AN INTRODUCTION TO LOGIC

BY

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Dedicated to the memory of Robert K. Clifton (1964 - 2002), my cherished former colleague and "onlie compiler and embellisher" of the lecture notes on which this book is based.

Logic is the beginning of wisdom, not the end.

Mr. Spock

CONTENTS

I. PROPOSITIONAL LOGIC

1. Statements, Inferences and Counterexamples

The *Oxford English Dictionary* defines *logic* as ``the branch of philosophy that deals with forms of reasoning and thinking in general, and especially with inference.... Also, the systematic use of symbolic techniques and mathematical methods to determine the forms of valid deductive argument.`` An *inference* is defined as ``the drawing of a conclusion from data or premises``` while an argument is defined to be ``a connected series of statements or reasons intended to establish a position.`` In the introduction to logic presented here, we shall simply take an *inference* or *argument* to be a list of *statements* starting with a number of statements called *premises* and ending with a statement called the *conclusion.* Here is an example of an inference in this sense[1]:

1. *Either this man's dead, or my watch has stopped*.
2. *This man is not dead.*
3. *Therefore, my watch has stopped.*

Statements 1 and 2 are premises, and 3 the conclusion.

We shall assume that the constituent statements of inferences are *assertive,* or *declarative* in the sense that each can be assigned exactly one of two *truth values* – *true* (t) or *false* (f) – as the case may be[2]. This is the *principle of bivalence* underlying *classical logic.* Granted this, it is natural to declare an inference *valid* if its conclusion is true in any case in which its premises are true. Thus, on the basis of our usual grasp of the meaning of the terms "or" and "not", the inference above would count as valid. A *counterexample* to an inference is a case in which all its premises are *true* but its conclusion is *false*. Thus an inference is *valid* provided it *has no counterexamples*, and *invalid* if it *has counterexamples*. As an example of an invalid inference, consider the following:

[1] With apologies to the late Groucho Marx.

[2] An example of a statement that would *not* fall into this category is "Would you please go to the store".

Either this man's dead, or my watch has stopped.

This man is dead.

Therefore, my watch has stopped.

This inference is *invalid*, because the situation in which this man is dead and my watch hasn't stopped constitutes a counterexample.

To investigate the validity of inferences we need first to consider how their constituent statements are formed, and how these are then to be assigned truth values. As the basic ingredients from which we shall fashion all such statements we shall take simple declarative sentences of the kind "It is snowing", "The geese are flying", etc. Such statements will be called *elementary* statements: we shall assume that elementary statements can be assigned truth values arbitrarily and entirely independently of one another.

From elementary statements we obtain *compound statements* by applying the *logical operations* "and", "or", "not", "if...then". In this way we obtain, e.g. statements such as "It is snowing and the geese are flying", "If it is snowing, the geese are not flying", etc. These operations are *truth functional* in the sense that the truth value of any compound statement built up from them is unambiguously determined by the truth values of its constitutive elementary statements.

We shall use capital letters *A, B, C,...* to denote elementary statements, and symbols

$\wedge$ for "and" (conjunction),

$\vee$ for "or" (disjunction),

$\neg$ for "not" (negation or denial),

$\rightarrow$ for "if...then" (implication).

The symbols *A, B, C,...* are called *statement letters*, and the symbols $\wedge$, $\vee$,$\neg$, $\rightarrow$, *logical operators* The operator "$\vee$" has the inclusive sense of "and\or"; it may also be understood as "unless".

Using these symbols, *statements* are obtained by starting with the statement letters. These count as the simplest kind of statement — what we have termed *elementary* statements. *Compound* statements are then produced from elementary statements by applying the logical operators $\wedge$, $\vee$, $\neg$, $\rightarrow$, using parentheses and brackets as necessary to eliminate ambiguity. So, for example, from the statement letters *A, B, C*, we obtain the compound statements $A \wedge B$, $(A \wedge B) \vee C$, $(A \rightarrow B) \vee C$, $\neg[(A \rightarrow B) \vee C]$, etc. We shall use letters p, q, r,... to denote arbitrary statements (elementary or compound).

Formally, a *statement*, or *propositional statement*, may now be defined by means of the following *rules of formation:*

1. Any statement letter is a statement.
2. If p and q are statements, so are $(p \wedge q)$, $(p \vee q)$, $(p \rightarrow q)$, $(\neg p)$.

Here p, q are called the *conjuncts* in the *conjunction* $(p \wedge q)$ and the *disjuncts* in the *disjunction* $(p \vee q)$. (Thus, for example, in the disjunction $(((\neg A) \vee B) \vee C)$ the disjuncts are $((\neg A) \vee B)$ and C.) Also p is called the *antecedent* and q the *consequent* in the *implication* or *conditional* $(p \rightarrow q)$.

Strictly speaking, a sequence of statement letters and logical operators which cannot be generated by repeated application of rules 1 and 2 above will not count as a statement. For example, $\neg A \rightarrow \vee B$ is obviously not allowed ("If not A, then or B" is gibberish), while $((\neg A) \vee B)$ is allowed. However, for the sake of brevity (and to preserve our sanity!) we shall bend the rules slightly and feel free to omit parentheses in statements when no ambiguity is likely to result. For example, instead of $((\neg A) \vee B)$ we shall write simply $\neg A \vee B$. For clearly the outside brackets are not needed if this statement is meant to be a complete statement and not a component of some larger statement (such as one disjunct of a larger disjunction). Also, by replacing $(\neg A)$ by $\neg A$ we are agreeing to understand the "not" operator as acting only upon the statement letter A and not upon the larger statement $A \vee B$. If, instead, we wanted to symbolize the statement "Neither A nor B" then we would have to write $\neg(A \vee B)$ so that the *scope* of the $\neg$ operator covers the entire disjunction. As a general rule of thumb, insert parentheses only when it would not otherwise be clear which statement is being negated, or which statements are being disjoined, conjoined, etc. (For example, $A \wedge B \vee C$ is *not* clear, while $A \wedge (B \vee C)$ is; $A \rightarrow \neg B \vee C$ is *not* clear, $(A \rightarrow \neg B) \vee C$ is; etc.)

Inferences involving statements will be written

$$\begin{array}{c} p \\ q \\ r \\ . \\ . \\ \hline s \end{array}$$

or

$$\begin{array}{c} p \\ q \\ r \\ . \\ . \\ \therefore s \end{array}$$

or $$p;\ q;\ r;\ \ldots\ \ \therefore s$$

Here p, q, r are the *premises,* and s the *conclusion,* of the inference.

2. Truth Tables and Testing Validity

Here are the rules for computing the truth values of compound statements:

- $A \wedge B$ is true if A and B are both true, and false if at least one of A and B is false.
- $A \vee B$ is true if at least one of A and B is true, and false if both A and B are false.
- $\neg A$ is true if A is false, and false if A is true.
- $A \rightarrow B$ is false when A is true and B is false, but true in all other cases.

The rule for negation enables the principle of bivalence to be restated as: *for any statement p, either p is true or ¬p is true.*

The least intuitive of these rules is the last one. The idea here is that we want a statement of the form $p \rightarrow q$ to be false exactly when the truth values of p and q constitute a *counterexample* to the validity of the inference from p to q, that is, when p is true and q is false. In all other cases, $p \rightarrow q$ shall be declared true. In particular, if p is false, then $p \rightarrow q$ is true *for any statement q.* This is known as the rule of *ex falso quadlibet*: anything follows from a falsehood.

For those to whom this seems odd, it may be made more palatable by considering the following down-to-earth example. Suppose that, upon leaving for work in the morning, I promise my wife "If I go to the store (p), then I will buy some milk (q)". When I arrive back from work in the evening, she asks me whether I picked up any milk, and I say No. Did I break my promise? That is: should $p \rightarrow q$ be declared false in this case; a case where, in fact, both p and q turned out to be false? Surely not: I would only have broken the promise if in fact I did go to the store but did *not* buy any milk (due to an oversight, or lack of money, or what have you).

Nevertheless, it must be admitted that our definition of the truth-conditions for the so-called 'material' conditional $\rightarrow$ fails to do justice to *all* our intuitions about how the "if...then..." construction in natural language functions. For example, we are being forced to declare that "If New York is a big apple, then grass is green" is true simply on the basis of its consequent being true (which it is). Even more oddly, we are also forced to declare the statement ``If 2+2 =5, then I am the Pope `` true simply because its premise is false. Bertrand Russell presented a tongue-in-cheek argument for the truth of this statement, which went as follows. Suppose that 2+2 = 5. Then 4= 5, so, subtracting 2 from each side, 2 = 3, and, subtracting one more , 1 = 2. But then, since the Pope and I are two, we are also one, and so I am the Pope!

A more sophisticated treatment of conditionals would involve discussing 'strict' conditionals, 'counterfactual' conditionals, etc. which are beyond the scope of our discussion here. .

We read "$p \rightarrow q$" variously as

- p *implies* q
- *if* p, *then* q,
- p *only if* q
- q *if* p

The above rules for computing truth values may be summed up in the form of *truth tables*.

A	B	$A \wedge B$	$A \vee B$	$\neg A$	$A \rightarrow B$
t	t	t	t	f	t
t	f	f	t	f	f
f	t	f	t	t	t
f	f	f	f	t	t

Each line under the first two (A,B) columns represents an assignment of truth values — a *(truth) valuation*. Here there are $2^2 = 4$ valuations. If we had *n* statement letters A_1 ,...,A_n there would be 2^n valuations.

A truth valuation assigns a unique truth value to each statement. For example, consider the statement $(\neg A \wedge (B \rightarrow C)) \vee (A \rightarrow B)$. Then its truth value under the valuation A t/ B f/C t is computed as follows:

A	B	C	$\neg A$	$B \rightarrow C$	$A \rightarrow B$	$\neg A \wedge (B \rightarrow C)$	$(\neg A \wedge (B \rightarrow C)) \vee (A \rightarrow B)$
t	f	t	f	t	f	f	f

So far we have laid down rules for forming compound statements from elementary statement letters. We also have rules for determining the truth values of any compound statement given the logical operators that occur in it and any truth valuation of its statement letters.

We can now give a formal definition of validity of an inference in propositional logic:

- **an inference is *valid* if, under any truth valuation of the statement letters occurring in it, whenever the premises of the inference are all true, so is its conclusion.**

We can also give a formal definition of a counterexample to an inference:

- **a *counterexample* to an inference is a truth valuation of its statement letters under which the premises are all true and the conclusion false.**

Clearly, then, it follows formally that *an inference is valid if it has no counterexamples and invalid if it has at least one counterexample.*

To illustrate, we test a few inferences for validity.

It is snowing and the geese are flying.
Therefore, it is snowing.

This has the form

$$\frac{A \wedge B}{A}$$

The inference is valid since, according to the truth table for $\wedge$, under any valuation of A and B, if the premise $A \wedge B$ is true, so is the conclusion A.

It is snowing or the geese are flying.
It isn't snowing.
Therefore, the geese are flying.

This has the form

$$\begin{array}{c} A \vee B \\ \neg A \\ \hline B \end{array}$$

Examining the truth table for possible counterexamples we find

		premises		conclusion
A	B	A	$A \vee B$	B
t	f	f	t	f
f	f	t	f	f

Notice that we only needed to examine the (two) cases in which the conclusion (B) is *false*, since counterexamples cannot arise in any other way. Since neither of *these* cases constitutes a counterexample, there are none, and the inference is, accordingly, *valid*. This is an example of the use of a *truncated* truth table - a truth table in which irrelevant lines are omitted - to test validity.

If it is snowing, the geese are flying.
It is snowing.
Therefore, the geese are flying.

This has the form

$$\begin{array}{c} A \rightarrow B \\ A \\ \hline B \end{array}$$

Examine the truth table for possible counterexamples (conclusion false): this gives rise to the truncated truth table

A	B	A	$A \rightarrow B$	B
t	f	t	f	f
f	f	f	t	f

Neither of these cases constitutes a counterexample, so the inference is *valid*.

If it is snowing, the geese are flying.

The geese are flying.

Therefore, it is snowing.

This has the form

$$\frac{\begin{array}{c}A \to B \\ B\end{array}}{A}$$

The following line in the truth table is a counterexample (in fact the only one)

A	B	A → B	B	A
f	t	t	t	f

The inference is, accordingly, *invalid*.

If it is snowing, the geese are flying.

If the geese are flying, the bears are restless.

Therefore, if it snowing, the bears are restless.

This has the form

$$\frac{\begin{array}{c}A \to B \\ B \to C\end{array}}{A \to C}$$

Now the only possible counterexamples arise when the conclusion $A \to C$ is false. This can happen just when A is true and C is false. Therefore we need merely examine the two lines in the truth table in which this occurs. These are the following:

A	B	C	$A \to B$	$B \to C$	$A \to C$
t	t	f	t	f	f
t	f	f	f	t	f

Since in neither of these lines are both premises $A \to B$ and $B \to C$ true, neither constitutes a counterexample, so there are none, and the inference is, accordingly, *valid*.

An inference is said to be *sound* if it is valid and all its premises are true. Clearly, the conclusion of a sound inference must also be true. But a valid inference need not necessarily be sound. Here is an example of a valid, but *unsound* inference:

Ants are mammals

Ants are mammals or groundhogs are mammals

We shall use the notation

$$p_1,\ldots,p_n \models q$$

to indicate that the inference from the statements $p_1,\ldots,p_n$ to the statement q is valid. Thus the validity of the first three inferences above may be symbolized:

$$A \wedge B \models A; \quad A \vee B, \neg A \models B; \quad A, A \to B \models B.$$

We read "$p_1,\ldots,p_n \models q$" as "$p_1,\ldots,p_n$ (jointly) *imply* q" or "q *is a logical consequence of* $p_1,\ldots,p_n$ `` or simply "q *follows from* $p_1,\ldots,p_n$" .[3]

We shall often write $p \Rightarrow q$ for $p \models q$. If $p \Rightarrow q$ holds, we shall say that p *entails* q.

Exercises.

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[3] Note that $p \models q$ is *not itself* a statement in our logical language, like $p \to q$, but rather a kind of 'meta-statement' or statement *about* statements—i.e. the statement that the argument from p as premise to q as conclusion is a valid argument. However, there is an obvious connection between the expressions $p \models q$ and $p \to q$, namely, the former holds exactly when the latter's truth-table has no f's.

A1. Use truth tables to determine whether the statements in the left column (jointly) imply the corresponding statement in the right column.

(a)	$A, B\rightarrow\neg C$	$\neg C$
(b)	$\neg(A\rightarrow B)$	$\neg B\rightarrow\neg A$
(c)	$A, [(A\vee B)\rightarrow B]$	$A\wedge B$
(d)	$A\rightarrow (\neg\neg B\rightarrow C)$	$(A\wedge\neg C)\rightarrow\neg B$
(e)	$A\rightarrow(B\vee C),\ A\rightarrow\neg B$	$A\rightarrow C$
(f)	$A\rightarrow\neg B,\ [(A\wedge C)\rightarrow B]$	$\neg[(\neg A\wedge C)\rightarrow A]$
(g)	$A\rightarrow(B\rightarrow C)$	$(A\wedge B)\rightarrow C$
(h)	$A\rightarrow(B\vee C)$	$(A\rightarrow B)\vee(A\rightarrow C)$
(i)	$\neg(A\wedge B)$	$\neg A\vee\neg B$
(j)	$\neg(A\vee B)$	$\neg A\wedge\neg B$
(k)	$(A\rightarrow B)\rightarrow B$	$A\vee B$
(l)	$A\rightarrow(A\rightarrow B)$	B
(m)	$(A\wedge B)\rightarrow C, \neg C, A$	$\neg B$
(n)	$A\rightarrow B, A\rightarrow\neg B$	$\neg A$
(o)	$\neg A\rightarrow A$	A

A2. Symbolize each of the following inferences and use truth tables to determine which of them are valid.

(a) Silas is either a knave or a fool. Silas is a knave; so, he's no fool.

(b) You may enter only if the Major's out. The Major is out. So you may enter.

(c) There will be a fire! For only if there's oxygen present will there be a fire. And of course there's oxygen present.

(d) If I'm right, then you're wrong. But if you're wrong, then I can't be right. Therefore you're wrong.

(e) If I'm right, then you're wrong. But if you're wrong, then I can't be right. Therefore, I can't be right.

(f) If they retreat provided we attack, then we attack. But they won't retreat. Therefore we attack.

(g) It's a duck if it walks and quacks like one. Therefore, either it's a duck if it walks like one or it's a duck if it quacks like one.

(h) You cannot serve both God and Mammon. But if you don't serve Mammon, you'll starve; if you starve, you can't serve God. Therefore, you can't serve God.

(i) If today's Friday, we must be in Toronto. Today is Friday, but we're not in Toronto. So we're in London.

(j) Computers can think only if they have emotions. If computers can have emotions then they can have desires as well. But computers can't think if they have desires. Therefore computers can't think.

A3. Symbolize this argument and use a truth table (truncated, if you like) to determine whether it is valid:

> If I'm right, then you're a fool. If I'm a fool, I'm not right. If you're a fool, I am right. So one or other of us is a fool!

A4. Symbolize the following argument and determine whether it is valid:

> Modern physics asserts that there is no such thing as absolute motion. If this is correct, then there is no such thing as absolute time, and our

> ordinary notions of time are wrong. So either our ordinary ideas about time or modern physics is mistaken.

A5. Translate the following two arguments into logical notation (defining your symbols). Then use a truth table (truncated, if you like) to determine whether the arguments are valid. For the invalid arguments (if any), supply all counterexamples.

(a) You will eat and either I will eat or we shall starve. Therefore, you and I will eat or we shall starve.*(assume that: starve = not eat)*

(b) We'll win! For if they withdraw if we advance, we'll win. And we won't advance!

A6. Using truth tables, determine whether the following arguments are valid.

(a) $A \to (H \wedge J)$

$J \to H$

$\neg J$

$\therefore \neg A$

(b) $(D \to \neg G) \wedge G$

$(G \vee [(A \to D) \wedge A]) \to \neg D$

$\therefore G \to \neg D$

(c) $(G \to H) \vee (\neg G \to H)$

$\therefore (\neg H \to \neg G) \vee (\neg H \to G)$

A7. Consider the following argument:

> This argument is unsound, for its conclusion is false, and no sound argument has a false conclusion.

Is this argument sound? ('Sound" means "Valid + True Premises".)

A8. Knaves always lie, knights always tell the truth, and in Camelot, where everybody is one or the other (but you can't tell which just by looking), you encounter two people, one of whom says to you: "He's a knight or I'm a knave." What are they? What if the speaker had said "He's a knight *and* I'm a knave." ?

A9. Politicians always lie, taxpayers always tell the truth, and in the US, where everybody is one or the other (but never both, as we all know!), you encounter two people, one of whom points to the other and grudgingly declares:

"I'm a taxpayer if and only if he is!"

What are they?

3. Tautologies, Contradictions and Satisfiability

Sometimes conclusions are obtainable *without using premises*. For example, consider the premiseless "inference"

Therefore, if it's snowing, it's snowing.

$$\frac{}{A \rightarrow A}$$

This "inference" is valid because in its truth table

A	A → A
t	t
f	t

the conclusion A → A is *always true*: there are *no* counterexamples.

A statement which, like A → A, is true in all possible cases is called (logically) *valid* or a *tautology*. The logically valid statement A → A is called the *law of self-implication.* Other important examples of logically valid statements are:

$A \vee \neg A$ *law of excluded middle*

$\neg\neg A \rightarrow A$ *law of double negation*

An inference with a valid conclusion is *always* valid, regardless of what its premises are. We shall use the symbol t to stand for a fixed tautology, which for definiteness we shall take to be the statement $A \rightarrow A$ (although any tautology would do). The symbol "t" is doing double duty: it indicates *both* a truth value *and* a particular statement. Notice that we then have

$$p \models t$$

for *any* statement p.

A list S of statements is said to be (jointly) *satisfiable* or *consistent* if there is *at least one* case in which all the members of S are true, and *unsatisfiable* or *inconsistent* if not. This concept is related to that of validity in the following way.

If $p_1,\ldots,p_n \models q$, *then the list* $p_1,\ldots,p_n,\neg q$ *is unsatisfiable, and conversely*.

For the unsatisfiability of the list $p_1,\ldots p_n,\neg q$ is just the assertion that $p_1,\ldots,p_n,\neg q$ are never simultaneously true, which amounts to asserting that $\neg q$ is false, i.e. q is true, whenever $p_1,\ldots,p_n$ are. In particular, it follows that if the list $p_1,\ldots,p_n$ is *unsatisfiable*, then $p_1,\ldots p_n \models q$ for *any* statement q. That is, *inconsistent premises yield any conclusion whatsoever*.

A *single* unsatisfiable statement (e.g. $A \wedge \neg A$) is called a *contradiction*. Thus a contradiction is a statement which is *always* false. Notice that contradictions are exactly the negations of tautologies. We shall use the symbol "f" to stand for a fixed contradiction, which for definiteness we take to be the statement $A \wedge \neg A$ (although, as in the case of "t", it matters not which particular contradiction we choose). Notice that we now have, for *any* statement p,

$$f \models p \ .$$

A statement is said to be *contingent* if it is neither a tautology nor a contradiction; so, a contingent statement is one which is true in at least one case, and false in at least one case (for example, $A \wedge B$). Any statement is either tautologous, contradictory, or contingent: we shall later develop an efficient technique for deciding which.

Notice that a list of statements $p_1,\ldots p_n$ is unsatisfiable if and only if

$$p_1,\ldots p_n \models f$$

and hence the assertions

$$p_1,\ldots p_n \models q$$

and

$$p_1,\ldots, p_n, \neg q \models f$$

are equivalent.

Exercises

B1. Classify the following statements as tautologous, contradictory or contingent:

(a) $(A\to B)\vee(B\to A)$ (b) $[(A\to B)\wedge B]\to A$ (c) $[(A\wedge B)\to C]\to[(A\to C)\vee(B\to C)]$

B2. Which of the following assertions is correct and why:

(a) There is a statement that implies every other statement.

(b) Any statement that follows from a satisfiable statement is satisfiable.

(c) Any statement implying a contingent statement is contingent.

(d) Any statement that follows from a contingent statement is contingent.

(e) Any statement that follows from a valid statement is valid.

(f) Any statement that implies a valid statement is valid.

(g) All contingent statements imply one another.

(h) No inference with a contradiction as conclusion can be valid.

(i) No statement implies its own negation.

(j) Each of the disjuncts of a valid disjunction is valid.

(k) An implication is valid precisely when the consequent follows from the antecedent.

(l) Any statement implied by its own negation is valid.

(m) Removing a premise from a valid argument cannot affect its validity.

(n) In a valid argument, the conclusion is always consistent with the premises; in a sound argument it is not. (Note: An argument is sound exactly when it is both valid and has true premises.)

B3. Determine which of the following five assertions are correct, justifying your answer.

(a) If a statement is not contingent, nor can its negation be.

(b) Every valid argument with a satisfiable set of premises has a satisfiable conclusion.

(c) If a conjunction is a tautology, so is each of its conjuncts.

(d) An invalid argument can always be made into a valid one by adding premises.

(e) The argument from p to q is valid if and only if $p \to q$ is valid.

B4. Circle the correct answer to each of the questions below.

(a) Identify the statement which is a contradiction in the following:

(i) $t \to t$ (ii) $t \to f$ (iii) $f \to t$ (iv) $f \to f$

(b) Identify the statement which is valid in the following:

(i) $A \wedge A$ (ii) $A \vee A$ (iii) $A \to A$ (iv) $(A \to A) \wedge A$

(c) Any argument with an unsatisfiable list of premises must be:

(i) valid and sound (ii) invalid and sound

(iii) valid and unsound (iv) invalid and unsound

B5. Provide a one to two sentence answer for these questions.

(a) Explain why each conjunct of a valid conjunction must itself be valid.

(b) Why is it that whenever the pair of statements $\{p, \neg c\}$ is satisfiable we can't write $p \models c$?

B6. Circle the correct answer to each of the questions below.

(a) Which of the following statements is valid?

(i) $A \wedge A$ (ii) $A \vee A$

(iii) $(A \rightarrow A) \wedge A$ (iv) $A \rightarrow A$

(b) Which of the following statements is a contradiction?

(i) t (ii) $f \rightarrow t$

(iii) $t \vee f$ (iv) $t \wedge (t \rightarrow f)$

(c) Any argument that concludes with a tautology must be:

(i) valid + sound (ii) valid + unsound

(iii) valid (iv) sound

(d) At least one of the disjuncts of a valid disjunction must be:

(i) valid (ii) sound

(iii) consistent (iv) contingent

(e) The consequent of an inconsistent conditional cannot be:

(i) unsatisfiable (ii) satisfiable

(iii) a conjunction (iv) inconsistent

B7. Indicate whether each of the following statements is true or false.

(a) If a statement is not valid, its negation must be.

(b) If a statement fails to logically imply another, it must imply the negation of that other.

(c) A statement that logically implies another cannot imply the negation of that other.

(d) If a set of statements is satisfiable, so is each statement in the set.

(e) If each statement in a set is satisfiable, so is the set.

(f) You cannot make a valid argument invalid by adding more premises.

(g) You cannot make an invalid argument valid by removing premises.

(h) Sound arguments can never have f as their conclusion.

B8. Indicate whether each of the following statements is true or false.

(a) A tautologous conjunction must have a tautologous conjunct.

(b) A contradictory disjunction must have a contradictory disjunct.

(c) If neither a statement nor its negation is valid, then both must be consistent.

