \Omega$$

and hence, putting $X = \coprod_{b \in B} \coprod_{b' \in B} U_{ab} \times U_{a'b'}$,

$$X \twoheadrightarrow 1 \xrightarrow{\text{true}} \Omega$$

$$= X \longrightarrow 1 \xrightarrow{\chi_{\hat{S}}\langle f\hat{a}, f\hat{a}' \rangle} \Omega.$$

Thus, since $X \twoheadrightarrow 1$ is epic,

$$\chi_{\mathfrak{S}}\langle f\hat{a}, f\hat{a}'\rangle = \text{true}.$$

But since $\langle a, a'\rangle \in R$ we have

$$\chi_R\langle \hat{a}, \hat{a}'\rangle = \text{true}$$

and so (6) follows in the case where $\langle a, a'\rangle \in R$.

An argument similar to the one just given establishes (6) in the case where $\langle a, a'\rangle \notin R$. Thus (6) holds for all $a, a' \in A$. This gives (*), and completes the proof. ■

Now we can prove the main result of the paper.

COROLLARY 1. *$\mathfrak{A} \cong_p \mathfrak{B}$ if and only if $\mathfrak{A} \equiv_T \mathfrak{B}$, and hence $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ if and only if $\mathfrak{A} \equiv_T \mathfrak{B}$.*

PROOF. If $\mathfrak{A} \cong_p \mathfrak{B}$, let P be a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ and let C be the complete boolean algebra of regular open subsets of P. At the beginning of this section we remarked that we can find a subfamily $\{U_{ab}: \langle a, b\rangle \in A \times B\}$ of C satisfying conditions (a)–(e) above, so it follows from the theorem that $\mathfrak{A}$ and $\mathfrak{B}$ are S_C-equivalent, whence $\mathfrak{A} \equiv_T \mathfrak{B}$.

Conversely, suppose that $\mathfrak{A} \equiv_T \mathfrak{B}$. Then there is an S-topos E such that $\mathfrak{A} \equiv_E \mathfrak{B}$, as guaranteed by the theorem, let $\{U_{ab}: \langle a, b\rangle \in A \times B\}$ be a subset of $\mathrm{Sub}(E)$ satisfying conditions (a)–(e). For each $0 \neq U \in \mathrm{Sub}(E)$ put

$$\tilde{U} = \{\langle a, b\rangle \in A \times B: U \leq U_{ab}\}.$$

We claim that

$$P = \{\tilde{U}: 0 \neq U \in \mathrm{Sub}(E)\}$$

is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

(1) *Each $\tilde{U} \in P$ is a one-one function.* For if $\langle a, b\rangle \in \tilde{U}$ and $\langle a, b'\rangle \in \tilde{U}$, then $U \leq U_{ab} \wedge U_{ab'} = 0$ if $b \neq b'$ by (d); so since $U \neq 0$ we must have $b = b'$. Thus $\tilde{U}$ is a function. In a similar way, now using (c), we can show that $\tilde{U}$ is one-one.

(2) *Each $\tilde{U} \in P$ is an isomorphism of its domain onto its range.* Suppose $\langle a, b\rangle \in \tilde{U}$, $\langle a', b'\rangle \in \tilde{U}$ and $\langle a, a'\rangle \in R$. Then, if $\langle b, b'\rangle \notin S$ it follows from (e) that $U \leq U_{ab} \wedge U_{a'b'} = 0$, so since $U \neq 0$ we must have $\langle b, b'\rangle \in S$. Similarly, if $\langle b, b'\rangle \in S$, we obtain $\langle a, a'\rangle \in R$.

(3) Suppose $\tilde{U} \in P$ and $a \in A$. By (a), we have $1 = \bigvee_{b \in B} U_{ab}$; hence

$$U = U \wedge \bigvee_{b \in B} U_{ab} = \bigvee_{b \in B} U \wedge U_{ab}.$$

Since $U \neq 0$, for some $b \in B$ we must have $V = U \wedge U_{ab} \neq 0$. Then $\tilde{V} \in P$, $\tilde{U} \subseteq \tilde{V}$ and $\langle a, b\rangle \in \tilde{V}$, whence $a \in \mathrm{dom}(\tilde{V})$.

Similarly, now using (b), for each $b \in B$ we get $\tilde{V} \in P$ such that $\tilde{U} \subseteq \tilde{V}$ and $b \in \mathrm{ran}(\tilde{V})$.

Thus P is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, and the proof is complete. ■

REMARK. The definition of $\tilde{U}$ in the preceding proof was suggested by considering the special case in which E is a Boolean extension $V^{(C)}$ containing an element f which is an isomorphism of $\hat{\mathfrak{A}}$ and $\hat{\mathfrak{B}}$ with probability 1. In this case we take $U_{ab} = [\![\langle \hat{a}, \hat{b}\rangle \in f]\!]$, so for each $U \in \mathrm{Sub}(V^{(C)}) \cong C$, $\tilde{U}$ is the set of pairs $\langle a, b\rangle \in A \times B$ (whose canonical images are) "forced" by U to be a member of f.

Finally, we observe that, for certain familiar kinds of topos E, E-equivalence is the *same* as isomorphism. Let us call a topos E *normal* if Sub(E) is (isomorphic to) a *topology* on a set. This condition is satisfied, for example, when E is a presheaf category $S^{D^{op}}$ where D is any small category, or when E is the category of sheaves over any topological space. We then have:

COROLLARY 2. *Let E be a normal S-topos. Then* $\mathfrak{A} \equiv_E \mathfrak{B}$ *if and only if* $\mathfrak{A} \cong \mathfrak{B}$.

PROOF. One way round is trivial. Conversely, suppose that $\mathfrak{A} \equiv_E \mathfrak{B}$ and let Sub(E) be a topology $\mathscr{T}$ on a set I. Let $\{U_{ab}: \langle a, b\rangle \in A \times B\}$ be a subfamily of $\mathscr{T}$ satisfying conditions (a)–(e) as guaranteed by the theorem. Since E is nondegenerate, I is nonempty and so we can choose a member $i_0 \in I$. Define a function $f: A \to B$ by putting $f(a) = b$ iff $i_0 \in U_{ab}$. It is now easy to verify, using conditions (a)–(e), that this defines an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. ■

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