(d) If a conditional is unsatisfiable, its consequent must be too.

(e) A contingent statement can logically imply both a statement and the negation of that statement.

(f) No subset of a set of satisfiable statements can be unsatisfiable.

(g) Every statement logically implies at least one other statement with which it is not equivalent.

(h) You can never make an invalid argument into a sound one by dropping some of its premises.

(i) You can never make a valid argument into an unsound one by adding more premises to it.

(j) Some statements are equivalent to every statement that logically implies them.

B9. Using truth tables (where necessary), decide if the following sets of sentences are satisfiable.

(a) $\{A \rightarrow B, B \rightarrow C, A \rightarrow C\}$

(b) $\{(J \rightarrow H) \rightarrow H, \neg J, \neg H\}$

(c) $\{A, B, C\}$

(d) $\{(A \wedge B) \vee (C \rightarrow B), \neg A, \neg B\}$

B10. True or False?

(a) A conjunction with one valid conjunct must itself be valid.

(b) An implication with a valid consequent must itself be valid.

(c) A disjunction with one unsatisfiable disjunct must itself be unsatisfiable.

(d) A sentence is valid iff its negation is unsatisfiable.

(e) An implication with a valid antecedent must itself be valid.

B11. Using truth tables, determine whether the following are valid.

(a) $(F \vee H) \vee (H \rightarrow \neg F)$ (b) $\neg A \rightarrow [(B \wedge A) \rightarrow C]$

B12. . Symbolize this set of sentences and determine whether the set is satisfiable:

> Either the witness was not intimidated, or if Flaherty committed suicide, a note was found. If the witness was not intimidated, then Flaherty did not commit suicide. If a note was found, then Flaherty committed suicide.

II. EQUIVALENCE

1. Equivalence and Bi-implication

Two statements are called (logically) *equivalent* if they take the *same* truth values in all possible cases. For example, consider the truth tables for the statements $A \rightarrow B$, $\neg B \rightarrow \neg A$:

A	B	$A \rightarrow B$	$\neg B \rightarrow \neg A$
t	t	t	t
t	f	f	f
f	t	t	t
f	f	t	t

Since $A \rightarrow B$ and $\neg B \rightarrow \neg A$ have the same truth value on every line of the table, they are equivalent.

We write $p \equiv q$ or $p \Leftrightarrow q$ to indicate that the statements p and q are equivalent. We may think of $\equiv$ as a kind of *equality* between statements. It is left as an exercise to the reader to show that for any statements p, q the assertion that $p \equiv q$ amounts to the same thing as:

$$p \models q \text{ and } q \models p$$

In connection with $\equiv$, we can define a new logical operator" $\leftrightarrow$ " called *bi-implication* (or 'if and only if') as follows:

A	B	$A \leftrightarrow B$
t	t	t
t	f	f
f	t	f
f	f	t

Thus $A \leftrightarrow B$ has value "t" exactly when A and B have the *same* truth value. It follows from this that $p \equiv q$ holds when and only when the statement $p \leftrightarrow q$ is valid. The statements p and q are called the *components* of $p \leftrightarrow q$.

Certain pairs of equivalent statements are known as *laws of equivalence logical laws.* The most important of these are the following:

$p \wedge q \equiv q \wedge p$ $\quad$ $p \vee q \equiv q \vee p$ $\quad$ *Commutativity*

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$ *Associativity*

$(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$ $(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$ *Distributivity*

$p \wedge p \equiv p$ $p \vee p \equiv p$ *Tautology*

$p \wedge f \equiv f$ $p \vee f \equiv p$

$p \wedge t \equiv p$ $p \vee t \equiv t$ } *Absorption*

$p \vee \neg p \equiv t$ *Law of Excluded Middle*

$p \wedge \neg p \equiv f$ *Law of Contradiction*

$p \equiv \neg\neg p$ *Law of Double Negation*

$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$ *De Morgan's Laws*

$p \rightarrow q \equiv \neg q \rightarrow \neg p$ *Law of Contraposition*

$p \equiv p \leftrightarrow t$ $\neg p \equiv p \leftrightarrow f$

These laws are easily established by means of truth tables.

We also have the following equivalences:

$$p \vee q \equiv \neg(\neg p \wedge \neg q)$$
$$p \wedge q \equiv \neg(\neg p \vee \neg q)$$
$$p \rightarrow q \equiv \neg p \vee q$$
$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

From these it follows that $\rightarrow$ and $\leftrightarrow$ are in a natural sense *expressible* in terms of $\{\wedge,\vee,\neg\}$ and hence in terms both of $\{\wedge,\neg\}$ and $\{\vee,\neg\}$. The question now arises as to whether *every* possible truth function, defined immediately below, is so expressible. The answer, as we shall see, is *yes.*

Exercises

A1. Which of the following pairs of statements are equivalent?

(a)	$(A\vee B)\wedge\neg A$	$\neg A\wedge B$
(b)	$A\rightarrow(A\wedge B)$	$A\wedge B$
(c)	$A\rightarrow(A\wedge B)$	$A\rightarrow B$
(d)	$\neg(A\leftrightarrow B)$	$A\leftrightarrow\neg B$
(e)	$\neg(A\leftrightarrow B)$	$(A\wedge\neg B)\vee(\neg A\wedge B)$
(f)	$A\leftrightarrow(B\leftrightarrow C)$	$(A\leftrightarrow B)\leftrightarrow C$
(g)	$A\vee(B\wedge C)$	$(A\vee B)\wedge C$
(h)	$A\rightarrow(A\rightarrow A)$	A
(i)	$(A\rightarrow A)\rightarrow A$	A

A2. (a) Indicate which of the following statements are <u>valid</u>:

(i) $f\leftrightarrow f$

(ii) $(t\downarrow f)\rightarrow(f\,|\,t)$

(iii) $\neg(p\rightarrow\neg p)$

(iv) $\neg(p\leftrightarrow\neg p)$

(v) $(p\rightarrow q)\rightarrow(q\rightarrow p)$

(vi) $(p\rightarrow q)\rightarrow(\neg q\rightarrow\neg p)$

(vii) $(\neg p \wedge \neg q) \leftrightarrow \neg(p \vee q)$

(vii) $(p \leftrightarrow (p \wedge q)) \leftrightarrow (q \leftrightarrow (p \vee q))$

(b) Which of the above statements (i)-(ix) are equivalent to each other?

A3. Using truth tables, show that:

(a) $A \leftrightarrow (B \leftrightarrow C) \equiv (A \leftrightarrow B) \leftrightarrow C$

(b) $\neg(A \leftrightarrow B) \equiv \neg A \leftrightarrow B \equiv A \leftrightarrow \neg B$

2. Truth Functions and Expressive Completeness

A *truth function H* of *n* statement letters $A_1,\ldots,A_n$ is an assignment, to each truth valuation of $A_1,\ldots,A_n$, of a truth value t or f, which we write as $H(A_1,\ldots,A_n)$. (For example, $H(A_1, A_2, A_3) = (A_1 \vee A_2) \leftrightarrow A_3$ defines a truth function, since truth values for A_1, A_2, and A_3 fix a truth value for $(A_1 \vee A_2) \leftrightarrow A_3$ via the truth tables for $\vee$ and $\leftrightarrow$.) This may be displayed this in the form of a truth table:

A_1	A_2	...	A_n	$H(A_1,\ldots,A_n)$
t	t	...	t	*
t	t	...	f	*
f	f	...	f	*

Here each * stands for a t or an f.

Each truth function *H* can be given what is called a *disjunctive normal form*. This is obtained as follows. Assume first that at least *one* of the entries in the H column is "t". For each valuation of $A_1,\ldots,A_n$ in which a "t" appears in the H column we form the conjunction A_1*

$\wedge$ ... $\wedge$ A_n^* where each A_i^* is A_i if the given valuation assigns t to A_i and $\neg A_i$ if not. Notice that this conjunction is true *precisely* under the given valuation and no other. Now we form the *disjunction* of all these conjunctions arising from the "t" cases of the given truth table. The resulting statement is called the *disjunctive normal form* (*d.n.f.*) of the given truth function. Clearly, its truth table is identical to that of the given truth function.

It remains to consider the case in which the given truth function always takes the value "f". Here we may take the disjunctive normal form to be, e.g., $A_1 \wedge \neg A_1$.

Since d.n.f.s contain only the logical operators $\wedge$, $\vee$, $\neg$, it follows from all this that *every possible truth function can be expressed in terms of* $\wedge$, $\vee$, $\neg$, and so *every statement is equivalent to one whose only logical operators are these*. We sum this up by saying that the set {$\wedge$, $\vee$ $\neg$} is *expressively complete*. Moreover, since $\wedge$ is expressible in terms of $\vee$, $\neg$ and $\vee$ in terms of $\wedge$, $\neg$, we may infer that *each of the sets* {$\wedge$,$\neg$} *and* {$\vee$ $\neg$} *is expressively complete*.

Before proceeding further let us determine a d.n.f. in a practical case. Suppose we are given, for instance, the truth table

A	B	C	H(A,B,C)
t	t	t	t
t	t	f	f
t	f	t	t
t	f	f	t
all remaining lines			f

The d.n.f. here is, writing $\underline{A}$ for $\neg A$ etc. and omitting the "$\wedge$"s,

$$ABC \vee A\underline{B}C \vee A\underline{BC}$$

Are there are *single* logical operations (involving just two statement letters) which are expressively complete? We shall see that there are exactly *two* of these.

We define the logical operators ("Sheffer strokes") "|" —*nand* — and "↓" — *nor* — by means of the following truth tables.

A	B	$A \mid B$	$A \downarrow B$
t	t	f	f
t	f	t	f
f	t	t	f
f	f	t	t

Clearly, $A \mid B \equiv \neg(A \wedge B)$ and $A{\downarrow}B \equiv \neg(A \vee B)$ (hence 'nand' is short for 'not and' and 'nor' short for 'not or'!).

First, we show that $\mid$ and $\downarrow$ are each expressively complete. To do this it suffices to show that $\neg$ and $\vee$ are both expressible in terms of $\mid$, and $\neg$ and $\wedge$ in terms of $\downarrow$. (Why?)

Clearly $A \mid A \equiv \neg(A \wedge A) \equiv \neg A$, so $\neg$ is expressible in terms of $\mid$. Now

$$A \mid B \equiv \neg(A \wedge B) \equiv \neg A \vee \neg B,$$

so

$$\neg A \mid \neg B \equiv \neg\neg A \vee \neg\neg B \equiv A \vee B.$$

Hence, recalling that $\neg A \equiv A \mid A$, we see that

$$A \vee B \equiv (A \mid A) \mid (B \mid B),$$

and so $\vee$ is expressible in terms of $\mid$.

Similarly, $\neg A \equiv A{\downarrow}A$ and $A \wedge B \equiv (A{\downarrow}A){\downarrow}(B{\downarrow}B)$. Therefore $\mid$ and $\downarrow$ are each expressively complete.

We next show that $\mid$ and $\downarrow$ are the *only* expressively complete logical operations on two statement letters.

For suppose that H(A,B) is expressively complete. If H(t,t) were t, then any statement built up using only H would take the value t when all its statement letters take value t. So $\neg A$ would not be expressible in terms of H. Therefore H(t,t) = f. Similarly, H(f,f) = t. So we obtain the partial truth table

A	B	H(A,B)
t	t	f
t	f	
f	t	
f	f	t

If the second and third entries in the last column are t,t or f,f, then H is | or ↓. If they are f,t, then H(A,B) ≡ ¬A; and if they are t,f, then H(A,B) ≡ ¬B. So in both of these cases H would be expressible in terms of ¬. But clearly ¬ is not expressively complete by itself, since the truth function t is not expressible in terms of it. So H is | or ↓ as claimed.

Exercises

B1. Find statements involving the operators ∧, ∨, ¬ and the statement letters A, B, C that have the following truth tables (1), (2), (3):

A	B	C	(1)	(2)	(3)
t	t	t	t	t	f
f	t	t	t	t	t
t	f	t	t	t	f
f	f	t	f	f	f
t	t	f	f	t	t
f	t	f	f	f	t
t	f	f	f	t	f
f	f	f	t	f	t

B2. The logical operator ⊻ called exclusive disjunction is defined by specifying that p⊻q is true when exactly one of p, q is true, and false otherwise.

(a) Show that p ⊻ q can be defined to be any of the following equivalent statements: (i) ¬(p ↔ q) ; (ii) (p ∨ q) ∧ ¬(p ∧ q); (iii) (p ∧ ¬q) ∨ (¬p ∧ q).

(b) Show that ⊻ is associative, in the sense that p⊻(q⊻r) ≡ (p⊻q)⊻r, for any statements p, q and r. (Use (a) and **A3.**)

(c) Show that {⊻,∧,t} is an expressively complete set.

(d) What are the truth conditions for $p_1 \veebar p_2 \veebar \ldots \veebar p_n$? (That is: when would you regard such an expression as true and when not? It turns out that the answer is: precisely when an *odd number* of the p's are true!)

B3. (a) Show that the pair {¬,→} is expressively complete.

(b) Show that the single truth function f(A,B,C) = (A∨B)→¬C is expressively complete. (Hint: One approach is to show that {¬, →} are both expressible in terms of the function f and invoke (a).)

B4. (a) Show that ↔ cannot be expressed in terms of → alone. (Hint: any statement containing exactly A, B, → takes value t in at least one case where A and B have opposite truth values.)

(b) Show that ∨ *can* be expressed in terms of → alone.

(c) Show that ∧ cannot be expressed in terms of → alone. (Start by showing that any statement containing just the logical operator → must take truth value t in at least two cases.)

B5. Find statements in ¬, ∨, ∧ that have the following truth functions f, g, h.

A	B	C	f(A, B, C)	g(A, B, C)	h(A, B, C)
t	t	t	t	f	t
f	t	t	t	t	f
t	f	t	f	t	f
t	t	f	f	t	f
f	f	t	f	f	f
t	f	f	f	t	t
f	t	f	f	t	t
f	f	f	t	t	t

B6. Show that the truth function h(A, B, C) determined by (A→B)→¬C is expressively complete.

B7. Find disjunctive normal forms for the following statements:

(a) $\neg(A \to B) \vee (\neg A \wedge C)$ (b) $A \leftrightarrow [(B \wedge \neg A) \vee C]$

B8. (a) Explain why {f,→} must be an expressively complete set.

(b) Out of the 16 possible binary logical operators one could define, exactly how many can be expressed in terms of → alone? (This one's moderately difficult!)

3. Arithmetical Representation of Statements and Logical Operations

Statements and logical operations can be nicely expressed within *binary arithmetic*, the arithmetic of 0 and 1. Binary arithmetic is the theoretical foundation for the construction of computers, and so is an important, if hidden, constituent of contemporary life. The significance of binary arithmetic for logic was first recognized by the English mathematician George Boole in the 1840s.

First, we describe the rules of binary arithmetic. We suppose given the two numbers 0,1 and two operations "+" (addition) and "." (multiplication) on them subject to the following rules (only one of them may be unfamiliar!):

$$0 + 0 = 1 + 1 = 0 \qquad 0.0 = 0.1 = 1.0 = 0$$
$$0 + 1 = 1 + 0 = 1 \qquad 1.1 = 1$$

We shall think of statements as determining *binary functions* (that is, functions taking just the values 0 and 1) as follows. Statement letters A,B,C,... will be regarded as *variables* taking values 0,1: we think of 1 as representing the truth value t and 0 as representing the truth value f. Then the operation "∧" corresponds to "." and the operation "¬" to the operation "1 + " of adding 1.

Given this, how do we interpret "∨" and "→"? We argue as follows.

$$\begin{aligned} A \vee B &\equiv \neg(\neg A \wedge \neg B) \\ &= 1 + (1 + A).(1 + B) \\ &= 1 + 1 + A + B + A.B \end{aligned}$$

$$= 0 + A + B + A.B$$

$$= A + B + A.B$$

And

$$A \rightarrow B \equiv \neg A \vee B$$

$$= 1 + A + B + (1 + A).B$$

$$= 1 + A + B + B + A.B$$

$$= 1 + A + A.B$$

In this way, any statement p gives rise to a binary function called its *binary representation* which we shall denote by the same symbol p. In that case, *tautologies* are those statements whose binary representations take only value 1, and *contradictions* those statements whose binary representations take only value 0.

When, for example, is $p \rightarrow q$ a tautology? Exactly when the corresponding binary representation $1 + p + p.q$ is constantly 1. But this is the case precisely when $0 = p + p.q = p.(1 + q)$, that is, when at least one of p and $1 + q$ is 0, in other words, if $p = 1$, then $1 + q = 0$, i.e. $q = 1$. But this means that the value of p never exceeds the value of q: we shall write this as $p \leq q$. It follows that

$$p \models q \quad \Leftrightarrow \quad p \rightarrow q \text{ is a tautology} \quad \Leftrightarrow \quad p \leq q$$

(where we have written "$\Leftrightarrow$" to indicate equivalence of assertions). That is, *in the binary representation,* $\models$ *corresponds to* $\leq$. By the same token,

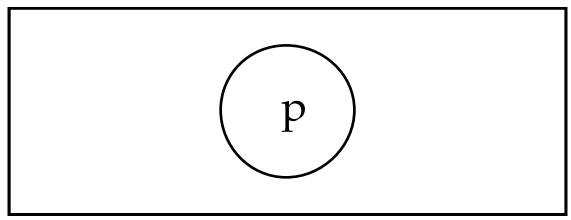
$$p \equiv q \quad \Leftrightarrow \quad p = q.$$

That is, *in the binary representation,* $\equiv$ *corresponds to* =.

The binary representation sheds light on expressive completeness. For example, the expressive completeness of $\{\wedge,\neg\}$ translates into the assertion that any binary function can be expressed in terms of the operations "." and "1 + ", while the expressive completeness of " | " translates into the assertion that any binary function can be expressed in terms of the single binary function $1 + x.y$.

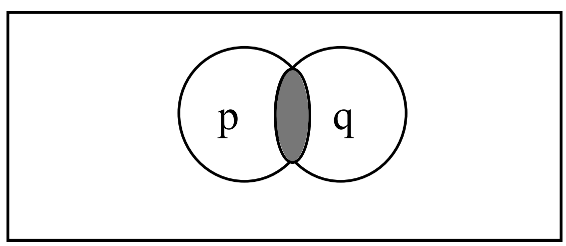
4. Venn Diagrams

Venn (or Euler) diagrams[4] are a convenient method of depicting logical relationships. We represent the various truth valuations of statement letters by points in a rectangle (which itself may be thought of as a kind of "logical space"). Then, for a given statement p, the collection of valuations making p true is represented by a circle within the rectangle:



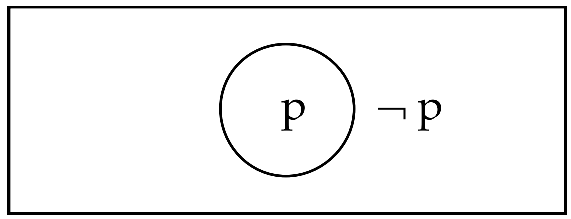
This circle is called the *region corresponding to* p, or simply *the region of* p.

It is clear that the region corresponding to any *tautology* is the *whole rectangle*, and that corresponding to any *contradiction* is the *empty region*. For a *conjunction* p $\wedge$ q the corresponding region is the shaded portion in the figure below, that is, the *intersection*

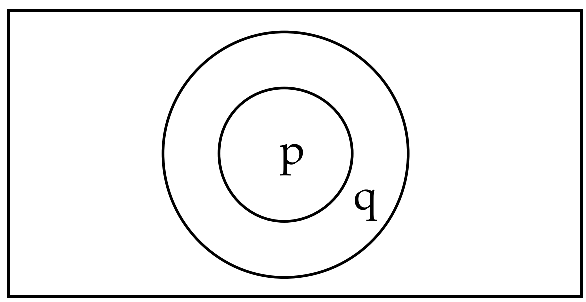


[4] The great Swiss mathematician *Leonhard Euler* introduced these diagrams in his ***Letters to a German Princess, On Different Subjects in Physics,*** a series of 234 letters written between 1760 and 1762 and addressed to Friederike Charlotte of Brandenburg-Schwedt and her younger sister Louise. In the 19th century the English mathematician *John Venn* refined Euler`s ideas and the diagrams have been associated with Venn ever since (at least in the Anglosphere)..

of the regions corresponding to p and q. For a *disjunction* p ∨ q the corresponding region is that covered by the *union* of both circles. For a *negation* ¬p the corresponding region is that lying outside the region of p: its *complement*.

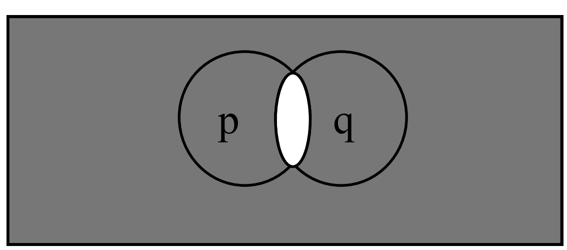


The relation of *logical implication* corresponds to the relation of *inclusion* between regions: p ⊨ q is equivalent to the region of p being



included in the region of q. Finally, the relation of *equivalence* corresponds to the relation of *identity* between regions: p ≡ q is equivalent to asserting that the regions of p and q *coincide.*

It is now straightforward to illustrate most of the logical laws presented on p. 27 by means of Venn diagrams. For example, consider the Venn diagram immediately below.



Clearly the shaded region — that corresponding to $\neg(p \wedge q)$ — is the union of the region outside that of p with the region outside that of q. This latter is the region corresponding to $\neg p \vee \neg q$. This verifies the first de Morgan law. The remaining laws may be similarly verified.

Exercises

C1. Show that the binary representation of $A \veebar B$ is A + B.

C2 (i) Find the binary representations of the following statements, (ii) using those representations classify each statement as valid, contradictory or contingent, and (iii) draw the Venn diagram corresponding to each statement:

(a) $A{\downarrow}B$ (b) $A \leftrightarrow B$ (c) $(A \rightarrow B) \vee (B \rightarrow A)$ (d) $(A \vee \neg A) \wedge (B \wedge \neg B)$

C3. Find the binary representations of the following statements and draw their Venn diagrams:

(a) $A \rightarrow (B \rightarrow B)$

(b) $(A{\downarrow}A){\downarrow}(A{\downarrow}A)$

(c) $(A \leftrightarrow B) \wedge (\neg A \leftrightarrow B)$

(d) $p \rightarrow (q \rightarrow r)$

(e) $[p \rightarrow (q \rightarrow r)] \wedge (s \vee \neg s)$

(f) $(C \veebar D)|(C \veebar D)$

C4. Find the binary representations of the following statements and draw their Venn diagrams:

(a) $(A \wedge \neg A) \rightarrow (A \vee \neg A)$

(b) $\neg[(A|A)|(A|A)]$

(c) $\neg[(A \leftrightarrow B) \vee (\neg A \leftrightarrow B)]$

(d) $(p \rightarrow q) \rightarrow r$

(e) $[(p \rightarrow q) \rightarrow r] \vee (s \wedge \neg s)$

(f) $(C \vee D){\downarrow}(C \vee D)$

C5. Find the disjunctive normal forms, binary representations and Venn diagrams for the following statements:

(a) $A \leftrightarrow (B \rightarrow \neg A)$

(b) $[(A \rightarrow B) \rightarrow A] \wedge \neg C$

III. TRUTH TREES

1. Introduction to Truth Trees

To test an inference for validity it suffices to conduct an exhaustive search for counterexamples. If none are found, then the inference is valid. *Truth trees* are an efficient and elegant device both for establishing validity of inferences, and for unearthing counterexamples, if such exist.

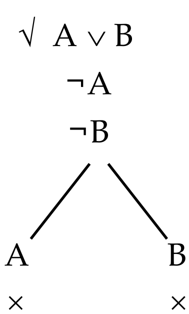
Consider, for example, the (valid) inference

$$\frac{A \vee B \quad \neg A}{B}$$

To obtain its tree form, we start by listing its premises and the *negation* of its conclusion:

$$\frac{A \vee B \quad \neg A}{\neg B}$$

These statements will be true in exactly the cases in which there are counterexamples to the original inference. Now we continue, generating an inverted tree-like structure:



Here the statements A, B, ¬A, ¬B, A ∨ B occupy positions, or as we shall call them, *nodes*[5]in the tree. The statement occupying the top node is a disjunction and requires analysis: A ∨ B is true in all those cases in which A is true and all those cases in which B is true, and in no other cases. We indicate this by writing A and B at the ends of a fork at the foot of the tree. At the same time we tick the statement A∨B, using "√", to indicate that all its t cases have been taken into account. *Ticking a statement*[6] *is, accordingly, equivalent to erasing it*. (**N.B.** *Once the use of the tree method has become familiar, we shall usually dispense with the*

[5] We shall often identify a node in a tree with the statement occupying it.

[6] We shall often use the locution"to tick a given node" as a synonym for "to tick the statement occupying the given node".

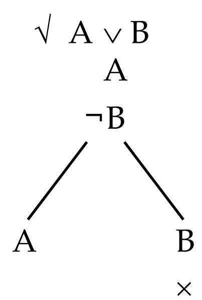
ticking of statements in trees.) Finally we write "×" at the foot of each path through the tree in which a statement occupies one node and its negation another. Such paths are called *closed*. In this particular tree all paths are closed; under these conditions the tree itself is said to be *closed.* And, as we shall see, the inference is then valid.

Why is this? Because the procedure was designed so that when we ticked a statement, we displayed all the possible ways in which that statement can be true. The various paths then represent all the ways in which the initial statements (i.e., the statements with which we began the tree) could possibly be true; that is, each path represents a potential counterexample to the original inference. In the case of a *closed* path, the possibility it represents does not really exist. Accordingly, if *all* paths are closed, then it is impossible for all the initial statements of the tree to be (simultaneously) true, in other words, there are no counterexamples to the original inference and so it is valid.

In contrast, observe what happens when we test an *invalid* inference, e.g.,

$$\frac{\begin{array}{c}A \vee B \\ A\end{array}}{B}$$

The tree looks like this:



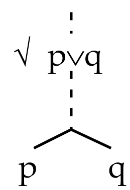
The left-hand path is not closed, that is, it is *open* and represents a genuine counterexample to the inference in question. To describe it, note which statement letters, *with or without* ¬, occupy nodes in the path. In this case they are A, ¬B, and the corresponding counterexample is that in which A is true and B is false:

A	B	A ∨ B	A	B
t	f	t	t	f

We next describe the various tree rules.

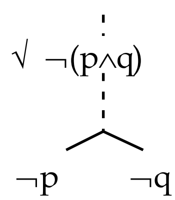
2. The Tree Rules

<u>Disjunction</u>



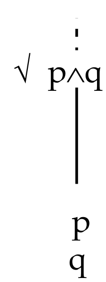
Tick a disjunction occupying a node and write the disjuncts at the end of a fork drawn at the foot of each open path containing the ticked node.

<u>Negated conjunction</u>



Tick a negated conjunction occupying a node and write the negations of the conjuncts at the end of a fork drawn at the foot of each open path containing the ticked node. (For: a conjunction is false exactly when some conjunct is false. Notice that, by de Morgan's law, this rule is nothing but the disjunction rule in disguise.)

<u>Conjunction</u>



Tick a conjunction occupying a node and write the conjuncts in a column at the foot of each open path containing the ticked node. (Justification: a conjunction is true exactly when both conjuncts are true.)

Negated disjunction

$$
\begin{array}{c}
\vdots \\
\surd\ \neg(p \vee q) \\
| \\
\neg p \\
\neg q
\end{array}
$$

Tick a negated disjunction occupying a node and write the negations of the disjuncts in a column at the foot of each open path containing the ticked node. (Justification: a disjunction is false exactly when both disjuncts are false.)

Implication

$$
\begin{array}{c}
\vdots \\
\surd\ p \rightarrow q \\
\vdots \\
\wedge \\
\neg p \qquad q
\end{array}
$$

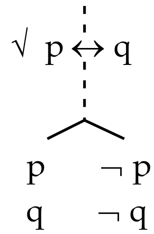
Tick an implication occupying a node and write the negation of the antecedent and the consequent at the ends of a fork drawn at the foot of each open path containing the ticked node. (For: an implication is true exactly when the negation of the antecedent is true, or the consequent is true, or both.)

Negated implication

$$
\begin{array}{c}
\vdots \\
\surd\ \neg(p \rightarrow q) \\
| \\
p \\
\neg q
\end{array}
$$

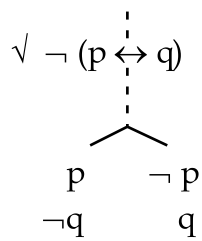
Tick a negated implication occupying a node and write the antecedent and the negation of the consequent in a column at the foot of each open path containing the ticked node. (For: an implication is false exactly when the antecedent is true and the consequent false.)

Bi-implication



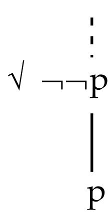
Tick a bi-implication occupying a node and draw a fork at the foot of each open path containing the ticked node. At the ends of each of these write in columns the components, and, respectively, the negations of the components, of the ticked node. (For: a bi-implication is true exactly when both components are true, or both are false.)

Negated bi-implication



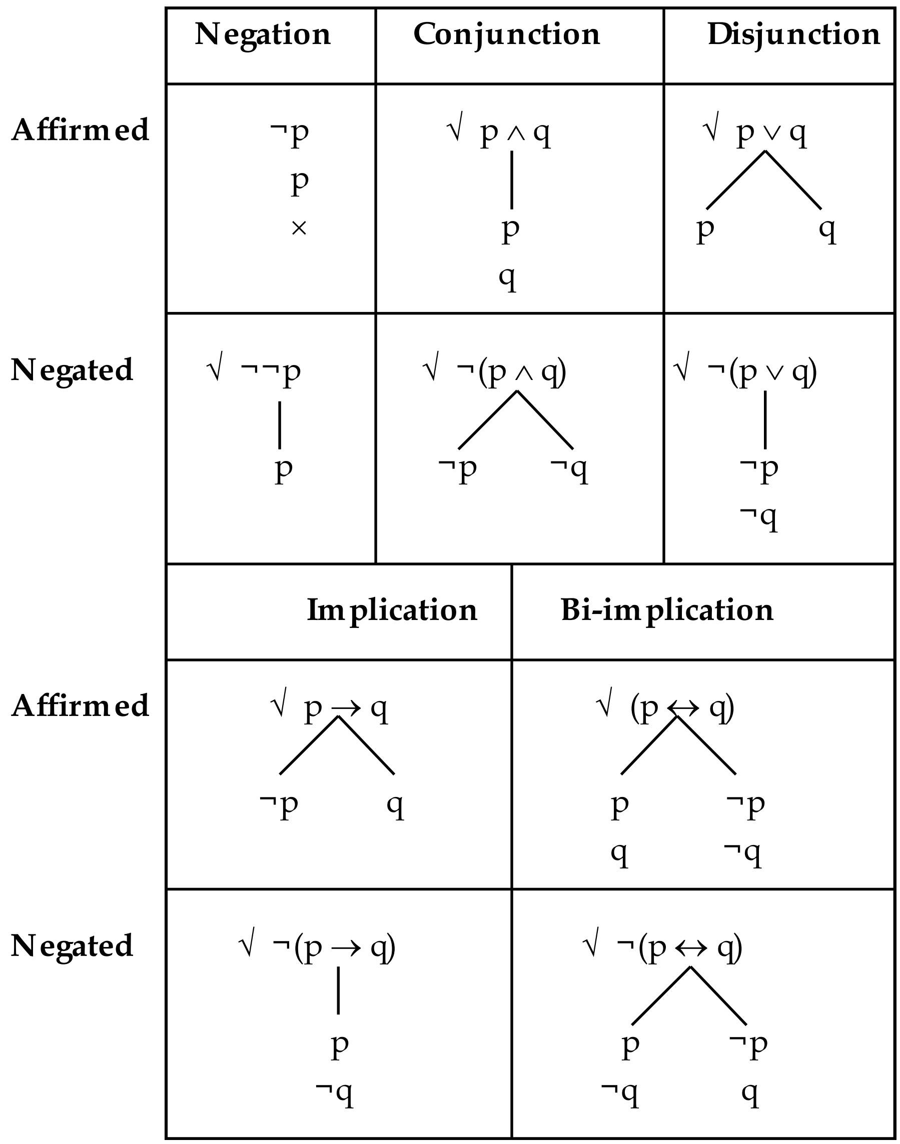
Tick a negated bi-implication occupying a node, and draw a fork at the foot of each open path containing the ticked node. At the ends of these write in columns the first component and the negation of the second, and, respectively, the negation of the first and the second. (For: a bi-implication is false exactly when one component is true and the other false.)

Double negation



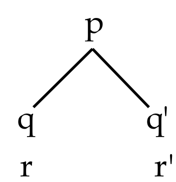
Erase double negations. (For: the negation of a statement is false exactly when the statement is true.)

We summarize these rules as follows:



When applying a tree rule of the form

p is called the *premise*, and {q,r} the *list of conclusions*, of the application. Similarly, when applying a tree rule of the form



p is called the *premise*, and {q,r}, {q',r'} the *lists of conclusions*, of the application.

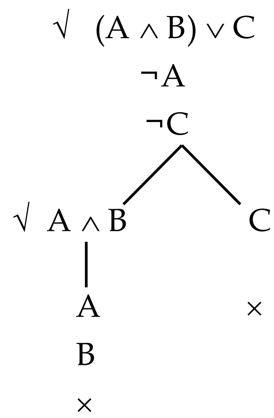
Note on the use of checkmarks in trees. We shall often omit these

3. Tree Test for Validity

To test an inference for validity, write its premises and the *negation* of its conclusion in a column and apply the tree rules to all unticked lines of open paths, ticking lines to which rules are applied, until the tree is *finished*, i.e. until the only unticked nodes in any remaining open paths are statement letters and their negations. A tree obtained in this way is called a [finished] tree *associated* with the given inference. If any such tree is *closed*, i.e. if all its paths are closed, the original inference is valid.

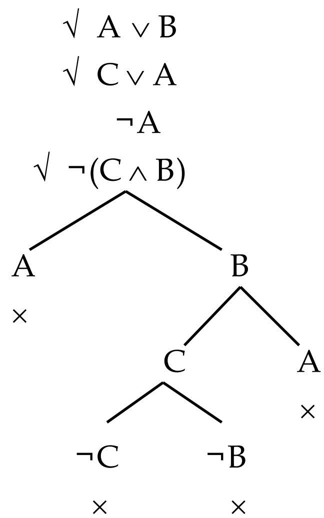
We now give some examples of the use of this test.

$$\frac{(A \wedge B) \vee C \quad \neg A}{C}$$



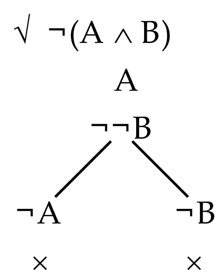
Tree closed, so inference valid.

$$\frac{A \vee B \quad C \vee A \quad \neg A}{C \wedge B}$$



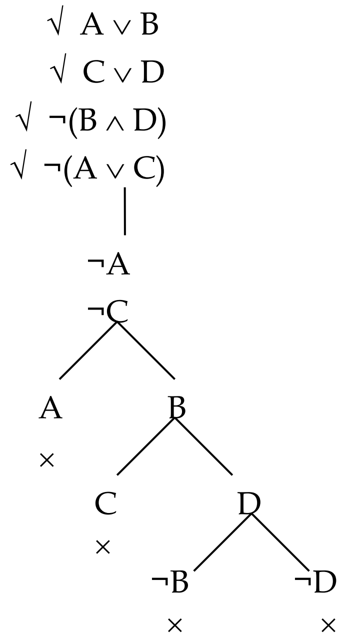
Tree closed, so inference valid.

$$\frac{\neg(A \wedge B) \quad A}{\neg B}$$



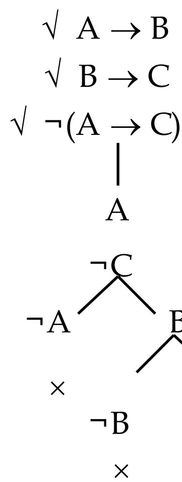
Tree closed, so inference valid.

A ∨ B
C ∨ D
<u>¬(B ∧ D)</u>
A ∨ C



Tree closed, so inference valid.

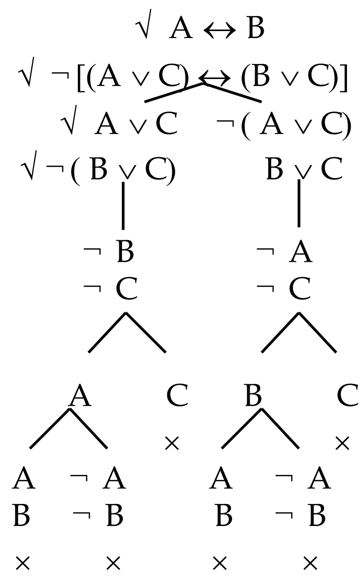
A → B
<u>B → C</u>
A → C



Tree closed, so inference valid.

$A \leftrightarrow B$

$(A \vee C) \leftrightarrow (B \vee C)$



Tree closed, so inference valid.

Exercises

A1. Use the tree method to determine whether the following arguments are valid. In the invalid cases, find all counterexamples:

(a) $(A \wedge B) \rightarrow C$

$\underline{\neg A \rightarrow D}$

$B \rightarrow (C \vee D)$

(b) $\underline{A \vee \neg(B \wedge C)}$

$(A \leftrightarrow C) \vee B$

(c) $(B \leftrightarrow \neg A) \rightarrow \neg C$

$(\neg B \wedge D) \vee (A \vee E)$

$\underline{(D \vee E) \rightarrow C}$

$A \rightarrow B$

(d) If Holmes has bungled or Watson is windy, Moriarty will escape. Thus Moriarty will escape unless Holmes bungles.

(e) Moriarty will not escape unless Holmes acts. We shall rely on Watson only if Holmes does not act. So if Holmes does not act, Moriarty will escape unless we rely on Watson.

(f) Moriarty will escape only if Holmes bungles. Holmes will not bungle if Watson's to be believed. So if Watson's to be believed, Moriarty won't escape.

A2. Use the tree method to determine which of the following inferences is valid. In the invalid cases, supply all counterexamples.

(a) $\underline{A \wedge (B \vee C)}$

$(A \wedge B) \vee (A \wedge C)$

(b) $(\neg A \vee B) \wedge (A \vee \neg B)$

$\underline{A \vee B}$

$A \wedge B$

(c) $\neg (A \wedge B) \vee C$

$\underline{A \vee D}$

$\neg B \vee (C \vee D)$

(d) $\underline{\neg(A \vee B) \vee C}$

$A \vee C$

A3. Use the tree method to determine which of the following inferences are valid. In the invalid cases find all counterexamples.

(i) $(A \wedge B) \rightarrow C$

$\underline{\neg A \rightarrow D}$

$B \rightarrow (C \vee D)$

(ii) $\underline{A \vee \neg(B \wedge C)}$

$(A \leftrightarrow C) \vee B$

(iii) $(B \leftrightarrow \neg A) \rightarrow \neg C$

$(\neg B \wedge D) \vee (A \vee E)$

$\underline{(D \vee E) \rightarrow C}$

$A \rightarrow B$

A4. Define the new logical operations [A, B, C] and A*B by

$$[A, B, C] = (A \rightarrow B) \rightarrow \neg C$$

$$A*B = A \rightarrow \neg B$$

Devise the simplest tree rules you can for these operations and their negations. Use the rules you have devised to determine which of the following inferences are valid:

(i) $\underline{[A, B, C]}$ (ii) $[A, B, C]$

$A*B$ $\underline{A*B}$

$C*B$

A5. Use the tree method to determine which of the following inferences are valid. In the invalid cases find all counterexamples.

(i) $(A \wedge B) \rightarrow C$ (ii) $\underline{(A \wedge B) \rightarrow C}$ (iii) $\underline{A \vee \neg(B \wedge C)}$

$\neg\underline{A \rightarrow D}$ $(A \rightarrow C) \vee (B \rightarrow C)$ $(A \leftrightarrow C) \vee B$

$B \rightarrow (C \vee D)$

A6. Define the logical operations {A, B, C}, A*B, and A•B by

$$\{A, B, C\} = (B \rightarrow A) \wedge (\neg B \rightarrow C)$$

$$A*B = A \rightarrow \neg B$$

$$A \bullet B = \neg(\neg A \rightarrow B)$$

Devise the simplest tree rules you can for these operations. Use these rules to test the validity of the following inferences:

(i) $\underline{A \bullet B}$ (ii) $\underline{A*B}$ (iii) $\underline{\{A, B, C\}}$

A7. Determine which of the following arguments are valid. In the invalid cases, supply <u>all</u> counterexamples.

(a) $A \rightarrow B$ (b) $(\neg q \vee r) \leftrightarrow \neg p$ (c) $\neg p \leftrightarrow (\neg q \vee r)$

$\neg B$ $q \vee \neg q$ $\neg q \wedge q$

$\underline{\neg A \rightarrow C}$ $\underline{\neg r \leftrightarrow p}$ $\underline{p \leftrightarrow \neg r}$

$\neg C \rightarrow B$ $\neg p \wedge (\neg q \rightarrow r)$ $(\neg q \rightarrow r) \wedge \neg p$

A8. Determine which of the following arguments are valid. In the invalid cases, supply <u>all</u> counterexamples.

(a)		(b)		(c)	
	$A \to B$		$(\neg q \vee r) \leftrightarrow \neg p$		It's a duck if it walks
	$B \to C$		$q \vee \neg q$		and quacks like one.
	$\underline{C \to D}$		$\underline{\neg r \leftrightarrow p}$		Either it's a duck if it walks like one
	$A \to D$		$p \to p$		or it's a duck if it quacks like one.

A9. Translate the following arguments into logical notation (indicating what elementary sentences your symbols refer to) and then determine whether each argument is valid. If not, indicate the total number of counterexamples.

(a) If Dumb knows that he's dumb, then he's dumb. If he knows that he's dumb, then he at least knows *something*. If Dumb knows something, then he's not dumb after all! Therefore, Dumb's not dumb.

(b) Canada's economy will fail if Quebec does separate. If Canada's economy won't fail, then the market will get the jitters if Quebec does separate. The market will get the jitters even if Quebec doesn't separate. So, the market will get the jitters and Canada's economy will fail.

A10. Use the tree method to determine whether the following argument is valid; if not, supply one counterexample.

Either scientists don't know what they are talking about, or the sun will eventually burn out and Earth will become dark and cold. If scientists don't know what they are talking about, then Mars is teeming with life. If Earth becomes dark and cold, then either the human race will migrate to other planets or will die out. Mars is not teeming with life, but the human race will not die out. Therefore, the human race will migrate to other planets only if Mars is teeming with life.

A11. Use the tree method to determine whether the following argument is valid. If not, supply the exact number of counterexamples.

It will not be the case that both the Representatives and the Senators will pass the bill. If either the Representatives or the Senators pass it, the voters will be pleased; but if both pass it, the President won't be pleased. The President won't be pleased if and only if Boehner rejoices. Therefore, Boehner won't rejoice.

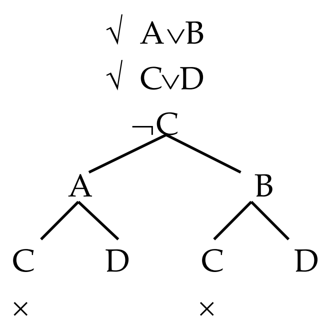
4. Further Applications of the Tree Method

A. Counterexamples from the associated tree. *Any open path remaining in a finished tree associated with an inference determines a counterexample to it (and so establishes its invalidity). And conversely, any counterexample is determined by an open path in any such tree.*

For example, here is an invalid inference:

$$\begin{array}{c} A \vee B \\ \underline{C \vee D} \\ C \end{array}$$

Consider the following finished tree associated with this inference:

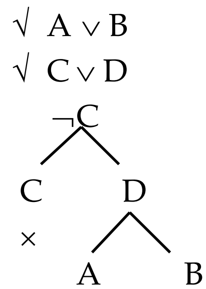


Each open path in this tree determines a counterexample to the given inference. For example, the left-hand open path, nodes of which are occupied by A, ¬C, D, but by neither B nor ¬B, determine as counterexamples all cases in which A, C, D are t, f, t respectively, regardless of the truth value of B. That is, we obtain two counterexamples A: t, B: t, C: f, D: t, and A: t, B: f, C: f, D: t. Similarly, the right-hand open path determines as counterexamples all cases in which B, C, D are t, f, t respectively, regardless of the truth value of A. In total we get the three distinct counterexamples

ABCD: ttft, tfft, ftft.

These are all the counterexamples to the given inference.

In this connection we observe that the open paths in the other finished tree associated with the above invalid inference, viz.,



of course determine exactly the same counterexamples as were obtained before.

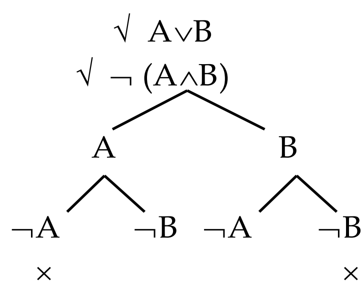
Recall that a set of statements is *satisfiable* or *consistent* if there is at least one case in which all the members of the set are true.

B. Tree test for satisfiability. *Given a set S of statements, start a tree with the members of S in a column. Then S is satisfiable precisely when there is an open path through the finished tree. Each open path determines a truth valuation that makes all the members of S true.*

We illustrate this by the following example. Consider the set of statements

{A ∨ B, ¬(A ∧ B)}.

The relevant finished tree is



There are two open paths in which the statement letters (negated or unnegated) A, ¬B; ¬A, B respectively, occupy nodes. Thus the valuations making the given set of statements true are AB:tf or ft.

C. Tree test for logical validity. *To determine whether a given statement is logically valid, start a tree with its negation. Then the given statement is logically valid precisely when the resulting finished tree is closed.*

For example, consider the statement (A ∧ B) ∨ (¬A ∨ ¬B). To test for logical validity, we construct the following tree:

$$\surd\ \neg\,[(A\wedge B)\vee(\neg A\vee\neg B)]$$

$$\surd\ \neg\,(A\wedge B)$$

$$\surd\ \neg\,(\neg A\vee\neg B)$$

$$\neg\neg A$$

$$\neg\neg B$$

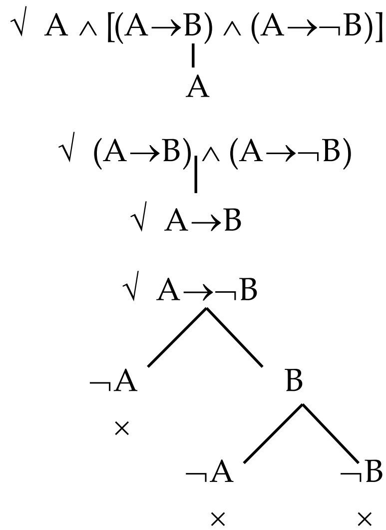
$$\neg A \qquad\qquad \neg B$$

$$\times \qquad\qquad \times$$

Since this (finished) tree is closed, the statement in question is logically valid.

D. Tree test for contradictions. *To test whether a given statement is a contradiction, start a tree with the (unnegated) statement. Then the statement is a* contradiction *precisely when the resulting finished tree is closed.*

For example, to test whether the statement A ∧ (A → B) ∧ (A→¬B) is a contradiction, construct the following tree:



Since this (finished) tree is closed, the given statement is a contradiction.

Exercises

B1. Use the tree method to determine which of the following sets of statements are satisfiable. In the latter cases, supply all the satisfying valuations.

(a) $A, B, \neg(A \wedge B)$

(b) $A, \neg B, \neg A \vee B$

(c) $A, \neg B, A \vee B$.

B2. Use the tree method to determine which of the following statements are <u>tautologies</u>.

(a) $\neg(A \wedge B) \vee A$

(b) $\neg A \vee (A \wedge B)$

(c) $\neg(A \wedge B) \vee A \vee B$

B3. Use the tree method to determine which of the following statements are <u>contradictions</u>.

(i) $\neg\left[\,[(A \to B) \to A] \to A\right]$ (ii) $\left[(A \to B) \to B\right] \wedge \neg A \wedge \neg B$

B4. In the land of knights and knaves, knights always state the truth and knaves falsehoods. Punch and Judy are two inhabitants of this land. From their assertions in each case use the tree method to deduce as much as you can about their statuses.

(i) *Punch*: Judy's a knight

Judy: We're not both knaves.

(ii) *Punch*: If Judy's a knave, we both are.

Judy: Either he's a knight, or I'm a knave.

B5. Use the tree method to determine which of the following pairs of statements are equivalent.

(i) $A \to (B \to C)$ $(A \wedge B) \to C$

(ii) $(A \leftrightarrow B) \leftrightarrow C$ $A \leftrightarrow (B \leftrightarrow C)$

(iii) $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$

(iv) $\neg(A \to (B \to \neg C))$ $\neg A \wedge (B \leftrightarrow C)$

(v) $\neg A \vee (B \to C)$ $A \wedge \neg B \wedge \neg C$

B6. Classify each of the following statements as tautologous, contradictory or contingent.

(a) $((A\to B)\to B)\to A$

(b) $\neg(p \wedge q)\vee p$

(c) $B\leftrightarrow(C\vee\neg C)$

(d) $(p\leftrightarrow(p\to q))\to q$

B7. Knaves always lie, knights always tell the truth, and in Camelot, where everybody is one or the other, you encounter two people, one of whom says to you:

(i) "He's a knight and I'm a knave." What are they?

(ii) What if that person had said: "If he's a knave, then so am I"?

(iii) How about if that person had said: "I'm a knight, and, then again, I'm not; though he's a knave if I am" ?

B8. Classify each of the following statements as tautologous, contradictory or contingent.

(a) $((A \rightarrow B) \rightarrow B) \rightarrow B$

(b) $\neg(p \vee q) \wedge q$

(c) $(B \wedge \neg B) \leftrightarrow (C \vee \neg C)$

(d) $\neg[(p \rightarrow q) \leftrightarrow p)] \vee q$

B9. Knaves always lie, knights always tell the truth, and in Camelot, where everybody is one or the other, you encounter three people, Lancelot, Arthur and Merlin, who say to you:

Lancelot: Merlin's a knave.

Arthur: Either Lancelot or Merlin is a knave.

Merlin: If I'm a knave, they are too.

What are they? (Hint: use the tree method. Let L (A , M) be the statement " Lancelot (Arthur, Merlin) is a knight". Then, for example. "Lancelot is a knave" is equivalent to $\neg L$. If Lancelot makes a statement A , then A is true if and only if Lancelot is a knight, that is, if L is true. Thus $L \leftrightarrow A$ must be true. Thus the three assertions made above make each of the statements $L \leftrightarrow \neg M$, $A \leftrightarrow (\neg L \vee \neg M)$, $M \leftrightarrow (\neg M \rightarrow (\neg A \wedge \neg L))$ true. Start a tree with these statements and work out the possible truth values of L, A, and M.)

B10. Circle the tautologies that occur below:

(a) $\neg\neg A \rightarrow A$ (b) $A \rightarrow (B \rightarrow A)$ (c) $A \leftrightarrow (B \vee \neg A)$

(d) $A \leftrightarrow (\neg A \wedge A)$ (e) $(A \rightarrow B) \vee (B \rightarrow A)$

B11. Circle the inconsistent sets of sentences that occur below:

(a) $B \rightarrow A$, B , $\neg A$ (b) $A \rightarrow \neg A$, $\neg A \rightarrow A$ (c) $A \leftrightarrow (B \vee \neg B)$, $\neg A$

(d) $A \rightarrow B$, $B \rightarrow C$, $A \wedge \neg C$ (e) $\neg (A \rightarrow (B \rightarrow C))$, C

B12. In the land of knights and knaves, where knaves always lie, knights always tell the truth, and everybody is either one or the other (clearly no one can be both!), you encounter two people, Dumb and Dumber, both of whom speak to you. In each case below, determine as much as you can about their individual identities.

(a) Dumb: 'If I'm a knave, we both are.' Dumber: 'He's a knight or I'm not.'

(b) Dumb: 'Dumber is a knight if and only if 2+2 is 4.' Dumber: 'Come on, 2+2 is *not* 4!'

B13. These puzzles concern a land populated by saints and sinners. Saints always tell the truth; sinners always lie. You are a traveler in this strange land and must try to identify those you meet as saints or sinners.

You encounter two people, Mutt and Jeff, one or both of whom speak to you. What can you deduce in each case, using the tree method, about whether they are saints or sinners?

1. *Mutt:* I'm a saint.

2. *Jeff*: Mutt is a saint.

3. *Mutt:* Jeff's a sinner.

4. *Jeff:* Either I'm a saint, or I'm not.

5. *Mutt*:I'm a saint, and, then again, I'm not.

6. *Jeff:* If Mutt is a sinner, so am I.

7. *Jeff:* Neither of us is a saint.

8. *Mutt:* We're not both saints.

9. *Mutt:* I'm a sinner if and only if Jeff's a saint.

10. *Jeff:* Mutt is a saint, and I'm a sinner.

11. *Mutt:* I'm a sinner unless Jeff's a saint.

12. *Mutt:* If either of us is a sinner, I am.

13. *Mutt*: Jeff's a sinner.

Jeff: We're not both sinners.

14. *Mutt:* I'm a saint if and only if Jeff's a sinner.

Jeff: Mutt is a sinner.

15. *Mutt:* Jeff's a saint.

Jeff: At least one of us is a sinner.

16. *Mutt:* I'm a saint if and only if Jeff is.

Jeff: Mutt is a saint.

17. *Mutt:* If I'm a sinner, we both are.

Jeff: Either he's a saint, or I'm a sinner.

18. *Mutt:* Jeff's a saint if and only if his brother is.

Jeff: Unfortunately, my brother's a sinner.

19. *Mutt:* Jeff and his brother are both saints.

Jeff: Well, I'm a saint, but my brother isn't.

At this point, you meet three curious looking people in the land of knights and knaves. What can you deduce about their status?

20. *Curly:* Larry's a sinner.

Moe: Either Curly or Larry is a sinner.

Larry: If I'm a sinner, they are too.

21. *Curly:* We're all saints.

Moe: Well, I'm a saint, but Larry's a sinner.

Larry: No, the other two are both sinners.

22. *Curly:* That Moe's a saint.

Moe: No, we're all sinners.

Larry: Curly, Moe, *and* their cousins are all sinners.

23. *Curly:* Well, at least we're not all of us sinners.

Moe: Curly is.

Larry: If Curly is, Moe is too.

24. *Curly:* If Moe's a saint, Larry is too.

Moe: Well, Larry's a sinner if Curly's one.

Larry: But Curly and Moe aren't both sinners.

25. *Curly:* If any of us are saints, Larry is.

Moe: But Larry's a sinner.

Larry: And I'm a sinner if and only if Moe's one.

26. *Curly:* If Moe's a sinner, Larry is too.

Moe: If Larry's a sinner, so is Curly.

Larry: If Moe's a saint, we all are.

B14. Determine which of the following sets of statements are (jointly) satisfiable, in each case describing the satisfying valuations:

a)	(b)	(c)
$\neg A \vee B$	$\neg(\neg B \vee A)$	$\neg D \vee B$
$B \vee \neg C$	$A \vee \neg C$	$A \vee \neg B$
$\neg B \vee \neg(C \vee D)$	$\neg B \vee \neg C$	$\neg(D \wedge A)$
		D

B15. Using the tree method, determine which of the following statements are tautologies. In the non-tautologous cases, supply all the truth valuations that make the statement false.

(i) $((A \rightarrow B) \rightarrow B) \rightarrow A$

(ii) $A \rightarrow (B \rightarrow (B \rightarrow A))$

B16. Using the tree method, determine which of the following sets of statements are satisfiable. In the satisfiable cases, supply all the satisfying valuations.

(i) $A \rightarrow B$, $B \leftrightarrow C$, $(C \vee D) \leftrightarrow \neg B$

(ii) $\neg(\neg B \vee A)$, $A \vee \neg C$, $B \rightarrow \neg C$

B17. Knaves always lie, knights always tell the truth, and in Camelot, where everybody is one or the other, you encounter three people, Lancelot, Arthur and Merlin, who say to you:

Lancelot: Merlin's a knave.

Arthur: Either Lancelot or Merlin is a knave.

Merlin: If I'm a knave, they are too.

Use the tree method to determine as much as you can about each person's identity.

B18. We return for one last visit to the land of Camelot where everyone is either a knight (always speaking the truth) or a knave (always uttering falsehoods). Sir Lancelot is searching for his mistress Queen Guinevere, and happens upon King Arthur and his band of merry men. When Lancelot asks of Guinevere's whereabouts, Arthur becomes jealous and is in no mood to give Lancelot a straight answer. So he instructs Merlin to cast a spell upon his men so that each, in turn, responds to Lancelot as follows:

Sir Karl the Pauper: Guinevere is in Camelot today.

Sir Loin of Beef: Sir Karl is a knight, but Sir Rob is most certainly a knave.

Sir Rob of Cliff Town: Hey, I'm a knave if and only if Sir Loin is!

Sir Lee Fellow: Yah! If any of us are knights, Sir Rob is.

Does Arthur succeed in hiding Guinevere's present whereabouts, or do his men inadvertently disclose her location to Arthur's rival in love? Use the tree method to find out.

B19. A certain island is populated entirely by heroes and scoundrels; the former always tell the truth, the latter invariably lie.

(a) You encounter four people on the island who say to you:

Dean: If I'm a hero, so is Stan.

Stan: If I'm a hero, so is Jerry.

Jerry: If I'm a scoundrel, Ollie isn't.

Ollie: Those others are all liars!

Determine as much as you can about their individual identities.

(b) Believe it or not, I once went to the Island myself in search of buried treasure. I don't remember the details too clearly, but I do recall encountering Dean and Jerry. Dean, I remember, told me:

"Jerry is a hero and there is buried treasure on the island", but I can’t quite recall what Jerry said. All I remember is that he said exactly one of the following:

“ Dean is a scoundrel and there's no buried treasure on the island”

or

“Dean is a scoundrel and there is buried treasure on the island”.

Nevertheless, I do remember being able to figure out whether treasure was buried on the island and the identity of both speakers. What were their identities? Was treasure buried on the island?

(c) Suppose you were to go to the Island of Heroes and Scoundrels and wished to find out whether or not there is gold on the island. You meet Dean (not knowing his identity) and you are allowed to ask him only one question, which must be answerable by ‘Yes’ or ‘No’. What question could you ask him that would allow you to figure out if there is buried treasure on the island? (This one’s tricky — and there may be more than one question that could do the job.)

B.20 Finally, here's a toughie. On a certain island, rumoured to contain buried treasure, live three gnomes, identical in appearance, of whom it is known that one invariably tells the truth, one always lies, and the third answers "yes" or "no" at random. You arrive on the island and, encountering the three gnomes, ask them a total of two questions, each addressed to one gnome at a time, and to which the answer is a simple "yes" or "no". What questions would you ask that would allow you to figure out if there is buried treasure on the island? (*Hint:* the answer to the first question must enable you to "eliminate" the gnome who answers at random. Also see Appendix B, section 2.)

5. Correctness and Adequacy of the Tree Method

We conclude this chapter with some arguments designed to justify the claims we have made concerning the use of trees in establishing validity and satisfiability.

First, let us call a tree rule R *correct* if whenever the premise of R is true under a given valuation, then all the statements in *at least one* of R's lists of conclusions are also true under the valuation. And let us call R *complete* if the converse holds, that is, the premise of R is true under a given valuation whenever all the statements in at least one of R's lists of conclusions is true under the valuation.

Clearly, *all the tree rules we have introduced are correct and complete in the above senses*.

Next, we observe that the process of constructing a finished tree *always terminates*. For the tree starts with a finite number of statements, each of which has finite length (taking the *length* of a statement to be the total number of symbols in it), and it grows 'downward' by a process of choosing an unticked statement occupying a node of an open path, ticking it and adding at the foot of the path some finite number of statements, each of which is *shorter* than the ticked one. Eventually the point must be reached at which all unticked statements occupying nodes of open paths have lengths 1 or 2 (i.e., are statement letters or their negations) and the process ends.

Given a set *S* of statements, let us say that a tree *starts with* S if it has S as its initial set of statements. Now we can establish the

Correctness of the tree method. *If a set* S *of statements is satisfiable, there will be an open (complete) path through any tree that starts with* S.

To prove this, observe first that, if all the statements occupying nodes in a path P of a tree are true under a given valuation, then P is open. For if there is a valuation making all statements occupying nodes in P true, then both a statement and its negation cannot both occupy nodes in P, otherwise the (alleged) valuation would have to make both a statement and its negation true—impossible. It follows that P cannot contain both a statement and its negation, which is just to say that path P is open.

Now suppose that under some valuation V all the members of S are true. Consider the following property of a tree **T**.

(*) **T** starts with S and contains a (complete) path P such that all statements occupying nodes of P are true under V.

By the observation above, any tree satisfying (*) contains an open path.

We claim that, if **T** has property (*), so does any tree **T*** obtained from **T** by applying a tree rule. For suppose that (a) all the statements occupying nodes in a certain path P through **T** are true under V and (b) we extend **T** to **T*** by applying a tree rule to one of its statements. Clearly we may assume that this statement is in P, for if not, then P is unaffected and is a complete path of **T***. Accordingly in the transition from **T** to **T*** the path P is extended to a new path, or extended and split into two new paths, by applying some tree rule. Since any tree rule is correct, all the statements occupying nodes in the new path, or all those occupying nodes in at least one of the new paths (each of which extends the path P), are true under V. But this shows that **T*** has property (*), as claimed.

It follows that *any* tree **T** starting with S has property (*), and hence contains an open path. For any tree **T** starting with S can be 'built up' (or rather, down!) by starting first with the tree with a single path consisting of the statements in S—which has property (*) by definition—and then applying tree rules, one after another (finitely many times), until tree **T** results. By the argument of the previous paragraph, at each stage of the 'tree building' process, property (*) is preserved, therefore the end result—the tree **T**—will have that property too (and so must contain an open path, which is what we needed to show).

As an immediate consequence of this, we obtain the

Inference correctness of the tree method. *If a* finished *tree associated with an inference is* closed, *then the inference is* valid.

Now we prove the converse of the above correctness result, that is, the

Adequacy of the tree method. *If there is an* open *path through a* finished *tree starting with a given set* S *of statements, then* S *is* satisfiable.

To prove this, let **T** be a finished tree starting with S and containing an open path P. We are going to show how to define a truth valuation V on the statement letters that figure in tree **T** such that the sentences in set S all come out true under V. Consider the single statement letters that occur in path P (not negated statement letters, just the non-negated elementary statements in P). Let V be the valuation that assigns all those statement letters value t, and all the statement letters that do *not* occur in path P (i.e. that occur somewhere else in the tree **T**) the truth value f. (If there are any other statement letters left out of this assignment, let them take any truth value you want.) We claim that *all statements occupying nodes of* P *are true under* V, not just the nodes containing statement letters.

To show this first notice that all statements of lengths 1 or 2 occupying nodes of P are true under V. For those of length 1 are statement letters and are accordingly true under V by definition. And any one of length 2 is a negation ¬A of a statement letter A; since P is open, A cannot occupy a node of P, and so is false under V. Thus ¬A is true under V.

Now suppose that, if possible, some statement occupying a node of P is false under V. Let p be such a statement of *shortest* length. Then by the above the length of p must be at least 3, so a tree rule, R say, may be applied to p. Since **T** is finished, some list L of conclusions obtained by applying R to p is already part of P. But each statement in L is *shorter* than p, and so must be true under V. Since R is complete, it follows that the premise p of the specified application of R is also true under V. Therefore the falsity of p is refuted, and the claim above follows.

Given the (now established) truth of the claim that all statements occurring in path P are true under V, it follows (in particular) that the initial statements in the set S are true under V (since that set clearly lies in all paths of the tree, P included). It follows, then, that the set S is satisfiable, and hence the tree method is adequate in the sense spelled out above.

As an immediate consequence, we obtain the converse of validity correctness, that is, the

Inference adequacy of the tree method. *If an inference is valid, then any finished tree associated with it is closed*.

It follows that the validity of propositional inferences id *decidable* in the sense that, given any such inference, one can determine in a finite number of steps whether the inference is valid or not. For one only has to generate a finished tree associated with the inference: this can be done in a finite number of steps. If the tree is closed, the inference is valid; if not, the inference is invalid.

IV. NATURAL DEDUCTION FOR PROPOSITIONAL LOGIC

1. Natural Deduction

Truth trees provide an efficient method for determining the validity of inferences. But they offer virtually no means for *constructing* (valid) inferences. We shall develop a system - called a *natural deduction system* - in which valid inferences in propositional logic can actually be exhibited.

A natural deduction system is a body of rules, known as *rules of inference,* which enables statements to be deduced , or derived from other statements. The key idea in a natural deduction system is that of a formal inference, or as we shall call it, a *derivation.* There are two types of derivations, *hypothetical* and *categorical. Hypothetical* derivations start with statements called *premises,* or *hypotheses,* and terminate with a statement called the *conclusion.* The derivation's assumptions act as assumptions from which the conclusion is then a formal consequence. Such derivations show that, if the premises are true, then the conclusion is also true; that is, the inference from premises to conclusion is valid. *Categorical* derivations, on the other hand, contain no assumptions. A categrorical derivation shows that its conclusion must be true outright, that is, must be a tautology.

A *derivation* in a natural deduction system is a sequence of *lines*, on each of which a statement is displayed[7]. Each statement in a derivation is either (a) a premise, or (b) derives, by means of a rule of inference from statements displayed on previous lines. The statement on the last line of a derivation is its *conclusion.* The line on which the conclusion of the derivation is displayed is preceded by the symbol $\therefore$, standing for *thereforee.* We shall say that a derivation is a *derivation of its conclusion from its assumptions*. For any list of statements $p_1, ..., p_n$ and any statement p, we say that p is *derivable from* $p_1, ..., p_n$ if there is a derivation with $p_1, ..., p_n$ as assumptions and p as conclusion. A *theorem* of a natural deduction system is a statement derivable without assumptions. Theorems are the conclusions of categorical derivations.

The system of natural deduction **P** for propositional logic we shall present[8] has two sorts of rules of inference: *strict rules of inference* and *rules of replacement.*

A strict rule of inference is a scheme of the form

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[7] We shall occasionally identify a line with the statement displayed on it.

[8] This system of natural deduction is due to I. M. Copi. See Copi and Cohen`s *Introduction to Logic,* 9th edution, Prentice-Hall, 1994.

$$\begin{array}{c} p \\ q \\ r \\ . \\ . \\ \hline s \end{array}$$

Here p, q, r, ... are the *premises* , and s the *conclusion* of the rule. The conclusion s is said to be *derived by means of the rule* from the premises p, q, r, ..

A *rule of replacement* is a scheme of the form

$$p \leftrightarrows q$$

In applying a rule of replacement $p \leftrightarrows q$ to a line in a derivation, p (q) *may be replaced by* q (p) *wherever they occur.*

The *strict rules of inference* for the system **P** are the following:

Modus ponens (MP)

$$\begin{array}{c} p \to q \\ p \\ \hline q \end{array}$$

Modus Tollens (MT)

$$\begin{array}{c} p \to q \\ \neg q \\ \hline \neg p \end{array}$$

Hypothetical Syllogism (HS)

$$\begin{array}{c} p \to q \\ q \to r \\ \hline p \to r \end{array}$$

Disjunctive Syllogism (DS)

$$\begin{array}{c} p \vee q \\ \neg p \\ \hline q \end{array} \qquad \begin{array}{c} p \vee q \\ \neg q \\ \hline p \end{array}$$

Constructive Dilemma (CD)

$$\begin{array}{c} (p \to q) \wedge (r \to s) \\ p \vee r \\ \hline q \vee s \end{array}$$

Expansion (Exp)

$$\begin{array}{c} p \to q \\ \hline p \to (p \wedge q) \end{array}$$

Simplification (Simp)

$$p \wedge q \qquad p \wedge q$$

Conjunction (Conj.)

$$p$$

$$\frac{}{p} \qquad \frac{}{q} \qquad\qquad \frac{q}{p \wedge q}$$

Addition (Add)

$$\frac{p}{p \vee q} \qquad \frac{q}{p \vee q}$$

It is easy to check, using truth tables, the validity of each of these rules of inference.

Here are a couple of examples of (hypothetical) derivations in **P** using just the strict rules of inference.

1. $A \rightarrow B$ *premise*
2. $A \vee (C \wedge D)$ *premise*
3. $\neg B \wedge \neg E$ *premise*
4. $\neg B$ *3, Simp*
5. $\neg A$ *1, 4, MT*
6. $C \wedge D$ *1, 6, DS*
7. $\therefore C$ *7, Simp*

1. $J \rightarrow K$ *premise*
2. $K \vee L$ *premise*
3. $(L \wedge \neg J) \rightarrow (M \wedge \neg N)$ *premise*
4. $\neg K$ *premise*
5. L *2, ,4, D*
6. $\neg J$ *5, 1, MT*
7. $L \wedge \neg J$ *5, 6, Conj.*
8. $M \wedge \neg N$ *3, 7, MP*
9. $\therefore M$ 8, *MP*

The *rules of replacement* for **P** are the following:

De Morgan's Laws (DM) $\neg(p \wedge q) \leftrightarrows \neg p \vee \neg q$
$\neg(p \vee q) \leftrightarrows \neg p \wedge \neg q$

Commutation (Com) $p \wedge q \leftrightarrows q \wedge p$

$$p \lor q \leftrightarrows q \lor p$$

Association (Assoc)

$$p \land (q \land r) \leftrightarrows (p \land q) \land r$$
$$p \land \lor (q \lor r) \leftrightarrows (p \lor q) \lor r$$

Distribution (Dis)

$$p \land (q \lor r) \leftrightarrows (p \land q) \lor (p \land r)$$
$$p \lor (q \land r) \leftrightarrows (p \lor q) \land (p \lor r)$$

Double Negation (DN)

$$p \leftrightarrows \neg\neg p$$

Contraposition (Cont)

$$p \rightarrow q \leftrightarrows \neg q \rightarrow \neg p$$

Material implication (Impl)

$$p \rightarrow q \leftrightarrows \neg p \lor q$$

Material equivalence (Equiv)

$$p \leftrightarrow q \leftrightarrows (p \rightarrow q) \land (q \rightarrow p)$$
$$p \leftrightarrow q \leftrightarrows (p \land q) \lor (\neg p \land \neg q)$$

Exportation (Expo)

$$(p \land q) \rightarrow r \leftrightarrows p \rightarrow (q \rightarrow r)$$

Tautology (Taut)

$$p \leftrightarrows p \land p$$
$$p \leftrightarrows p \lor p$$

It is readily checked by truth tables that in each of these rules the statements on either side of $\leftrightarrows$ are logically equivalent.

Here are a couple of examples of derivations using the rules of replacement.

1. $A \rightarrow \neg B$ *premise*
2. $\neg(C \land \neg A)$ *premise*
3. $\neg C \lor \neg\neg A$ *2, DM*
4. $\neg C \lor A$ *3, DN*
5. $C \rightarrow A$ *4, Impl*
6. $\therefore$ $C \rightarrow \neg B$ *5, 1, HS*

1. $(M \rightarrow N) \land (O \rightarrow P)$ *premise*

2. $\neg N \vee \neg P$	*premise*
3. $\neg(M \wedge O) \rightarrow Q$	*premise*
4. $M \rightarrow N$	*1, Simp*
5. $(O \rightarrow P) \wedge (M \rightarrow N)$	*1, Com*
6. $O \rightarrow P$	*5, Simp*
7. $\neg N \rightarrow \neg M$	*4, Cont*
8. $\neg P \rightarrow \neg O$	*6, Cont*
9. $(\neg N \rightarrow \neg M) \wedge (\neg P \rightarrow \neg O)$	*7, 8, Conj*
10. $\neg M \vee \neg O$	*9, 2, DS*
11. $\neg(M \wedge O)$	*10, DM*
12. $\therefore$ Q	*11, MP*

It is easy to see that the conclusion of any derivation in **P** is a logical consequence of its premises. For each of the rules of inference leads from true statements to statements, so, if the derivation stats with true premises, all of the statements displayed on the lines of the derivation, and in particular its conclusion, must also be true.

It is in fact the case - although we shall not prove it here - that **P** is *complete* in the sense that, if $p_1,...p_n \models q$, then q is derivable in **P** from $p_1,...p_n$.

The system **P** only allows us to construct *hypothetical* derivations. In order to construct *categorical* derivations we need to add a new rule allowing us simply to introduce a statement into s derivation rather than deriving it from a previous line. The new rule is:

Self-implication (SI) $p \rightarrow p$

An application of this rule in a derivation simply amounts to introducing a new line, and writing down $p \rightarrow p$ in it, where p is any statement.

We write **P*** for the system **P** augmented by the rule SI. The conclusion of a categorical derivation in **P*** is, as we recall, called a *theorem* (of **P***).

Here a couple of examples of categorical derivations in **P*.**

1. $p \to p$ *SI*
2. $\neg p \vee p$ *1, MI*
3. $\therefore\ p \vee \neg p$ *2, Comm*

1. $p \to p$ *SI*
2. $\neg p \vee p$ *1, MI*
3. $\neg q \vee (\neg p \vee p)$ *2, Add*
4. $(\neg q \vee \neg p) \vee p$ *3, Assoc.*
5. $(\neg p \vee \neg q) \vee p$ *4, Comm*
6. $\neg p \vee (\neg q \vee p)$ *5, Assoc*
7. $p \to (\neg q \vee p)$ *6, Impl*
8. $\therefore\ p \to (q \to p)$ *7, Impl*

Exercises

A1. Construct derivations, using just strict rules of inference, for the following valid inferences:

(i). $(F \to G) \wedge (H \to I)$
$J \to K$
$(F \vee J) \wedge (H \vee L)$
$\therefore\ G \vee K$

(ii) $(\neg M \wedge \neg N) \to (O \to N)$
$M \to N$
$\neg N$
$\therefore\ \neg O$

(iii) $(K \vee L) \to (M \vee N)$
$(M \vee N) \to (O \wedge P)$
K
$\therefore\ O$

(iv) $(Q \to R) \wedge (S \to T)$
$(U \to V) \wedge (W \to X)$
$Q \vee U$
$\therefore\ R \vee V$

(v) $W \rightarrow X$

$(W \wedge X) \rightarrow Y$

$(W \wedge Y) \rightarrow Z$

$\therefore \quad W \rightarrow Z$

(vi) $A \rightarrow B$

$C \rightarrow D$

$A \vee C$

$\therefore \; (A \wedge B) \vee (C \wedge D)$

(vii) $(E \vee F) \rightarrow (G \wedge H)$

$(G \vee H) \rightarrow I$

E

$\therefore \quad I$

(viii) $(N \vee O) \rightarrow P$

$(P \vee Q) \rightarrow R$

$Q \vee N$

$\neg Q$

$\therefore \quad R$

A2, Construct derivations, using any of the rules of inference, including the rules of replacement, for the following valid inferences:

(i) $(D \wedge \neg E) \rightarrow F$

$\neg(E \vee F)$

$\therefore \quad \neg D$

(ii) $(J \vee K) \rightarrow \neg L$

L

$\therefore \neg J$

(iii) $R \vee (S \wedge \neg T)$
$(R \vee S) \to (U \vee \neg T)$
$\therefore\ T \to U$

(iv) $[(Y \wedge Z) \to A] \wedge [(Y \wedge B) \to C]$
$(B \vee Z) \wedge Y$
$\therefore\ A \vee C$

(v) $[H \vee (I \vee J)] \to (K \to J)]$
$L \to [I \vee (J \vee H)]$
$\therefore\ (L \wedge K) \to J$

(vi) $(P \to Q) \wedge (P \vee R)$
$(R \to S) \wedge (R \vee P)$
$\therefore\ Q \vee S$

(vii) $J \vee (\neg J \wedge K)$
$J \to L$
$\therefore\ (L \wedge J) \leftrightarrow J$

(viii) $(R \vee S) \to (T \wedge U)$
$\neg R \to (V \to \neg V)$
$\neg T$
$\therefore\ \neg V$

A3 Show that the following statements are theorems of the system **P* :** **(i)** $(p \to q) \to (\neg q \to \neg p)$; **(ii)** $p \leftrightarrow \neg\neg p$; **(iii)** $p \to (q \to p)$; **(iv)** $(p \wedge q) \to p$: **(v)** $p \to (p \vee q)$; **(vi)** $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$; **(vii)** $\neg(p \wedge q) \leftrightarrow (\neg p \wedge \neg q)$; **(viii)** $(p \to q) \vee (q \to p)$.

2. Indirect Deduction

The tree method establishes the validity of an inference by starting a tree with the premises and the negation of the conclusion, and then showing that the tree closes. There is an analogous method using derivations of establishing validity called *indirect deduction.* Indirect deductions are derivations of the form

Premises
Negation of conclusion *AID*
.
.
.

Contradiction

Here *AID* stands for *assumption for indirect deduction.*

A indirect deduction **D** establishes the validity of the inference from premises $p_1, \ldots, p_n$ to conclusion q. For **D** shows that $p_1, \ldots p_n, \neg q \models f$, and we have seen that this is equivalent to $p_1, \ldots p_n \models q$.

Here are a couple of examples of indirect deductions.

The following indirect deduction establishes the validity of the inference

$$\frac{\begin{array}{c} A \rightarrow B \\ \neg A \vee \neg B \end{array}}{\neg A}$$

1. $A \rightarrow B$ *premise*
2. $\neg A \vee \neg B$ *premise*
3. $\neg\neg A$ *AID*
4. A *5, DN*
5. B *4, 1, MP*
6. $A \wedge B$ *4, 5, Conj*
7. $\neg(A \wedge B)$ *2, DM*
8. $(A \wedge B) \wedge \neg(A \wedge B)$ *6, 7, Conj* Contradiction

The premiseless indirect deduction below shows that the statement $(p \wedge q) \rightarrow p$ is a theorem of **P**:

1. $\neg((p \wedge q) \rightarrow p)$ *AIP*
2. $\neg(\neg(p \wedge q) \vee p)$ *1, Impl*
3. $\neg\neg(p \wedge q) \wedge \neg p$ *2, DM*
4. $(p \wedge q) \wedge \neg p$ *3, DN*
5. $p \wedge q$ *4, Simp*
6. p *5, Simp*
7. $\neg p$ *4, Simp*
8. $p \wedge \neg p$ *6, 7, Conj* Contradiction

Exercises

B1. Construct indirect deductions for the following inferences:

(i) $C \rightarrow (\neg B \vee D)$
$C \rightarrow \neg D$
$\therefore$ $C \rightarrow \neg B$

(ii) $D \rightarrow (B \wedge C)$
$A \rightarrow (\neg B \wedge \neg E)$
$\therefore$ $(A \wedge D) \rightarrow F$

(iii) $(C \rightarrow A) \rightarrow (B \rightarrow D)$
$\neg(C \wedge A) \rightarrow D$
$\therefore$ $\neg D \rightarrow \neg B$

B2. Show that $(p \rightarrow q) \vee (q \rightarrow p)$ is a theorem of **P** by constructing a premiseless indirect proof of it.

V. BASIC CONCEPTS OF SET THEORY

1. Sets and Membership

The concept of validity (which we shall call *propositional validity*) that we have employed up to now is restricted in that it does not cover a large class of arguments which are clearly logically correct. Consider, for example, the following argument:

1. All Cretans love all animals.

2. All horses are animals.

3. Epimenides is a Cretan.

∴ *4. Someone loves all horses.*

This argument, while patently not propositionally valid, is still, given the usual reading of the terms "all" and "some", logically correct. Its correctness also of course derives from our grasp of the *grammatical structure* of the statements constituting it, which involve in an essential way *predicates* or *properties*—"(is a) horse", "(is an) animal", "Cretan"—and *relations* —"loves".

The concepts of property and relation are closed tied to the concept of a *set.* We are all familiar with the basic idea of a set, also called a *class* or *collection*. As examples, we may consider the set of all coins in one's pocket, the set of all human beings, the set of all planets in the solar system, etc. These are all *concrete* sets in the sense that the objects or individuals constituting them—their *elements* or *members*—are material things. In mathematics and logic we wish also to consider *abstract* sets whose members are not necessarily material things, but abstract objects such as numbers, lines, ideas, names, etc. We shall use the term *set* to cover concrete and abstract sets, as well as sets which contain a mixture of material and abstract elements.

We shall typically use lower case letters *a, b, c, ..., x, y, z* to stand for objects or individuals and upper case letters *A, B, C, ..., S, T, ... X, Y, Z* to stand for sets.

Given objects *a, b,* we write $a = b$ if *a* and *b* are identical and $a \neq b$ if not. We assume that, for any object *a,* the statement $a = a$ is true and the statement $a \neq a$ is false.

If *S* is a set, and *a* is an element of *S*, we say that *a belongs to S*, and write

$$a \in S.$$

If *b* does not belong to *S*, we write $b \notin S$.

In any discussion of the properties of, or relations between objects or individuals, we shall assume that we are given a set to which all the objects we wish to consider belong. This set is called the *universal set, universe of discourse,* or simply *universe* underlying the discussion and will be denoted by *U*. It is important to remember that the universal set will not always be the same but will vary with the discussion: it can, in fact, be any set whatsoever. For example, if we are discussing the properties of the natural number system, we may take *U* to the set of all natural numbers. When discussing people, we shall want *U* to be the set of all human beings. When discussing family sizes, we may take *U* to be the set consisting of all human beings and all natural numbers. In the argument above, *U* may be taken to be the set consisting of all of all animals and all human beings.

Once a universal set has been specified, we can consider *predicates* and *relations* defined on it. Suppose, for instance, that the universal set *U* is the set of all people. Then examples of predicates defined on *U* are the expressions

x is female   *x is male*   *x is Canadian*

and examples of relations on *U* are the expressions

x is taller than y   *x is married to y.*

Here *x* and *y* are being used as *variables* which are understood to *range* over *U*. This means that, when the variables in each expression are replaced by names of elements of *U* (in the case at hand, names of human beings), a statement having a definite *truth value* is obtained. (If the resulting statement is true, the elements are said to *satisfy* the predicate or relation in question.) For example, if in the expression *x is Chinese* we replace *"x"* by "*Arnold Schwarzenegger"* we obtain the false statement

Arnold Schwarzenegger is Chinese,

while if in the expression *x is taller than y* we make the same substitution for *x* and replace "*y*" by "*Danny de Vito"* we obtain the true statement

Arnold Schwarzenegger is taller than Danny de Vito.

The most direct way of specifying a set is to list its elements explicitly. Thus, for example,

{*Romeo,Juliet*}

denotes the set whose elements are Romeo and Juliet. And

{*Juliet*}

denotes the set whose sole member is Juliet. Following card-players' terminology, for any individuals *a*, *b*, the set {*a*} is called the *singleton* of *a*, and the set {*a*, *b*} the *doubleton* of *a*, *b*.

The formation of sets by explicit listing of elements however, of no use when the number of members of the set we are trying to specify is infinite, or finite but excessively large. To specify such sets we must instead state the *characteristic property* that an object must have to be a member of the set. *Predicates*, or *properties* are used for this purpose. For example, consider the predicate *x is Canadian*. This predicate is defined whenever x is a human being, in other words, over the universe of discourse consisting of all people. Then we write

$$\{x\text{: } x \textit{ is Canadian}\}$$

for the set of all people who are Canadian. Similarly,

$$\{x\text{: } x \textit{ is divisible by } 2\}$$

denotes the set of all numbers that are even.

In general, given a universe of discourse U, a predicate P is said to be *defined* over U if, whenever, $x \in U$, it makes sense to ask whether x satisfies P. Thus the predicate *x is Canadian* is defined over the universe of discourse is consisting of all human beings but not over the universe of discourse consisting of all natural numbers. If P is a predicate defined over a universe of discourse U, we customarily write $P(x)$ for the assertion that the object x (from U) *satisfies* the predicate P. If P is a predicate defined over U, we write

$$\{x\text{: } P(x)\}$$

to denote the set of all elements of U such that $P(x)$. This set is called the set *determined* by P. Notice that we then have

$$\text{for any } a,\ a \in \{x: P(x)\} \Leftrightarrow P(a).$$

We shall introduce the convenient abbreviation $\forall$ for *any, all, or every.* Thus $\forall x$ will stand for *for all* x, and, for any statement p involving the variable x, $\forall x$p will stand for, *for all x,* p holds. Thus the assertion above may be written

$$\forall a\ [a \in \{x: P(x)\} \Leftrightarrow P(a)].$$

It is also convenient to have a notation for the *empty* set, that is, the set which has *no* members. We use the symbol $\varnothing$ to denote this set..In general, we can define $\varnothing$ to be the set $\{x:\ P(x)\}$ where P is any predicate, such as $x \neq x$, which is not satisfied by any individual. For definiteness we define

$$\varnothing = \{x:\ x \neq x.\}$$

Notice that then we have,

$$\forall x\ (x \in \varnothing \Leftrightarrow x \neq x)$$

from which it follows that

$$\forall x\ (x \notin \varnothing).$$

In other words, $\varnothing$ has no members and so is truly empty.

Two sets A, B are said to be *equal*, and (as usual) we write $A = B$ if they have the same members, that is, if

$$\forall x\ (x \in A \Leftrightarrow x \in B).$$

If the sets A and B are determined by predicates P and Q defined over a common universal set U, that is, if A is $\{x: P(x)\}$ and B is $\{x: Q(x)\}$, then

$$A = B \Leftrightarrow \forall x\ [P(x) \Leftrightarrow Q(x)].$$

That is, two sets are equal exactly when their determining predicates are equivalent. This observation is constantly employed in establishing the equality of sets.

Notice that, for any set A,

$$A = \varnothing \Leftrightarrow \forall x\ (x \notin A)$$

For

$$\begin{aligned} A = \varnothing &\Leftrightarrow \forall x\ (x \in A \Leftrightarrow x \in \varnothing) \\ &\Leftrightarrow \forall x\ (x \in A \Leftrightarrow x \neq x) \\ &\Leftrightarrow \forall x\ (x \in A \Leftrightarrow \mathrm{f}) \\ &\Leftrightarrow \forall x\ (x \notin A) \end{aligned}$$

If A and B are sets, we say that A is a *subset* of B, or that A is *included* or *contained* in B and write

$$A \subseteq B$$

if every member of A is a member of B, that is, if

$$\forall x\ (x \in A \Rightarrow x \in B).$$

For example,

$$A = \{1, 2, 3\} \subseteq \{0, 1, 2, 3\} = B.$$

Notice that this is *not* the same as $A \in B$, since the elements of B are 0, 1, 2, 3 and A is not one of these.

Clearly

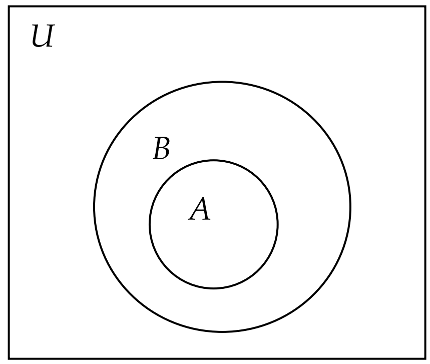
$$A = B \Leftrightarrow (A \subseteq B \wedge B \subseteq A).$$

If A and B are determined by predicates P and Q defined on a universal set U, then

$$A \subseteq B \Leftrightarrow \forall x\ [P(x) \Rightarrow Q(x)].$$

Each predicate P defined on a universal set U determines a subset of U, namely $\{x: P(x)\}$. And conversely, each subset A of U determines a predicate defined on U, namely the predicate $x \in A$. In view of this predicates defined on a universal set and subsets of that set "amount to the same thing".

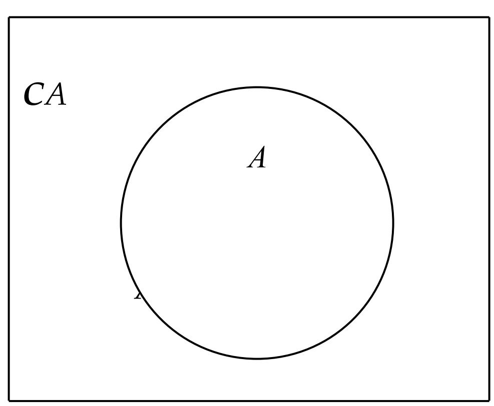
As is the case for propositional logic, it is often convenient to depict relationships between sets by means of Venn diagrams. For example, the diagram below depicts the relation $A \subseteq B$. The universal set U is represented by the square and the sets A and B by the regions within the square.



If A is a set, considered as a subset of a universal set U, its *complement* **C**A is defined by

$$\mathbf{C}A = \{x\colon x \notin A\}.$$

In the diagram below, A is represented by the circle. It can be seen from this diagram that **C**A *depends on the universal set* U



For example, if A is the set of positive natural numbers, and U the set of all natural numbers, then $\mathbf{C}A$ is {0}, while if U is the set of all integers, then A is the set {..., –2 ,–1, 0}.

If A and B are sets, their *union* $A \cup B$ and *intersection* $A \cap B$ are defined by

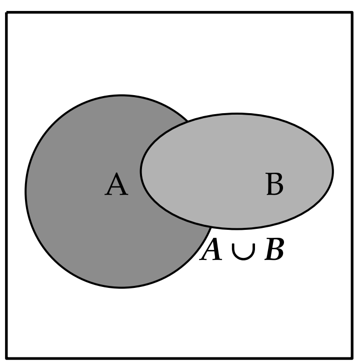
$$A \cup B = \{x: x \in A \vee x \in B\}, \quad A \cap B = \{x: x \in A \wedge x \in B\},$$

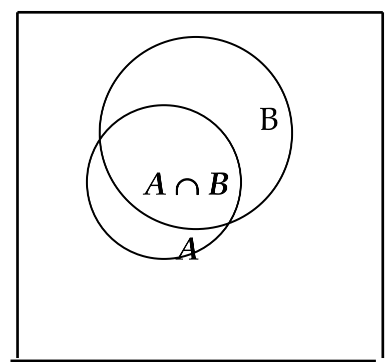
For example,

$$\{1,2,3\} \cap \{2,3,4\} = \{2,3\}, \ \{1,2,3\} \cup \{2,3,4\} = \{1,2,3,4\}, \ \{1,2,3\} \cap \{0,4\} = \varnothing$$

$$\{x: x \leq 0\} \cap \{x: x \geq 0\} = \{0\}$$

A





Two sets A and B are said to be *disjoint* if $A \cap B = \varnothing$.

Most of the laws of equivalence stated in section 1 of Chapter II have direct counterparts in set theory in which $\cup$ corresponds to $\vee$, $\cap$ to $\wedge$, **C** to $\neg$, U to t, and $\varnothing$ to f. For example, given a universal set U, the counterpart of the Law of Excluded Middle is

$$A \cup \mathbf{C}A = U.$$

To prove this we have to show that

$$\forall x\ (x \in A \cup \mathbf{C}A \Leftrightarrow x \in U).$$

Now because we are assuming that U is the universal set, every element x is automatically in U, so we are simply required to prove

$$\forall x\ (x \in A \cup \mathbf{C}A).$$

But this follows from the observation that

$$x \in A \cup \mathbf{C}A \Leftrightarrow x \in A \text{ or } x \in \mathbf{C}A \Leftrightarrow x \in A \text{ or } \neg(x \in A) \Leftrightarrow \text{t}.$$

The counterpart of the Law of Contradiction $\text{p} \wedge \neg\text{p} \equiv \text{f}$ is

$$A \cap \mathbf{C}A = \varnothing.$$

To prove this we have to show that

$$\forall x\ (x \in A \cap \mathbf{C}A \Leftrightarrow x \in \varnothing.).$$

But this follows from the observation that

$$x \in A \cap \mathbf{C}A \Leftrightarrow x \in A \wedge x \in A \Leftrightarrow \text{f} \Leftrightarrow x \in \varnothing.$$

The counterpart of the first de Morgan law $\neg(\text{p} \wedge \text{q}) \equiv \neg\text{p} \vee \neg\text{q}$ is

$$\mathbf{C}(A \cap B) = \mathbf{C}A \cup \mathbf{C}B.$$

To prove this we have to show that

$$\forall x\ (x \in \mathbf{C}(A \cap B) \Leftrightarrow x \in \mathbf{C}A \cup \mathbf{C}B.$$

For this we argue as follows:

$$x \in \mathbf{C}(A \cap B) \Leftrightarrow \neg(x \in A \cap B) \Leftrightarrow \neg(x \in A \wedge x \in B) \Leftrightarrow \neg(x \in A) \vee \neg(x \in B)$$
$$\Leftrightarrow x \in \mathbf{C}A \vee x \in \mathbf{C}B \Leftrightarrow x \in \mathbf{C}A \cup \mathbf{C}B$$

Here the step in red is justified by the first de Morgan Law itself.

Exercises

A1. Show that, for sets A, B, $A = B \Leftrightarrow (A \subseteq B \wedge B \subseteq A)$.

A2. Prove that $\varnothing$ is a subset of every set.

A3. Draw Venn diagrams to illustrate the relations **(i)** not $A \subseteq B$, **(ii)** A and B are *disjoint*.

A4. Prove that, for any subsets A, B, C of a universal set U: **(i)** $\mathbf{CC}A = A$ **(ii)** $A \subseteq A$; **(iii)** $A \cup \mathbf{C}A = U$; **(iv)** $A \cap \mathbf{C}A = \varnothing$; **(v)** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$; **(vi)** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; **(vii)** $A \cap (B \cap C) = (A \cap B) \cap C$; **(viii)** $A \cup (B \cup C) = (A \cup B) \cup C$; **(ix)** $\mathbf{C}(A \cap B) = \mathbf{C}A \cup \mathbf{C}B$; **(x)** $\mathbf{C}(A \cup B) = \mathbf{C}A \cap \mathbf{C}B$. Draw Venn diagrams to depict these.

A5. For subsets A, B of a universal set U, prove that the following are equivalent: **(a)** $A \subseteq B$, **(b)** $\mathbf{C}B \subseteq \mathbf{C}A$, **(c)** $A \cup B = B$, **(d)** $A \cap B = A$, **(e)** $A \cap \mathbf{C}B = \varnothing$ **(f)** $\mathbf{C}A \cup B = U$.

A6. Let $A - B$—the *relative complement of B in A*—denote the set $\{x: x \in A \wedge x \notin B\}$. Draw a Venn diagram to depict $A - B$. **(i)** Prove that $A - B = A - (A \cap B)$, $A = (A \cap B) \cup (A - B)$. **(ii)** Are the following always true? $(A - B) \cup B = A$, $(A - B) - B = A$.

A7. Define the *symmetric difference* of two sets A, B by $A \Delta B = (A - B) \cup (B - A)$. Prove that:

(i) $A \Delta B = B \Delta A$

(ii) $A \Delta \varnothing = A$

(iii) $A \Delta A = \varnothing$

(iv) $A \Delta (A \Delta B) = B$

(v) if A and B are disjoint, then $A \cup B = A \Delta B$.

2. Relations

The argument with which this chapter began contained an example of a *binary relation* between individuals, to wit *x loves y.* Logic (and everyday life) teems with relations. Here are some examples:

- the *marriage* relation between people
- the *sisterhood* relationship between people
- the relation of a person being *taller than* another
- the relation of a number being *greater than* another
- the relation of a star being *farther away* from the Sun than another.

In each of these examples the related things are of the same sort, persons, numbers, stars, etc. But relations can also relate things of *different* sorts, for example,

- the *occupamcy* relation between people and dwellings of a person being an *occupant* of a dwelling
- the *nationality* relation between people and countries of a person being a national of a country
- the *incidence* relation between lines and points in a plane of a line passing through a point
- the *membership* relation between individuals and sets.

We have seen that predicates defined on a given set are naturally correlated with subsets of the set. But how are *relations* to be presented in terms of sets? The answer is to introduce the idea of an *ordered pair* of individuals. An ordered pair is an object built from two individuals given in a specified *order.* Thus given any two individuals $a, b,$ we shall suppose that we can form an object (a, b) called the *ordered pair* with *first component a* and *second component b*. The idea is to represent binary relations as *properties of ordered pairs.* For example, the relation *x loves y* is to be regarded as the property - write it as **Love -** of ordered pairs (a, b) of people defined by **Love** $((a, b))$ if and only if a loves $b.$

If a and b are distinct, the ordered pair (a, b) will be held to be different from the ordered pair (b, a). It follows that (a, b) cannot be the same as the doubleton set $\{a, b\}$, since always $\{a, b\} = \{b, a\}$. Generally speaking, ordered pairs (a, b) and (c, d) are said to be *equal* precisely when their first and second components are *pairwise identical*, that is, if $a = c$ and $b = d.$ Thus

$$(a, b) = (c, d) \Leftrightarrow a = c \wedge b = d.$$

Now, given two sets $A, B,$ we may form the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. This is called the *Cartesian product* of A and B and is denoted by $A \times B$. From the definition of $A \times B$ it follows immediately that

$$(x, y) \in A \times B \Leftrightarrow x \in A \;\&\; y \in B.$$

Cartesian products arise frequently in mathematics, and implicitly in logic. For instance, since each point in the Euclidean plane can be identified by an ordered pair of coordinates, the plane itself can be described as the Cartesian product of two lines. (This fact was essentially known to Descartes—hence the term "Cartesian".)

Now, as we have suggested, a binary relation is to be regarded as a *property of ordered pairs*. In other words, a binary relation is *a predicate defined on a Cartesian product of two sets.* Since a predicate defined on a set amounts to the same thing as a subset of that set, it follows that *a binary relation is essentially just a subset of a Cartesian product of two sets.* This is best illustrated by an example.

Consider the binary relation of *sisterhood* : *y is x's sister.* Writing **F** for the set of females, and **H** for the set of all people, the sisterhood relation may be identified with the set S of ordered pairs (x, y) in which x is a male or a female, y is a female, and y is x's sister. Thus S is a subset of $\mathbf{H} \times \mathbf{F}$: we naturally say that S is a *relation between* **H** *and* **F**. In general, a subset of the Cartesian product $A \times B$ of two sets A and B is called a (binary) *relation between A and B*. In the case at hand, the susterhood relation S is clearly also a subset of $\mathbf{H} \times \mathbf{H}$: it is accordingly natural to say that *S is a relation on* **H**. Generally, a subset of a Cartesian product $A \times A$ is called a (binary) *relation on A*.

If R is a binary relation, it is common practice to write Rab or aRb for $(a, \mathrm{b}) \in R$. aRb is read "b bears the relation R to a".

If A, B, C, D are sets such that $\mathrm{A} \subseteq C$ and $B \subseteq D$, then any relation between A and B is also a relation between C and $D.$

We occasionally encounter relations linking more than two terms. For example, consider the set P of points on a given straight line. On P we have the *ternary* relation B B of *betweenmess* which holds of three points a, b, c when *b is between a* and c. Or consider the set **N** of natural numbers. On **N** we have the *quaternary* relation which holds of four numbers m, n, p, q when $mn = pq.$

In general, we may wish to consider, for each natural number $n \geq 2$, the idea of an *n-ary* relation which relates n terms. This can be done in set theory by introducing ordered triples (a, b, c), ordered quadruples (a, b, c, d)—in general, for any $n \geq 2$, ordered n-tuples $(a_1, ..., a_n)$. As in the case of ordered pairs, all we need to know about these is that

$$(a_1,\ldots, a_n) = (b_1,\ldots, b_n) \Leftrightarrow a_1 = b_1 \wedge \ldots \wedge a_n = b_n.$$

Given n sets $A_1, \ldots, A_n$, the set of all ordered n-tuples $(a_1,\ldots, a_n)$ with $a_1 \in A_1,\ldots, a_n \in A_n$ is called the *Cartesian product of* $A_1, \ldots, A_n$ and denoted by $A_1 \times \ldots \times A_n$. If all the A_i's are identical with a fixed set A, then $A_1 \times \ldots \times A_n$ is written A^n and called the n^{th} (Cartesian) *power of* A.

A subset of $A_1 \times \ldots \times A_n$ is called a *relation among* $A_1, \ldots, A_n$ and a subset of a Cartesian power A^n an *n-ary relation on A.*

If R is an n-ary relation, it is customary to write $Ra_1\ldots a_n$ for $(a_1, \ldots, a_n) \in R$.

Exercises

B1. Prove that, for any set A, $A \times \varnothing = \varnothing$,

B2. If $C \neq \varnothing$, prove that $A \subseteq B \Leftrightarrow A \times C \subseteq B \times C$. Why must we insist that $C \neq \varnothing$?

B3. Prove that:

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$,

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$,

(iii) $A \times (B - C) = (A \times B) - (A \times C)$.

(iv) $(A \times B) - (C \times D) = ((A - C) \times B) \cup (A \times (B - D))$.

3. Composite, Inverse, and Kinship Relations

Kinship relations are very familiar and provide particularly illustrative examples of relations. Let us write **Humans** for the set of human beings, **Males** for the set of males, and **Females** for the set of females. Then we have the following familiar kinship relations:

- the *parenthood* relation P on **Humans***:* $aPb \Leftrightarrow b$ is a *parent* of a
- the *childhood* relation C on **Humans**: $aCb \Leftrightarrow b$ is a *child of a*
- the *fatherhood* relation F between **Humans** and **Males***:* $aFb \Leftrightarrow b$ is the *father of a*
- the *motherhood* relation M between **Humans** and **Females***:* $aMb \Leftrightarrow b$ is the *mother of a*
- the *brotherhood* relation B between **Humans** and **Males***:* $aBb \Leftrightarrow b$ is a *brother of a*
- the *sisterhood* relation S between **Humans** and **Females***:* $aSb \Leftrightarrow b$ is a *sister of a*
- the *sibling* relation S^* on **Humans**: $aS^*b \Leftrightarrow b$ is a *sibling of a*
- the *unclehood* relation U between **Humans** and **Males***:* $aUb \Leftrightarrow b$ is an *uncle of a*
- the *aunthood* relation A between **Humans** and **Males***:* $aA\,b \Leftrightarrow b$ is an *aunt of a*

All of the above relations may also be regarded as relations on **Humans.**

Now the last two of these relations are defined in terms of the others. For example, is a parent's sister. Thus

$$aAb \Leftrightarrow \textit{for some } x\ (aPx \wedge xSb)$$

We express this by saying that A is the *composite* of P and S. We write this composite as PS. Thus we have $A = PS$. Similarly, $U = PB$.

In general, suppose we are given sets A, B, C , and relations R between A and B, and S between B and C. We define the *composite relation RS* between A and C to be the set of all pairs (*a, c*) with $a \in A$, $c \in C$ such that, for some $b \in B$, we have *aRb* and *bSc.* In other words,

$$a(RS)c \Leftrightarrow \textit{for some } b\ (aRb \wedge bSc)$$

The relations *P* and *C* of parenthood and childhood are *inverse* to one another in that $aPb \Leftrightarrow bCa$. In general, if *R* is a relation between sets *A* and *B*, the *inverse* relation R^{-1} between *B* and *A* is defined to be the set of all pairs (*b, a*) such that *aRb.* Thus we have

$$b\, R^{-1}a \Leftrightarrow = aRb.$$

The inverse relation R^{-1} may be regarded as the relation *R* `viewed in a mirror``. In the case of the relations *P* and *C* of parenthood and childhood we have $P^{-1} = C$ and $C^{-1} = P$.

Exercises

C1. Recall the kinship relations *P, C, F, M, S** introduced a few pages back. **(i)** Identify the composite relations *PC, PP, PF, PM, MF, FM, MM, FF.* **(ii)** The composite relation *PS*C* has a familiar name. What is it?

C2. Let *R* and *S* be relations between sets *A* and *B*. Prove that $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ and $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

C3. Let *R* be a relation between *A* and *B,* and *S* a relation between *B* and *C.* Prove that $(R^{-1})^{-1} = R$ and $(RS)^{-1} = S^{-1}R^{-1}$.

4. Equivalence Relations

The idea of *equivalence* is of universal importance: in fact all abstractions met with in everyday life involve this idea. For instance, a hitchhiker seeking a ride in a passing vehicle will ignore all the properties of such a vehicle except its mobility: as far as he or she is concerned, all moving vehicles are equivalent, regardless of type.

This idea of equivalence is given precise expression in set theory through the concept of *equivalence relation*. An *equivalence relation* on a set A is a relation R on A satisfying the following conditions for all a, b, c in A:

(i) *reflexivity:* aRa,

(ii) *symmetry:* $aRb \Leftrightarrow bRa$

(iii) *transitivity:* $(aRb \wedge bRc) \Rightarrow aRc$.

As examples of equivalence relations we have:

- The *identity relation* I_A on any set A consisting of all ordered pairs of the form (a, a) with a in A.
- The relation S on the set of all human beings defined by $aSb \Leftrightarrow a$ and b have the same parents.
- The relation R on the set of natural numbers defined by $mRn \Leftrightarrow m$ and n have the same remainder when divided by 2.
- The relation R on the set of straight lines in a plane defined by $\ell P \ell' \Leftrightarrow \ell$ and ℓ' are parallel.
- The relation of *logical equivalence* on the set of all statements of propositional logic.

If R is an equivalence relation on a set A, and $a \in A$, the *equivalence class* of R *containing a*, written a_R, is the set comprising all members of A which bear the relation R to a, that is, $a_R = \{x: aRx\}$.For example, the equivalence classes of the first three relations above are, respectively. all sets of the form $\{a\}$ for $a \in A$; all families of siblings; the set of even numbers and the set of odd numbers.

Exercises

D1. Certain relations are "almost" equivalence relations in being symmetric and transitive but not reflexive. Which of the kinship relations on p.96 are "almost" equivalence relations in this sense?

D2. Prove that, for any equivalence relation, and any two equivalence classes X and Y, either $X = Y$ or X and Y are disjoint.

D3. Let R be a relation on a set A. Prove that **(i)** R is symmetric $\Leftrightarrow R = R^{-1}$ **(ii)** R is reflexive $\Leftrightarrow$[9] $I_A \subseteq R$; **(iii)** R is irreflexive $\Leftrightarrow R \cap I_A = \varnothing$; **(iv)** R is transitive $\Leftrightarrow RR \subseteq R$.

5. Rankings and orderings

The idea of a *ranking*, or *ordering*, is another important concept in everyday life. Whenever we make a comparison, for example, when we say that something is smaller than, or heavier than, or less interesting than something else, we are implicitly employing the idea of ranking or ordering with respect to the property in question.

In set theory this idea is captured by making the following definition. A *strict ranking relation*, or simply a *strict ranking* on a set A is a relation R on A which is both

[9] Recall that I_A is the *identity* relation on A consisting defined by $xI_Ay \Leftrightarrow x = y$.

- *transitive:* $(a\ R\ b$ and $b\ R\ c) \Rightarrow a\ R\ c$

  and

- *irreflexive:* not $a\ R\ a$,

A strict ranking R is *total* it satisfies the condition of

- *totality:* if $a \neq b$, then $a\ R\ b$ or $b\ R\ a$.

One of the most familiar strict ranking relations (which is actually total) is the *(strict) less than* relation < on the set **N** of natural numbers. For this reason it is customary to use the symbol < to denote an arbitrary strict ranking or ordering relation.

Examples.

(i) The relations of being *taller than, shorter than, older than,* etc., on the set of human beings are all strict ranking relations. Each is total.

(ii) Consider the set L of geographical locations in a given country. The relation *due north of* is a (non- total) strict ranking on L.

Now consider such relations as *no taller than, no heavier than, no larger than,* etc. Writing R for any one of these relations, it is clear that R is transitive and

- *reflexive* $a\ R\ a$

An arbitrary relation R on a set A which is reflexive and transitive is called a *ranking* on A. If in addition R is

- *antisymmetric* $(a\ R\ b$ and $b\ R\ a) \Rightarrow a = b$,

then R is called an *ordering* on A. Naturally, a ranking or ordering having the property of totality is called a *total* ranking or ordering.

A set given together with a ranking or ordering on it is called a *ranked,* or *ordered set.*

One of the most familiar ranking relations (actually a total ordering) is the *equal to or less than* relation $\leq$ on the set $\mathbf{N}$ of natural numbers. For this reason it is customary to use the symbol $\leq$ to denote an arbitrary ranking or ordering relation.

If A is a ranked set with ranking relation $\leq$, and a, b are elements of with $a \leq b$, we shall say that a is *(ranked) below b.*

Examples.

(i) Consider the set A of soldiers in an army. Each soldier is assigned a *military rank*: private, sergeant, lieutenant, major, colonel, general, etc. The set M of military ranks is (totally) ordered by convention, placing the rank of sergeant below that of lieutenant, that of major below that of colonel, etc. The assignment of military ranks to members of A then leads to a total ranking on A in which all the sergeants are ranked below all the lieutenants, etc.

(ii) The *equal to or less than* relation $\leq$ on the set **N** of natural numbers is a total ordering on **N.**

(iii) The relation $\mid$ of *divisibility* on **N,** defined by $m \mid n \Leftrightarrow m$ is a divisor of n, is a non-total ordering on **N.**

(iv) The *entailment* relation $\vDash$ on the set S of all statements of propositional logic is a non-total ranking on S.

(v) Given a collection **S** sets, the relation $\subseteq$ of *inclusion* between the sets in **S** is an ordering on **S.** This relation may or may not be total. For example, the inclusion relation collection $\{\varnothing, \{1\}, \{1, 2\}\}$ is totlal, but the inclusion relation on the collection $\{\varnothing, \{1\}, \{2, 3\}\}$ is not.

Exercises

E1. If R is a ranking (or ordering) on a set A, prove that its inverse R^{-1} is also a ranking (or ordering) on A.

E2. If R is a relation on a set A, write $\not R$ for the relation on A defined by $a \not R\, b \Leftrightarrow \neg(a\ R\ b)$. Prove that R s a strict ranking $\Leftrightarrow \not R$ is a ranking. Illustrate when R is a relation such as *taller than* and the like.

E3. If R is a relation on a set A, write $R^{\uparrow}$, $R^{\downarrow}$ for the relations defined by $x R^{\uparrow} y \Leftrightarrow x R y \vee x = y$; $x R^{\downarrow} y \Leftrightarrow x R y \wedge x \neq y$. Prove that **(i)** if R is a strict ranking, then $R^{\uparrow}$ is a ranking; **(ii)** if R is an ordering, then $R^{\downarrow}$ is a strict ranking. Illustrate when R is a relation such as *taller than* and the like.

E4. Suppose that R is a ranking on a set A. Prove that the relation S defined by $aSb \Leftrightarrow (aRb \wedge bRa)$ is an equivalence relation on A. What are the equivalence classes of this equivalence relation when R is the preordering specified in **(i)?**

6. Functions and Operations

Intuitively, a *function* from a given set A to a given set B is a device which assigns a unique element of B to each element of A. In set theory this idea is given a precise formulation in terms of *relations*. Thus we define a *function from A to B* to be a relation f between A and B possessing the following property:

for any $a \in A$, there is a *unique* $b \in B$ for which afb.

In this situation we write $f: A \to B$. A is called the *domain*, and B the *codomain*, of f. For $x \in A$, we also write $f(x)$ or fx for the unique element b of B such that xfb: $f(x)$ is called the *value* of f at x. A function $f: A \to A$ is called an *operation* on A.

Clearly, if $f: A \to B$ and $g: A \to B$, then

$$f = g \text{ if and only if } f((x) = g(x) \text{ for all } x \in A.$$

Examples. (i) The fatherhood relation F on the set H of all human beings defined by

$$aFb \Leftrightarrow b \text{ is the father of } a$$

is an operation on H.

(ii) The relation R between the set H of human beings and the set $\mathbb{N}$ of natural numbers defined by

$$aRn \Leftrightarrow n \text{ is the number of children of } a$$

is a function from H to $\mathbb{N}$.

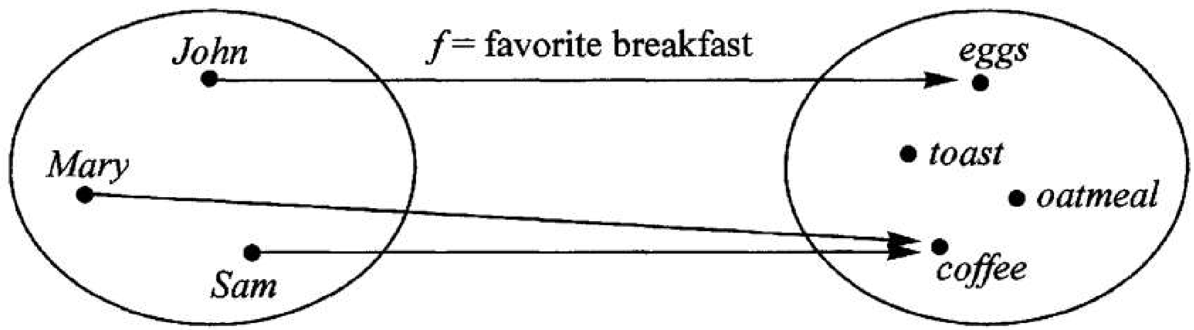
(iii) The relation R on $\mathbb{N}$ defined by

$$mRn \Leftrightarrow n = m^2$$

is an operation on $\mathbb{N}$.

(iv) For any set A, the identity relation on A is an operation on A. As such, it is called the *identity operation* on A and denoted by 1_A. Clearly $1_A(x) = x$ for any $x \in A$.

It is often helpful to depict functions by diagrams, as in



Suppose that $f: A \to B$ and $g : B \to C$. For each $x \in A$, we have $f(x) \in B$ and so $g(f(x)) \in C$. The set of all ordered pairs of the form $(x, g(f(x))$ for $x \in A$ clearly defines a function from A to C: it is called the *composite* of g and f and is written $g \circ f$. [10]Thus $g \circ f: A \to C$ and, for each $x \in A$, $(g \circ f)(x) = g(f(x))$.

Functions or operations can have *more than one variable.* For example, the operation of addition on the set of natural numbers and the operation of conjunction on the set of all statements are *binary* operations in that both involve *two* variables. we shall confine our attention to functions and operations with a single variable.

Exercises

F1. Let $f: A \to B$. Prove that $f = 1_B \circ f = f \circ 1_A$.

F2. Give examples of operations f, g on the set $\{0, 1\}$ such that $f \circ g \neq g \circ f$.

F3. A function $f: A \to B$ is said to be *one-to-one* if, for any x, y in A, $f(x) = f(y) \Rightarrow x = y$.
(i) Which of the functions in the examples above are one-to-one?

---

[10] In fact, if we regard f and g as *relations*, then $g \circ f$ is identical with the composite relation fg.

(ii) If $f: A \to B$ and $g: B \to C$ are both one-to-one, prove that $g \circ f$ is also.

F4. Let $f: A \to B$ be a function, and X a subset of A. The *image* of X under f is the set $f[X]$ consisting of all elements of B of the form $f(x)$ with x in X. The function $f: A \to B$ is said to be *onto* B if $f[A] = B$.

(i) Draw a "mapping" diagram to illustrate $f[X]$.

(ii) If $X, Y \subseteq A$, prove that $X \subseteq Y \Rightarrow f[X] \subseteq f[Y]$.

(iii) If $X, Y \subseteq A$, prove that $f[X \cup Y] = f[X] \cup f[Y]$. Does this remain true when "$\cup$" is replaced by "$\cap$"?

(iv) If $f: A \to B$ and $g: B \to C$ are both onto, prove that $g \circ f: A \to C$ is also.

F5. Let $g: A \to B$ be a function, and Y a subset of B. The *preimage* of Y under g is the set $g^{-1}[Y] = \{x: g(x) \in Y\}$.

(i) Draw a "mapping" diagram to depict $g^{-1}[Y]$.

(ii) If $Y, Z \subseteq B$, prove that $g^{-1}[Y \cup Z] = g^{-1}[Y] \cup g^{-1}[Z]$, $g^{-1}[Y \cap Z] = g^{-1}[Y] \cap g^{-1}[Z]$, and $g^{-1}[B - Y] = A - g^{-1}[Y]$.

(iii) Prove that, for any $X \subseteq A$, $X \subseteq g^{-1}[g[X]]$. If g is one-to- one, prove that $X = g^{-1}[g[X]]$.

(iv) Prove that, for any $Y \subseteq B$, $g[g^{-1}[Y]] \subseteq Y$,. If g is onto, prove that $g[g^{-1}[Y]] = Y$.

VI. QUANTIFICATIONAL LOGIC

1. Predicates, Relations and Quantifiers in Logic

Let us recall the argument with which we began the previous chapter, namely

1. All Cretans love all animals.

2. All horses are animals.

3. Epimenides is a Cretan.

∴ *4. Someone loves all horses.*

In order to symbolize this argument (and others like it) we need to enlarge our logical vocabulary. Thus, as in algebra, it is natural to introduce *variables* x,y,z,... to refer to arbitrary individuals, and then to write, for example, "Ax" for "x is an animal", "Cx" for "x is a Cretan", "Hx" for "x is a horse", "Lxy" for "x loves y", and "e" for "Epimenides". The symbols A, C and H are *predicate symbols*, L is a *relation symbol*, and e a *name*. Finally we introduce two symbols ∀ and ∃ called the *universal* and *existential quantifier*, respectively: the expression "∀x" will symbolize the phrase "for all (or any) x", and "∃x" the phrases "for some x", or, equivalently, "there exists x".

To put our argument in symbolic form, we first write it in the following way:

1'. For any individual x, *if* x *is a Cretan, then for any individual* y, *if* y *is an animal, then* x *loves* y.

2'. For any individual x, *if* x *is a horse, then* x *is an animal.*

3'. Unchanged.

∴ *4'. For some individual* x, *for all individuals* y, *if* y *is a horse, then* x *loves* y.

Now 1'- 4' can be symbolized directly in terms of our enlarged logical vocabulary thus:

$$1''.\ \forall x[Cx \to \forall y(Ay \to Lxy)]$$

2". $\forall x(Hx \rightarrow Ax)$

3". Ce

4". $\therefore \exists x \forall y(Hy \rightarrow Lxy)$.

We frequently need to assert that two names refer to the *same*, or *different*, things, as, for instance, in the (correct) inference

1. Yesterday I was home

2. Monday I was out.

$\therefore$ *3. Yesterday was not Monday.*

In order to symbolize this inference we need to introduce into our logical vocabulary a symbol for *identity* or *equality* of individuals. For this we employ the usual equality sign =, which we agree is to be written in between variables or names, as in x = y, a = x or a = b. Writing "m" for "Monday", "n" for "yesterday", and "Hx" for "I was home on day x", we still lack a way of symbolizing statement 3. Now our inference may be symbolized

$$\frac{\begin{array}{c} Hn \\ \neg Hm \end{array}}{\neg(n = m)}$$

The logical system associated with the enlarged vocabulary of variables, predicate and relation symbols, identity symbol, names, and quantifiers is called *quantificational logic (with identity).* Statements formulated within this vocabulary will be called *quantificational statements*.

Let us be precise. A *vocabulary* for quantificational logic (or simply a *quantificational* vocabulary) consists of the following symbols:

- *variables* x, y, z, ... We shall assume that there are as many of these as required.

- *names* (or *constant symbols*) m, n, .. Again, we shall assume that there are as many of these as required.
- *predicate symbols* P, Q, ... (possibly none of these)
- *relation symbols* R, S, ... (again, possibly none of these). Each relation symbol is assigned a natural number $d \geq 2$ called its *multiplicity.* A relation symbol of multiplicity d will be called *d-ary.* A 2-ary relation symbol will be called *binary.*
- *identity symbol* =
- *logical operators* ∧, ∨, ¬, →, ↔
- *quantifiers* ∀, ∃
- *parentheses* (,)

Statements in quantificational logic are defined as follows.

1. The following are statements: **(i)** any predicate symbol followed by a name (for example Pm); **(ii)** any expression of the form m = n, where m and n are names; **(ii)** any d-ary relation symbol followed by d names (for example Rmno...). Statements formed under this rule are called atomic statements, and statements formed under clause **(ii) identity statements.** [11]

2. If p and q are statements, so are (¬p), (p ∧ q), (p ∨ q), (p → q), (p ↔ q).

3. If p is a statement, write p[n/x] for the result of replacing in p a particular name n by a variable x that does not appear in p. Then both (∀x) p[n/x] and (∃ x)p[n/x] are statements. For simplicity we shall usually write these as (∀x)p(x) and (∃x)p(x) respectively. These are known as *quantified statements*[12].

4. Nothing counts as a statement unless its being so follows from rules **1** to **3.**

We usually abbreviate ¬(x = y) to x ≠ y.

Starting with a quantified statement (∀x) p(x), if we suppress the quantifier (∀x) we obtain the expression p(x). An expression obtained in this way is called a *formula.* It is important to remember that, in general, are not statements. Any formula p(x) can be converted into a statement by substituting m for x at each of its occurrences in p(x). The resulting statement is written p(m). Thus, for example, starting with the quantified statement (∀x)Rxn, p(x) is Rxn and p(m) is Rmn.

To illustrate the precise mode of operation of rule **3,** consider the process of forming the statement (∀x)(∃y)Rxy. Starting with the sentence Rmn, we first replace "n" by "y" to obtain the expression (not a statement) Rny. To this expression we prefix "(∃y)" to obtain (by rule **3**) the statement (∃y)Rmy. In this latter statement we now replace "m" by "x" (which does not appear in it), thereby obtaining the expression (∃y)Rxy (again, not a statement). To this expression we prefix "(∀x)", thus finally obtaining the desired statement (∀x)(∃y)Rxy.

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[11] Note that, for the construction of atomic statements to go through our vocabulary must include sufficiently many names We have already allowed for this.

[12] Again, to allow the formulation of quantified statements to go through we require the presence of sufficiently many names.

As with propositional statements, we shall feel free toomit parentheses in quantificational statements when there is no danger of ambiguity. In particular we shall omit the parentheses around $(\forall x)$ and $(\exists x)$.

Exercises

A1. Introduce an appropriate vocabulary and use it to symbolize the following statements:

(a) Some know all.

(b) Some know all who know them.

(c) Some know all who know themselves.

(d) All who know some know some who know all.

(e) No one who knows someone Susie knows, knows all who know Susie.

(f) All who know everyone Susie knows know some who know Susie.

(g) Only Susie knows Susie.

(h) Susie is the only one who knows everyone who knows her.

A2. Introduce an appropriate vocabulary and use it to symbolize the following statements:

(a) I like myself.

(b) Someone likes me.

(c) No one likes me.

(d) Everyone likes someone.

(e) I like myself and no one else.

(f) Someone likes everyone who likes me.

(g) I dislike anyone who dislikes me.

(h) Someone likes everyone I dislike.

(i) Someone who likes me likes everyone.

(j) Everyone who likes someone I like likes no one I dislike.

(k) I am the only one disliked by everyone.

A3. Introduce an appropriate vocabulary and use it to symbolize the following statements:

(a) Rob is taller than everyone.

(b) Everyone is taller than someone.

(c) No one's taller than everyone.

(d) Everyone's taller than everyone *else*.

(e) Rob is taller than no more than one person.

(f) No two people are taller than each other.

(g) If anyone's taller than Rob, Rob is.

(h) Someone is taller than everyone Rob isn't taller than.

(i) Someone taller than Rob is taller than everyone taller than Rob.

(j) Everyone taller than someone is taller than someone taller than everyone.

A4. Introduce an appropriate vocabulary and use it to symbolize the following statements:

(a) Some love anyone.

(b) No one loves all who love them.

(c) No lonely lover loves any other lonely lover.

(d) Some love all who love themselves.

(e) Some lonely loners love only themselves.

(f) All lovers love some who love all.

(g) Nobody who loves somebody loves somebody who loves nobody.

(h) All who love all love all lovers.

(i) Only the lonely love only the lonely.

2. Interpretations, Validity and Satisfiability

Let us recall the definition of valid inference in propositional logic:

- **an inference is *valid* if, under any truth valuation of the statement letters occurring in it, whenever the premises of the inference are all true, so is its conclusion.**

We want to extend this definition to quantificational logic. In order to do this we must extend to quantificational statements the concept of truth valuation and the concepts of truth (and falsity) under a truth valuation. We shall make use of some simple concepts from set theory.

For quantificational statements the concept of truth valuation is replaced by that of an *interpretation* of a quantificational vocabulary An *interpretation* **I** of a quantificational vocabulary consists of:

(1) A *nonempty set* A called the *domain* or *universe* of **I**;
(2) an assignment, to each name m, of a definite *element* of *A*, which we shall denote by $m^{\mathbf{I}}$ and call the *interpretation under* **I** of m;

(3) an assignment, to each predicate symbol P, of a definite *subset* of *A*, which we shall denote by $P^{\mathbf{I}}$ and call the *interpretation under* **I** of P;
(4) an assignment, to each *d*-ary relation symbol R, of a definite *d*-ary *relation* on A, which we shall denote by $R^{\mathbf{I}}$ and call the *interpretation under* **I** of R

Once an interpretation of a quantificational vocabulary has been fixed, we can assign a definite *truth value* (t or f) to each quantificational statement based on that vocabulary by assigning predicate and relation symbols the meanings specified by the given interpretation, and giving the identity symbol, the logical operators and the quantifiers their customary meanings. Thus the identity symbol = is understood to mean *the same as;* logical operators $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$ are understood as meaning *and, or, not, if...then* and *if and only it,* respectively; the quantifier $\forall$ is understood as meaning *for all* (or *for every* or *for any*); and the quantifier $\exists$ as meaning *there exists* (or *there is* or *for some*). Truth values of quantificational statements under a given interpretation **I** are calculated by meanings of the following rules:

- Pm is true under **I** if and only if $m^{\mathbf{I}} \in P^{\mathbf{I}}$
- m = n is true under **I** if and only if $m^{\mathbf{I}} = n^{\mathbf{I}}$.
- Rmno... is true under **I** if and only if $Rm^{\mathbf{I}}n^{\mathbf{I}}o^{\mathbf{I}}$... holds
- $\neg$p is true under **I** if and only if p is false under **I**
- p $\wedge$ q is true under **I** if and only if both p and q are true under **I**
- p $\vee$ q is true under **I** if and only if p is true under **I** or q is true under I
- p $\rightarrow$ q is true under **I** if and only if either p is false under **I** or q is true under **I**
- p $\leftrightarrow$ q is true under **I** if and only if p and q have the same truth value under **I**

To state the rules of interpretation for statements with quantifiers we require some new notation. For a name n and an element *a* of the domain of **I**, write **I**[n/*a*] for the interpretation which coincides with **I** *except in assigning the element a to the name* n. Thus the interpretation of n under **I**[n/*a*] is *a*. Now the rules for interpreting statements with quantifiers are the following. Here we suppose that, for some name m occurring in p, ∀xp(x) is ∀xp[n/x] and ∃xp(x) is ∃xp[n/x].

- ∀xp(x) is true if under **I** and only if, for *any a* in the domain of **I,** p is true under **I**[n/*a*]
- ∃xp(x) is true under **I** if and only if, for *some a* in the domain of **I,** p is true under **I**[n/*a*]

An informal way of understanding how truth values are assigned to quantified statements is this. Given a formula p(x), and an element *a* of the domain of the interpretation **I,** let us say that p(*a*) is true under **I** if *p* is true under **I**[n/*a*]. Then A universally quantified statement ∀xp(x) is true under **I** exactly when p(*a*) is true under **I** for any element *a* of the domain of **I,** in other words no matter what element of the domain x stood for. An existentially quantified statement ∃xp(x) is true if there is an element of the domain such that, if x stood for it, p(x) Is true.

In practice, the calculated of truth values under interpretations is carried out by "translating" formal quantificational statements into meaningful, informal, assertions whose truth values can be easily determined. The following example illustrates the procedure.

Suppose our vocabulary contains a binary relation symbol L and a name m. Consider the interpretation **I** whose universe is the set of natural numbers {1,2,3,...}, m is interpreted as the number 1, and L is interpreted as the "less than" relation <. Let us determine the truth values of the statements

$$(1)\ \exists xLmx \quad (2)\ \forall xLmx \quad (3)\ \exists x(Lmx \wedge Lxm) \quad (4)\ \forall x\forall y(Lxy \rightarrow x \neq y)$$
$$(5)\ \forall x\exists yLxy$$

under I. To do this we use **I** to "translate" these statements into assertions possessing truth values as follows. The "translation" of (1) is:

There is a number x such that 1 < x

which is obviously *true*. The translation of (2) is

For all numbers x, 1 < x,

which is clearly *false* since it is not the case that 1 < 1. The translation of (3) is

There is a number x such that 1 < x *and* x < *1,*

which is also obviously *false.* The translation of (4) is

For any numbers x, y, *if* x< y, *then* x *is unequal to* y,

which is clearly *true.* Finally the translation of (5) is

For any number x, *there is some number* y *such that* x < y,

which is obviously *true*.

When a statement is true under an interpretation, we shall also say that it *holds.*

Now we can define an inference to be *valid* if, for any interpretation **I**, whenever all the premises of the inference are true under **I**, so is its conclusion. In that case, the *invalidity* of an inference can be established by producing an interpretation in which the *premises* of the inference are *true,* but the *conclusion* is *false.* Such an interpretation is called a *counterexample* to the inference.

A statement p is *valid* if it is true under any interpretation. We note that, because the domain of any interpretation is nonempty, the statement $\exists x(x = x)$ is valid. This is called the *existence principle.*

A set of statements *S* is *satisfiable* if there is an interpretation under which all the statements in *S* are true.

Two statements are *equivalent* if they have the same truth value under any interpretation. As for propositional statements, we write $p \Leftrightarrow q$ or $p \equiv q$ for the assertion that the (quantificational) statements p and q are equivalent.

The most important pairs of equivalent statements involving quantifiers are the known as the *quantifier interchange laws,* namely

$$(1)\ \neg\exists x p(x) \Leftrightarrow \forall x \neg p(x).\ (2)\ \neg\forall x p(x) \Leftrightarrow \exists x \neg p(x)$$

The correctness of (1) follows from the observation that to deny the existence in a set of an element having a particular property is equivalent to asserting that every element of that set lacks that property. The correctness of (2) follows from the observation that to deny that every element of a non-empty set has a particular property is equivalent to asserting the existence of an element of the set having that property.

Exercises

B1. Symbolize the following arguments and determine whether they are valid. If not, specify one counterexample.

(i) There's something that's tasty if it's a chocolate bar. So there's a tasty chocolate bar.

(ii) Some like it hot and some don't. Those who like it hot like Marilyn Monroe, and those who don't *don't* like her! Therefore, everybody either likes her or doesn't.

(iii) If anyone is taller than Rob, Gurpreet is. If Gurpreet is taller than Rob, anyone is. So it isn't the case that there's someone taller than Rob and someone not.

B2. Consider the domain consisting of <u>points</u> and <u>straight lines</u> in a given <u>plane</u>, with the following vocabulary for describing it:

Px = x is a Point   Lx = x is a straight Line   Oxy = (point) x lies On (line) y

Determine the truth value of each of the following statements, providing a brief justification of your answer in each case.

(a) $\forall x \forall y[(Lx \wedge Ly) \rightarrow \exists z(Pz \wedge Ozx \wedge Ozy)]$

(b) $\forall x \forall y[(Px \wedge Py \wedge x \neq y) \rightarrow \exists z \forall w[(Lw \wedge Oxw \wedge Oyw) \rightarrow z=w]]$

B3. Knaves always lie, knights always tell the truth, and in Camelot, where everybody is one or the other, you encounter some people, among them King Arthur who says to you:

"Exactly one out of every two of us is a knave"

Choose names for any other people you might need to refer to and specify an interpretation (i.e. 'case') in which Arthur is a **knight** (if indeed it's possible for him to be a knight, given what he says). Also, specify an interpretation in which Arthur is a **knave** (again, only if that's possible).

B4. Consider the following scenario involving three objects, two predicates F and G, and a relation R:

Domain = {1,2,3} F = {2,3} G = {1} R = {(1,1),(2,2),(1,3),(2,3),(3,3)}

Which of the following is true, and which is false, under this scenario?

(i) $\forall x(Fx \rightarrow Gx)$

(ii) $\exists x(Fx \rightarrow Gx)$

(iii) $\exists x(Fx \vee Gx)$

(iv) $\forall x \exists y \neg Ryx$

(v) $\exists y \forall x(Rxy \rightarrow Gx)$

B5. Consider the domain consisting of the positive whole numbers {1,2,3,...} with the following vocabulary for describing it:

Ex: x is even Ox: x is odd $x<y$: x is less than y

$x>y$: x is greater than y $x \neq y$: x is unequal to y

Determine the truth value of each of the following statements, providing a brief justification of your answer in each case.

(a) $\forall x[Ox \rightarrow \exists z(Ez \wedge (x>z))]$

(b) $\exists x \forall y((x \neq y) \rightarrow (x<y))$

B6. Determine whether the following sets of statements are satisfiable. For each of the satisfiable sets, supply an interpretation in which all the statements are true.

(a) $\exists x \forall y Pxy$

$\forall x \forall y \exists z(Pxz \wedge Pxy)$

(b) $\forall x \exists y Pyx$

$\forall x \forall y(Pxy \rightarrow \neg Pyx)$

$\forall x \forall y \forall z((Pxy \wedge Pyz) \rightarrow Pxz)$

(c) $\forall x \forall y \forall z ((Pxy \wedge Pyz) \rightarrow Pxz)$

$\forall x \neg Pxx$

$\exists x \exists y(Pxy \wedge Pyx)$

B7. Symbolize the following arguments, and determine whether they are valid. If not, supply a counterexample.

(a) A person is famous if and only everyone has heard of him or her. So, all famous people have heard of each other.

(b) Tweety bird hates cats. No cats hate Tweety bird. Sylvester is a cat. Therefore, Tweety Bird hates someone who hates him. [use:- Hxy: x **hates** y; t: **Tweety bird**; Cx: x is a **cat**; s: **Sylvester**)

(c) Logic students are taller than business students. Exactly one out of every pair of students is a logic student. So some student is taller than some other.

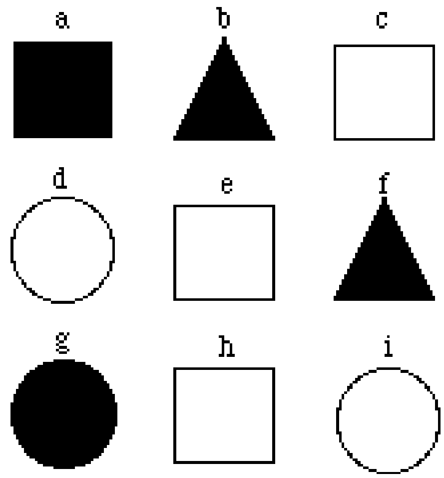
B8. Symbolize the following arguments, and determine whether they are valid. If not, supply a counterexample.

(a) Exactly one professor lives in Talbot College. Professor Bell lives in Talbot College. Professor Bell is a toposopher. Therefore, every professor who lives in Talbot College is a toposopher. [For translation, assume domain is **professors** and use: Cx = x lives in **Talbot College**, n = **Bell**, Tx = x is a **toposopher**.]

(b) Exactly one out of every pair of balls is red. Exactly one ball is red. So exactly one ball isn't red. [Assume domain is **balls**.]

(c) Stallone can outgun everybody who can outgun anyone he can. Therefore, Stallone can outgun himself and no one else! [For translation, assume domain is **persons**.]

B9. Here is a small world:



and symbols for describing it (with any variables restricted to ranging over the above nine inhabitants):

Domain = The shapes with names a, b, c, d, e, f, g, h, i

Sx = x is a **square**   Lxy = x is **directly left** of y   Cxy = x is in the **same column** as y

Tx = x is a **triangle**   Axy = x is **directly above** y   Rxy = x is in the **same row** as y

Bx = x is **black**

Interpreting each of the formulae below as a statement about this world, state whether it is true or false of the world, and, if it is false, briefly state why (referring to any of the shapes above by name, if you need to):

(i) ∀xTx

(ii) $\neg\exists xLxx$

(iii) $\neg\exists x\neg(Tx\vee Sx)$

(iv) $\forall x(Tx\rightarrow Bx)$

(v) $\forall y\exists xLxy$

(vi) $\neg\exists x\exists y(Tx\wedge Ty\wedge Cxy)$

(vii) $\exists x[Tx\wedge\forall y((Rxy\wedge x\neq y)\rightarrow\neg Ty)]$

(viii) $\forall x((Sx\wedge\exists yAxy)\rightarrow\neg Bx)$

(ix) $\forall x\forall y((Sx\wedge Sy\wedge x\neq y)\rightarrow(Cxy\rightarrow(\neg Bx\wedge\neg By)))$

(x) $\forall y\exists x(Bx\wedge Sx\wedge x\neq y)$

B10. Using the indicated key: symbolize (1) through (4); translate (5) through (8) into clear English (not just logical jargon); and say whether each of these statements (1)-(8) is true or false, briefly justifying your answer.

Key –

Domain: statements   Ixy = x logically implies y   Exy = x is logically equivalent to y

(1) Every statement implies some statement or other.

(2) Some statements are equivalent to anything that implies them.

(3) Statements with the same implications are equivalent.

(4) Some statements imply all and only what implies them.

(5) $\forall x\exists y\neg Ixy$

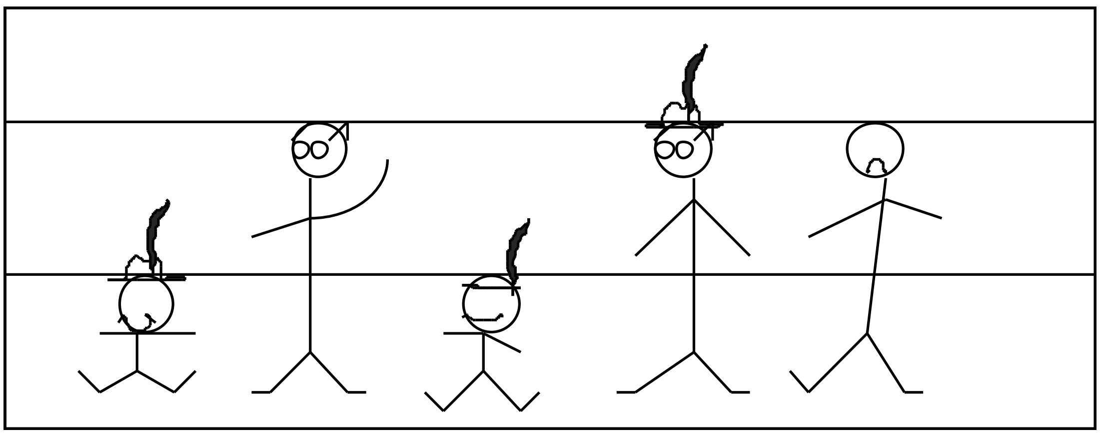
(6) $\forall x\forall y(Ixy\vee Exy)$

(7) $\exists x \forall y(Ixy \rightarrow Exy)$

(8) $\exists x \forall y(Ixy \rightarrow \exists z \neg Iyz)$

B11. Here is a small world:

← Left Right→



a b c d e

and a vocabulary describing it (variables restricted to the five inhabitants):

Fx : x wears a feather

Gx : x wears glasses

Hx: x wears a hat

Ixy: x is identical to y

Lxy: x is left of y

Rxy: x is right of y

Txy: x is taller than y

Which of the following quantified formulae are true and which are false of this small world? If a formula is false, describe a minimal change of the world that would make it true (e.g. take somebody's feather away, move people around, etc.. but don't move anybody into or out of the world).

1. $\forall x(Fx \rightarrow \exists yRxy)$
2. $\exists x \forall y (\neg Ixy \rightarrow Lxy)$
3. $\forall x \forall y [\neg Ixy \rightarrow (\neg Txy \wedge \neg Tyx)]$
4. $\forall x(Gx \rightarrow \exists y(Gy \wedge Rxy)$
5. $\forall y(\exists x (Fx \wedge Lyx) \rightarrow Hy)$

B12. Here is another small world:

Domain = {Mum, Pop, Junior}

F = {Pop, Junior}

G = {Mum}

R = {(Pop, Pop), (Mum, Mum), (Mum, Junior), (Pop, Junior)}

Which of the following is true, and which false, when interpreted as assertions about this world:

(a) $\forall x(Fx \leftrightarrow Gx)$

(b) $\exists x(Gx \to Fx)$

(c) $\exists x \exists y\ (Fx \wedge Gy)$

(d) $\forall x[\exists y\ Ryx \to Fx]$

(e) $\exists x[\forall y Gy \vee Fx]$

(f) $\exists x \forall y\ \neg Rxy$

(g) $\forall x(\exists y Rxy \to \exists z \neg Rxz)$

(h)$\forall x \forall y(Fx \vee Gy)$

(i) $\exists x(Fx \wedge \neg Gx \wedge \exists y Rxy)$

(j) $\neg \forall x \forall y Rxy \vee \exists x \exists y Rxy$

(k) $\exists y \forall x(Fx \wedge Gy)$

B13. For each of the following statements, provides an interpretation which makes it <u>true</u>, and one which makes it <u>false</u>.

(a) $\exists x \forall y \forall z \neg Rzyx$

(b) $\forall x\exists yRxy\rightarrow\exists y\forall xRxy$

(c) $\forall y\exists z(Pz\wedge z\neq y\wedge Qy)$

B14. Here is a small world (a quack optometrist's eye chart!):

D <u>e</u> ***f*** e D **<u>f</u>** *D* **E** *<u>d</u>* **F** **<u>e</u>**

and symbols for describing it (with any variables restricted to ranging over the above eleven inhabitants):

Dx: x **is the letter d (or D)**

Ex: x **is the letter e (or E)**

Fx: x **is the letter f (or F)**

Cx: x **is capitalized**

Bx: x **is bold-faced**

Ux: x **is underlined**

Ix: x **is italicized**

Rxy: x **is right of** y

Lxy: x **is <u>directly</u> left of** y

Interpreting each of the formulae below as a statement about this world, state whether it is true or false of the world:

(i) $\forall xUx$

(ii) $\neg\exists x(Bx\wedge Ix)$

(iii) $\forall x(Fx\rightarrow Bx)$

(iv) $\forall x[Bx\rightarrow\exists y(Ryx\wedge Uy)]$

(v) $\exists x\exists y[Lxy\rightarrow(Ux\vee Uy)]$

(vi) $\forall x(\neg Cx\leftrightarrow\exists yLyx)$

(vii) $\exists x(Ex\wedge Cx\wedge Bx\wedge\forall y[(Ey\wedge Cy\wedge By)\rightarrow y=x])$

(viii) $\exists x(Dx\wedge\forall y[(Dy\wedge Ryx)\rightarrow\neg Cy])$

B15. Here is ANOTHER quack optometrist's eye chart:

<u>**f**</u> *h* **E** <u>*d*</u> **F** <u>**E**</u> **D** <u>e</u> *f* e **d**

and symbols for describing it:

Dx: x **is the letter d (or D)** Cx: x **is capitalized** Rxy: x **is <u>somewhere</u> right of** y

Ex: x **is the letter e (or E)** Bx: x **is bold-faced** Lxy: x **is <u>directly</u> left of** y

Fx: x **is the letter f (or F)** Ux: x **is underlined**

Ix: x **is italicized**

Interpreting each of the formulae below as a statement about this world, state whether it is true or false of the world:

(i) $\forall xCx$

(ii) $\neg\exists x(Ux\wedge Ix)$

(iii) $\forall x(Fx\rightarrow Bx)$

(iv) $\exists x\forall y(Lyx\rightarrow(Fy\wedge By))$

(v) $\forall x[Bx\rightarrow\exists y(Ryx\wedge Uy)]$

(vi) $\exists x\exists y[Lxy\rightarrow(Ux\vee Uy)]$

(vii) $\forall x(\neg Cx\leftrightarrow\exists yLyx)$

(viii) $\exists x(Ex\wedge Cx\wedge Bx\wedge\forall y[(Ey\wedge Cy\wedge By)\rightarrow y=x])$

(ix) $\exists x(Dx\wedge\forall y[(Dy\wedge Ryx)\rightarrow\neg Cy])$

(x) $\exists x(\exists yLxy\wedge\forall zRxz)$

B16. Prove that the sentence $\exists x[Px \wedge \forall y(Py \rightarrow y = x)]$ is true under an interpretation **I** if and only if the set P^I contains exactly one element. Formulate sentences which are true under an arbitrary interpretation ***I*** if and only if P^I contains **(i)** at most one element, **(ii)**

at most two elements, **(iii)** at least two elements, **(iv)** exactly two elements, **(v)** at most three elements, **(vi)** at least three elements, **(vii)** exactly three elements.

3. Tree Rules for Quantifiers

In order to be able to employ the tree method to test inferences within quantificational logic for validity we need to formulate new tree rules governing the quantifiers. In the case of ∀ we shall be guided, as before, by the usual meaning of generality, namely, that whenever we assert that *all* individuals under consideration have a certain property, then, given any individual, *that* individual has, or, as we shall sometimes say, *instantiates* the property. We call this the principle of *universal instantiation*. The corresponding tree rule may be formulated thus:

> **UI**. Given a statement of the form ∀xp(x) occupying a node of an open path of a tree,
> (1) if a name m appears in the path, write p(m) at its foot unless that statement already occupies a node of the path (in which case, writing p(m) once more in the path would be redundant);
> (2) if *no* name appears in the path, choose some name m and write p(m) at its foot[13].
> *Do not tick the line* ∀xp(x).

In writing p(x) we have indicated that the statement p contains an occurrence of the variable x; this done, we have written p(m) for the result of substituting "m" for "x" at each occurrence of the latter in p.[14]

Let us observe this rule in action. Consider the inference:

---

[13] Here, yet again, the presence of sufficiently many names is required.

[14] Strictly speaking, by "occurrence of x" here we mean *free* occurrence, that is, an occurrence of x *not* within a context of the form "∀xq(x)" or "∃xq(x)". We shall always tacitly assume that this is the kind of occurrence in question.

1. Juliet loves all who love Romeo.

2. Romeo loves himself.

∴ *3. Juliet loves herself.*

This inference may be symbolized as follows, using "r" as a name for Romeo, "j" for Juliet, and writing "L" for "loves":

1. $\forall x(Lxr \rightarrow Ljx)$
2. Lrr

∴ 3. Ljj.

As in the case of propositional trees, we start off with the premises of the argument followed by the negation of its conclusion, and then continue so as to obtain a closed tree in the following way:

1.	$\forall x(Lxr \rightarrow Ljx)$		
2.	Lrr		
3.	$\neg Ljj$		
4.	√ $Lrr \rightarrow Ljr$		(UI applied to 1)
5.	$\neg Lrr$	Ljr	(from 4)
6.	×	√ $Ljr \rightarrow Ljj$	(UI applied to 1 again)
7.	$\neg Ljr$	Ljj	(from 6)
	×	×	

In this example we used the **UI** rule twice to obtain lines 4 and 6:

1. $\forall x(Lxr \rightarrow Ljx)$	$\forall x p(x)$
4. $Lrr \rightarrow Ljr$	$p(r)$
6. $Ljr \rightarrow Ljj$	$p(j)$

Both applications were made to the same node, 1, and in both the variable v was "x", and p(x) the statement "$\forall x(Lxr \rightarrow Ljx)$". The two applications differed, however, in respect of the name substituted for x: in the first case it was "r" and in the second "j". In the first case we obtained p(r) by substituting "r" for "x" in p(x), and in the second p(j) by substituting "j" for "x" in p(x).

From the fact that we had to apply **UI** *twice* to the same statement 1. it should now be apparent why we do not tick a statement to which **UI** has been applied. Indeed, in this example we had to continue to apply it with every name actually appearing in the path in question before the path (and the tree) finally closed.

In general, given an inference in quantificational logic, any quantificational tree whose initial statements are the premises of the inference, followed by its conclusion, is said to be *associated* with the inference. As for propositional logic, the use of the tree method for establishing validity of inferences in quantificational logic is that an inference is valid provided that some tree associated with the inference closes. This, the property *of inference correctness* for quantificational tree, will be proved in the next section.

Let us now consider an example of an application of **UI** in which no names are initially given. Here the tree method will be used to test *satisfiability* rather than validity. Consider the conditions:

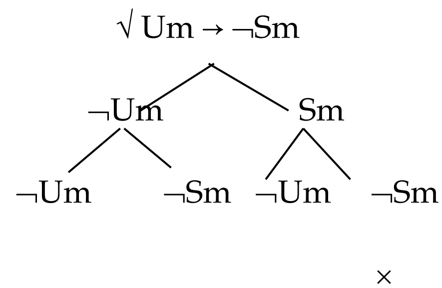
All unicorns are speedy. *No unicorns are speedy.*

Using the obvious notation, these apparently conflicting hypotheses concerning unicorns are expressible as the statements occupying the first two nodes of the following tree, which tests their *satisfiability*:

$$\forall x(Ux \rightarrow Sx)$$

$$\forall x(Ux \rightarrow \neg Sx)$$

$$\surd\ Um \rightarrow Sm$$



The third node here results by applying **UI** to the first node, at the same time introducing the new name m. Once this name has been introduced into the path, it must be used in any subsequent application of **UI** in that path, in particular, in the application yielding the fourth line from the second.

We note that the tree is finished since no further applications of **UI** can be made, and it has 3 open full paths. It is easily seen that each of these open paths determines an *interpretation* under which all the statements occupying lines in it are true. The elements of the domain of the interpretation associated with an open path correspond to the names appearing in that path. In our example, there is only one such name—"m"—present, so that the domain of each interpretation has exactly one element, which we take to be named by "m". Since the statement ¬Um occurs in each path, the statement "m is not a unicorn" holds under each interpretation. In the second open path the statement ¬Sm appears, so the statement "m is not speedy" holds in the associated interpretation. The third open path contains the statement Sm, so "m is speedy" holds in the associated interpretation. On the other hand, the first path contains neither Sm nor ¬Sm, so in this case there are *two* associated interpretations, one in which a is speedy and another in which a is not speedy. Since the object named by "m" is the *sole* individual in each domain, we see that in each of these interpretations the statement ¬Ua has the stronger meaning that *nothing* is a unicorn. Thus under each interpretation no unicorns exist, so that any assertion about all unicorns, including our two conditions above, automatically come out true, and are therefore jointly satisfiable there (contrary to what one might naively expect).

In general , each open full path in a finished quantificational tree determines an interpretation under which all the statements on it, and in particular the initial statements of the tree are true. This is the *tree test for satisfiability of quantificational statements.*

We now require a rule for the existential quantifier. This is the rule of *existential instantiation*:

EI Given an unticked statement of the form ∃xp(x) occupying a node of an open path, check to see whether it contains a node occupied by a statement of the form p(m). If so, do nothing. If not, choose a name n *that has not been used anywhere in the path* and write the statement p(n) at its foot. When this has been done for every open path in which the statement ∃xp(x) occupies a node, tick the node occupied by the given statement:

√ ∃xp(x)
|
p(n) (n new)

It is important to observe in applying this rule that the name n introduced *not be already present in the path*. This is imperative because we want n to name an individual *about which we assume nothing except that it satisfy* p; individuals that have already been named may have properties that conflict with this supposition. For example, consider the following (true) premises:

Someone is Canadian	∃xCx
Nixon is not Canadian	¬Cn.

Were we allowed to use the old name n instead of being forced to introduce a new one, we would be able to generate a closed tree from these premises:

√ ∃xCx
¬Cn
Cn
×

where we have (incorrectly!) applied **EI** to the first node to obtain the third. This would mean that the premises are not jointly satisfiable, in other words, that from the assertion "someone is Canadian", we would be able to infer "Nixon is Canadian". Using the same line of reasoning, we would in fact be able to infer "everyone is Canadian". Incorrectly applied, **EI** can lead to absurdities such as these.

Correctly applied, on the other hand, **EI** leads in our example to

$$\begin{array}{l} \surd\ \exists x Cx \\ \quad \neg Cn \\ \quad\ Cm \end{array}$$

where m is a *new* name, denoting, as it were, an "typical Canadian", whose identity is not further specified.

We also have the

RULE FOR NEGATED QUANTIFICATION	
If a statement beginning with ¬∀x (or ¬∃x) occupies a node of an open path, tick it and write at the feet of all open paths containing that node the same statement with ∃x¬ in place of ¬∀x (or with ∀x¬ in place of ¬∃x) in front.	$\surd\ \neg\forall x\ p(x)$ | $\exists x\neg\ p(x)$ $\surd\ \neg\exists x\ p(x)$ | $\forall x\neg\ p(x)$

This rule is justified by the quantifier interchange laws stated in section **2**.

Armed with these new rules for quantifiers, let us return to the inference with which this chapter began, and see if an associated tree closes. Here is one:

∀x[Cx→∀y (Ay→Lxy)]

∀x(Hx→Ax)

Ce

√ ¬∃x∀y (Hy→Lxy)

√ Ce→∀y (Ay→Ley)

¬Ce   ∀y (Ay→Ley)

×   ∀x¬∀y (Hy→Lxy)

√ ¬∀y (Hy→Ley)

√ ∃y¬ (Hy→Ley)

√ ¬ (Hm→Lem)

Hm

¬Lem

√ Hm→Am

¬Hm   Am

×   √ Am → Lem

¬Am   Lem

×   ×

We see that the tree does close.

Let us now illustrate how the tree method may be applied to generate counterexamples to *invalid* quantificational inferences. Consider the following inference: *Something's round, something's square, so something's round and square.* This may be symbolized

∃xRx

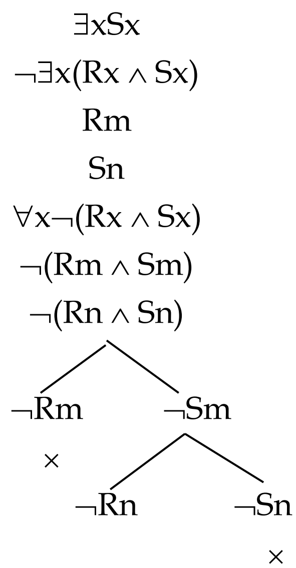
∃xSx

―――――

∃x(Rx ∧ Sx)

Here is a finished tree associated with the inference:

∃xRx



There is one open full path through this tree. It determines the following interpretation **I:**

Domain	{1, 2}
Interpretation of a, b	1, 2
Interpretation of R	{1}
Interpretation of S	{2}

I is a counterexample to the original inference because both ∃xRx and ∃xSx are true under it but ∃x(Rx ∧ Sx) is not.

The last quantificational tree rules to be introduced are the *rules for identity and nonidentity.* These are

RULE FOR IDENTITY

If an open path contains a node occupied by a statement of the form m = n and also a node occupied by a statement p in which one of the names m, n appears one or more times, write at the foot of the path a statement q obtained by replacing one or more of the occurrences of that name in p by the other name, provided that q does not already occupy a node in that path:

m = n
p
|
q

RULE FOR NONIDENTITY (OR DIVERSITY)

Close any path that contains a node occupied by a statement of the form m ≠ m

m ≠ m
×

The rule for identity expresses the idea that identicals can always be substituted for identicals, while the rule for nonidentity is a way of expressing the reflexivity of identity, the fact that m = m.

The rule for identity enables us to close the tree associated with the valid inference

Hm
¬Hn
―――
¬(m = n)

which was considered early in the chapter. Thus

Hm
¬Hn
¬¬(m=n)
m = n

$$Hn$$
$$\times$$

Here we have obtained the fourth line by applying the rule for identity to the first and third lines.

From these rules we can derive the *four basic laws of identity*, viz., *substitutivity, reflexivity, symmetry, and transitivity*.

Substitutivity

$$\frac{\begin{array}{c} p(m) \\ m = n \end{array}}{p\,(n)}$$

Here the associated closed tree is:

$$\begin{array}{c} p(m) \\ m = n \\ \neg p(n) \\ p(n) \\ \times \end{array}$$

where we have used the rule for identity to obtain the fourth line from the first two.

Reflexivity

$$m = m$$

The associated closed tree is simply

$$\begin{array}{c} m \neq m \\ \times \end{array}$$

Symmetry

$$\frac{m = n}{n = m}$$

Here the associated closed tree is

$m = n$
$n \neq m$
$n \neq n$
$\times$

in which the third statement is obtained from the first two by the rule for identity, and closure from the rule for nonidentity.

Transitivity

$$\begin{array}{c} m = n \\ \underline{n = o} \\ m = o \end{array}$$

In this case the associated closed tree is

$m = n$
$n = o$
$m \neq o$
$m \neq n$
$\times$

where the last statement arises from the second and third by the rule for identity.

It follows that, from a formal point of view, = *is an equivalence relation.*

The tree associated with the existence principle $\exists x(x = x)$ also closes:

$\neg\exists x(x = x)$
$\forall x\neg(x = x)$
$m \neq m$

Remark on unfinishable quantificational trees. The quantificational trees we have introduced so far have all been finished. It should be noted, however, that, unlike propositional trees, which can always be finished, *quantificational trees can, in principle, go on forever*. Here is an example.

$\forall x \exists y Rxy$ (*)
|
$\exists y Rmy$
|
Rmn
|
$\exists y Rny$
|
Rno
|
$\exists y Roy$
|
Ros
⋮

Clearly the repeated application of **UI** to (*) can continue indefinitely.

Exercises

C1. Using the tree method, determine which of the following inferences are valid.

(i) $\underline{\forall x(Px \to Qx)}$
$\forall xPx \to \forall xQx$

(ii) $\underline{\forall xPx \to \forall xQx}$
$\forall x(Px \to Qx)$

(iii) $\underline{\exists x(Px \wedge Qx)}$
$\exists xPx \wedge \exists xQx$

(iv) $\underline{\exists xPx \wedge \exists xQx}$
$\exists x (Px \wedge Qx)$

(v) $\underline{\forall x(Px \vee Qx)}$
$\forall xPx \vee \forall xQx$

(vi) $\underline{\forall xPx \vee \forall xQx}$
$\forall x(Px \vee Qx)$

(vii) $\underline{\forall x(Px \vee Qx)}$
$\forall xPx \vee \exists xQx$

(viii) $\underline{\forall xPx}$
$\exists xPx$

C2. Symbolize the following arguments, and, using the tree method, determine whether they are valid (always try to choose natural predicate letters and name letters!):

(a) All logicians are neurotic. No vegetarians are neurotic. Therefore, no vegetarians are logicians.

(b) Every Greek who loathes a Trojan is feared by all. Achilles loathes a Trojan, so, if Achilles is Greek, everybody fears him.

(c) Alma has a brother who has no brother, so she's no one's brother.

(d) I'll be home before four o'clock. Therefore there's a time before four o'clock that I'll be home before.

C3. Use the tree method to determine whether the following argument is valid:

$\underline{\forall xFx \to \forall xGx}$

$\exists x\forall y(Fx \to Gy)$

C4. Using the tree method, establish the validity of the statements (a) $\exists x(\exists yPy \to Px)$ and (b) $\exists x(Px \to \forall yPy)$. What does each statement mean in a given interpretation?

C5. Symbolize the following arguments (using the given symbols), and determine whether they are valid using the tree method:

(a) If anyone can learn physics, you can. Anyone who can learn logic can learn physics. Dr. Rob can learn logic. So *you* can learn physics! [Use — Px: x can learn physics; u: you; Lx: x can learn logic; d: Dr. Rob]

(b) No acrobats are clumsy. Therefore, if Alma is a waiter, then if all waiters are clumsy, Alma is no acrobat. [Use — Ax: x is an acrobat; Cx: x is clumsy; Wx: x is a waiter; m: Alma]

(c) All dogs are cats. Therefore, whoever loves a dog loves a cat. [Use — Dx: x is a dog; Cx: x is a cat; Lxy: x loves y]

C6. Symbolize the sentences in the following arguments and, using the tree method, determine which ones are valid.

(i) Everything has a cause. If the world has a cause, then there is a God. Hence, there is a God.

(ii) If everyone litters, the world will be dirty. Hence, if you litter, the world will be dirty.

(iii) Everybody loves a lover. Romeo loves Juliet. Therefore, I love you.

(iv) Any barber in Seville shaves exactly those men in Seville who do not shave themselves. Hence, there is no barber in Seville.

C7. Symbolize the following (sets of) sentences and, using the tree method, determine in each case whether they are satisfiable.

(i) Any reasonable person can understand logic and is fit to vote. But Joe doesn't understand logic and yet is fit to vote.

(ii) There is a barber who shaves exactly those who do not shave themselves.

(iii) Everybody loves all lovers. You love yourself, but you don't love me.

(iv) There are at least three objects in this box, and exactly one out of every two is black.

C8. Using the tree method, determine which of the following arguments is valid:

(a) There is someone who is going to pay for all the breakages. Therefore, each of the breakages is going to be paid for by someone.

(b) No student in the statistics class is smarter than every student in the logic class. Hence, some student in the logic class is smarter than every student in the statistics class.

(c) Any person who is not mad can understand logic. None of Wagner's sons can understand logic. No mad persons are fit to vote. Therefore, none of Wagner's sons is fit to vote.

C9. Demonstrate the validity or invalidity of each of the following two arguments by first translating them using the given symbols, and then doing their trees.

For translation, use:

Cx = x is a **chimpanzee**	Bx = x **will get a banana**	f = **fred**
Sxy = x **can solve** y	Txy = x is **trying harder than** y	n = **barney**
Px = x is a **problem**		

(i) Not all chimpanzees are trying equally hard. No chimpanzee tries harder than himself. Therefore there are at least two chimpanzees.

(ii) Fred and Barney can solve exactly the same problems. If Fred can solve even one problem, then he will get a banana. Fred will not get a banana. Therefore Barney can't solve any of the problems, and he won't get a banana either.

C10. Using the tree method, determine which of the following pairs of statements are equivalent:

(i) $\forall x\, Px \rightarrow \forall x Qx$ $\quad\quad$ $\forall x \exists x\, (Px \rightarrow Qy)$

(ii) $\exists x Px \rightarrow \exists x Qx$ $\quad\quad$ $\exists y \forall x\, (Px \rightarrow Qy)$

C11. Define $\exists!x$ by writing $\exists!xPx$ for $\exists x[Px \wedge \forall y(Py \rightarrow y=x)]$. State, in simple language, the meaning of $\exists!xPx$.

Determine which of the following inferences are valid:

(i) __________

$\forall x \exists! y\, (x = y)$

(ii) $\underline{\exists!x\, Px}$

$\exists x \forall y (Py \leftrightarrow y=x)$

(iii) $\underline{\exists ! x(Ax \wedge Bx)}$

$\exists ! x Ax \wedge \exists ! x Bx$

(iv) $\underline{\forall x \forall y\ (x = y)}$

$\exists ! x\ (x = x)$

C12. Let p be a sentence and Q a predicate. Establish the validity of the following sentences. **(i)** $\forall x(p \wedge Qx) \leftrightarrow (p \wedge \forall xQx)$, **(ii)** $\forall x(p \vee Qx) \leftrightarrow p \vee \forall xQx$, **(iii)** $\exists x(p \wedge Qx) \leftrightarrow (p \wedge \exists xQx)$, **(iv)** $\exists x(p \vee Qx) \leftrightarrow (p \vee \exists xQx)$, **(v)** $\forall x(p \rightarrow Qx) \leftrightarrow (p \rightarrow \forall xQx)$, **(vi)** $\forall x(Qx \rightarrow p) \leftrightarrow (\exists xQx \rightarrow p)$, **(vii)** $\exists x(p \rightarrow Qx) \leftrightarrow (p \rightarrow \exists xQx)$, **(viii)** $\exists x(Qx \rightarrow p) \rightarrow (\forall xQx \rightarrow p)$.

C13. Determine which of the following sentences are valid. For each sentence which is not valid, provide an interpretation in which it is false. **(i)** $\exists x(Px \rightarrow \forall yPy)$, **(ii)** $\forall x\forall y\forall z[(Rxy \wedge Ryz) \rightarrow Rxz]$, **(iii)** $(\exists xPx \wedge \exists xQx) \rightarrow \exists x(Px \wedge Qx)$, **(iv)** $\forall x(Px \vee Qx) \rightarrow (\forall xPx \vee \forall xQx)$.

C14. Which of the following sets of sentences are satisfiable? In each of the satisfiable cases, supply an interpretation under which all of the sentences are true. **(i)** $\forall xPx$, $\forall x[Px \rightarrow \exists yRxy]$, $\exists x\exists y\neg Rxy$. **(ii)** $\forall x\exists yRxy$, $\forall x\exists y\neg Rxy$. **(iii)** $\forall x\neg Rxx$, $\forall x\forall y\ \forall z((Rxy \wedge Ryz) \rightarrow Rxz)$, $\exists x\exists y\exists z(Rxy \wedge Ryz \wedge Rzx)$.

C15. Demonstrate the validity or invalidity of each of the three arguments below by first translating them using the given symbols, and then doing their trees.

(a) Tweety bird despises cats. No cats despise Tweety bird. Sylvester is a cat. Therefore, Tweety bird despises someone who despises him. (use:- Dxy: x **despises** y; t: **Tweety bird**; Cx: x **is a cat**; s: **Sylvester**)

(b) Any good logic teaching assistant helps all and only those who don't help themselves. Hence there aren't any good logic teaching assistants! (use:- Gx = x **is a good logic teaching assistant**; Hxy = x **helps** y)

(c) I'll finish this exam before four o'clock. For any pair of times, one later than the other, there is a time in between them. So there's a time before four o'clock that I'll finish this exam before. (use:- Ex = **I'll finish this exam <u>at</u> time** x; x<y = **time** x **is earlier than time** y; f = **four o'clock**)

C16. Demonstrate the validity or invalidity of each of the three arguments below by first translating them using the given symbols, and then doing their trees.

(a) Ben loves cats. No cats love Ben. Whitey is a cat. Therefore, Ben loves someone who doesn't love him. (use:- Lxy: x **loves** y; n: **Ben**; Cx: x **is a cat**; w: **Whitey**)

(b) There's a set containing all and only those sets which are not members of themselves. Therefore, every set is a member of itself. (use:- Sx = x **is a set**, x ∊ y = x **is a member of** y)

(c) Everyone loves lovers. Romeo loves Juliet. So Fred loves Wilma. (assume domain = persons and use:- Lxy = x **loves** y x; r = **Romeo**; j = **Juliet**; f = **Fred**; w = **Wilma**)

C17. The philosopher Leibniz defined a *good* person as one who loves everybody. Adopting the definition $\forall x(Gx \leftrightarrow \forall yLxy)$, which of the statements below follow? Use the tree method.

(a) All good people love somebody or other.

(b) All good people love themselves.

(c) Nobody who is not good loves anybody.

(d) Someone is loved by all good people.

(e) Somebody loves all good people.

(f) Everyone good is loved by somebody or other.

(g) All good people love all good people.

(h) Everybody is loved by somebody or other.

(i) If there are any good people, then everybody is loved by somebody.

(j) Everyone who is good loves everyone who is not good.

4. **Correctness of the Quantificational Tree Method.**

We have mentioned above that quantificational trees have the property of *inference correctness*, namely, that an inference in quantificational logic is valid provided that some tree associated with the inference closes. Here we offer a proof of this fact.

First, let us call a tree rule R *correct* if whenever the premise of R is true under a given interpretation, then all the statements in *at least one* of R's lists of conclusions are also true under the interpretation. Now all the tree rules we have introduced - *apart from* **EI** - are correct in this sense. In the case of the propositional rules, this should be clear. The correctness of the rule UI, follows from the observation that if its premise ∀xp(x) is true under a given interpretation, its conclusion p(a) is evidently also true under that interpretation. The correctness of the negated quantification rules is an immediate consequence of the quantifier interchange laws. Finally the rules for identity are correct. The rule for nonidentity is a single premise rule telling us to close paths containing statements denying self-identity. Since the premise of this rule is false under any interpretation, correctness of the follows from the principle of *ex falso quodlibet.* An the correctness of the correctness of the rule of identity follows from the fact that the result of substituting equals for equals must be true under any interpretation in which both premises are true.

Given a set *S* of statements, let us say that a tree *starts with* S if it has S as its initial set of statements. Now we can establish the

Correctness of the quantificational tree method. *If a set* S *of* quantificational *statements is satisfiable, there will be an open (complete) path through any tree that starts with* S.

To prove this, observe first that, if all the statements occupying nodes in a path **P** of a tree are true under a given interpretation, then **P** is open. For if there is an interpretation making all statements occupying nodes in **P** true, then both a statement and its negation cannot both occupy nodes in **P**, since otherwise both would have to be true under the interpretation, which is impossible. It follows that **P** cannot contain both a statement and its negation, which is just to say that path **P** is open.

Now suppose that under some interpretation **I** all the members of S are true. Consider the following property of a tree **T**.

(*) **T** starts with S and contains a (complete) path **P** such that all statements occupying nodes of **P** are true under **I**.

By the observation above, any tree satisfying (*) contains an open complete path.

We claim that, if **T** has property (*), so does any tree **T*** obtained from **T** by applying a quantificational tree rule. For suppose that (a) all the statements occupying nodes in a certain path **P** through **T** are true under **I** and (b) we extend **T** to **T*** by applying a tree rule to one of its statements. Clearly we may assume that this statement is in **P**, for if not, then **P** is unaffected and is a complete path of **T***. Accordingly in the transition from **T** to **T*** the path **P** is extended to a new path, or extended and split into two new paths, by applying some tree rule. It the applied tree rule is correct (i.e. , if it is any rule apart from EI), then since all the statements occupying nodes in the extended path, or all those occupying nodes in at least one of the new paths (each of which extends the path **P**), are true under **I**. But this shows that **T*** has property (*), as claimed.

It remains to consider the case when the applied rule is **EI**. Here the premise is ∃xp(x) and a new statement p(n) is appended to **P,** where n is a name not already appearing in **P.** Now ∃xp(x) is (∃*x*)p[m/x] for some name m and, as a statement occupying a node of **P**, it is true under **I.** Accordingly, there is some element *a* in the domain of **I** for which p is true under **I**[m/*a*]. Now **I**[n/*a*] assigns the same truth values to statements occupying nodes of **P** and the same truth value to p(n) as **I**[m/*a*] does to p.

(This is because p(n) is the result of substituting n for m in p.) Thus all statements in the extended path are true under **I**[n/*a*]. So in this case, too, T* has the property (*).

It follows that *any* tree **T** starting with S has property (*), and hence contains an open path. For any tree **T** starting with S can be 'built up' (or rather, down!) by starting first with the tree with a single path consisting of the statements in S—which has property (*) by definition—and then applying tree rules, one after another until tree **T** results. By the argument of the previous paragraph, at each stage of the 'tree building' process, property (*) is preserved, therefore the end result—the tree **T**—will have that property too (and so must contain an open path, which is what we needed to show).

As an immediate consequence of this, we obtain the

Inference correctness of the tree method. *If a tree associated with an inference is* closed, *then the inference is* valid.

For if the inference is invalid, then the set S of statements composed of the inference`s premises together with the negation of its conclusion is satisfiable, and so any tree associated with the inference contains an open complete path. So if there is a closed tree associated with the inference, it cannot be invalid, and so must be valid.

Thus we have justified the claim that, in order to determine whether an inference is valid in quantificational logic, we need only check whether the associated tree closes. But note that, because, as we have seen, quantificational trees cannot always be finished, we will not in general be able to determine whether the tree associated with a given inference closes. It follows that we cannot use the same argument as for propositional logic, whose trees are always finishable, to infer that the validity of inferences in quantificational logic is decidable. In fact it can be shown that validity of quantificational inferences is. in general, *undecidable.* What this means is that no computer, however powerful, can be programmed to decide whether an arbitrary quantificational inference is valid or not.

For propositional logic, we proved the *inference adequacy* of the tree method, namely that if an inference is valid, then any finished tree associated with it is closed. This also holds for quantificational logic, but the proof is too involved to go into here.

5. Many-Sorted Logic

In English (and other languages) there are different quantifiers for different types of domain, for example, various *universal* quantifiers.

Domain	*Places*	*Times*	*People*	*Things*
Quantifier	*Everywhere*	*Always*	*Everyone*	*Everything*

It is convenient to introduce similar devices into our formal logical notation. The method is best illustrated by an example.

Consider the following vocabulary:

Px: x is a person
Qx: x is a politician
Tx: x is a time
Fxyz: x can fool y at (time) z

Then the statement

There is someone who can fool only himself and all politicians all of the time.

may be symbolized in our customary notation as

$$\exists x[Px \wedge \forall y[Py \rightarrow [\forall z(Tz \rightarrow Fxyz) \rightarrow (x = y \vee Qy)]]].$$

This rather involved expression may be simplified by introducing different sorts of letter to indicate individuals satisfying P ("persons") or T ("times"). Thus, if we agree to use letters x,y for persons, and letters t,u for times, the statement above assumes the simpler "many sorted" form:

$$\exists x \forall y[\forall t[Fxyt \rightarrow (x = y \vee Qy)]].$$

The advantage here is that we no longer need to employ explicit predicates to restrict the "range" of the variables. Notice that in order to transcribe this many-sorted statement back into its original "one-sorted" form we need to replace "$\exists x$" by "$\exists x(Px \wedge ...)$", "$\forall y$" by "$\forall y(Py \to ...)$" and "$\forall t$" by "$\forall z(Tz \to ...)$".

6. Operation symbols

Relationships such as motherhood or fatherhood have the property that each individual determines a specific, unique, individual (one's mother or father, respectively) with respect to which it stands in that relationship. In other words these relations define *operations.* The introduction of devices called *operation symbols* into our vocabulary will enable us to give symbolic expression to this fact.

Thus consider, for example, the relation M of motherhood on the domain of discourse consisting of all persons. We introduce the *operation symbol* g to stand for "mother of", so that gx is to be read "mother of (person) x". Then there are two equivalent ways of expressing the statement "y is the mother of x", viz.,

$$Mxy \text{ and } y = gx.$$

Thus, for example, if "m" names Liza Minnelli, then "gm" names Judy Garland.

Names and variables are noun-like *terms*, and operation symbols may be applied to terms of this sort to yield new terms. Thus we may write, for example,

ggx = mother of mother of x = maternal grandmother of x.

Similarly, if in addition we introduce the operation symbol f for "father of", then

gfx = mother of father of x = paternal grandmother of x

etc.

In general, we may introduce a operation symbol in connection with a relation R precisely when R has the two following properties:

Existence: for any x, there exists y such that Rxy
Uniqueness: for any x,y,z, if Rxy and Rxz, then y = z.

When these conditions are satisfied, then for any x there is a *unique* y such that Rxy, and so we can introduce an operation symbol f with the meaning that, for any x, fx denotes this uniquely determined y. Thus, for any x and y, the following conditions are equivalent:

y = fx and Rxy.

Accordingly we now suppose that in addition to names, predicate symbols and relation symbols, our logical vocabulary includes *operation symbols* f, g, h,... . The *terms* of our logical vocabulary are now defined as follows.

(i) Any variable or name standing alone is a term.
(ii) If f is an operation symbol, and t is a term, then ft is a term.
(iii) Nothing is a term unless it follows from **(i)** and **(ii)** that it is so.

In this enlarged vocabulary, a *name* will now be any term which does not contain variables, and a *simple name* will be a name in the original sense, i.e., a name that does not contain operation symbols.

We also extend the idea of an *interpretation* to operation symbols and names in the enlarged sense by the clause:

An interpretation **I** with domain A assigns, to each operation symbol f, an operation $f^{\mathbf{I}}$ on A. If t is a name, and f an operation symbol, $(ft)^{\mathbf{I}} = f^{\mathbf{I}}(t^{\mathbf{I}})$.

Example. Suppose our logical vocabulary has one predicate symbol *P*, one binary relation symbol *R*, two operation symbols *f, g*, and two simple names *m, n*. Let ***I*** be the interpretation of this vocabulary whose domain is the set *H* of human beings and in which the interpretation of *P* is the set of females, that of *R* is the parenthood relation, that is, the set of pairs (*x, y*) for which *x* is a parent of *y*, and those of *f/g* are the operation on *H* assigning to each human being his or her father/mother. The interpretations of *m* and *n* will be two arbitrary but fixed human beings *a* and *b*.

Then, for example, the sentence

$$m = gfn$$

is true under ***I*** just when *a* is *b*'s paternal grandmother. And the sentence

$$m \neq n \wedge Rfmn \wedge Rgmn \wedge Pm$$

is true just when *m* is *n*'s sister.

Operation symbols may also be employed in trees, where such terms as fa, fma, etc. are counted as names. However, in doing this we must at the same time insist that when the **EI** rule requires us to introduce a new name, it must be a simple one, i.e. a new letter not already used. To illustrate, we establish the validity of the inference

$$\frac{\forall x\,(fm = x)}{\forall x\,(fx = m)}$$

(An example of this form of argument in English is: "Everybody's Adam's father", therefore "Adam's everybody's father".) The tree for this inference is

$$\forall x\ (fm = x)$$
$$\surd\ \neg\forall x\ (fx = m)$$
$$\surd\ \exists x\ (fx \neq m)$$
$$fn \neq m$$
$$fm = fn$$
$$fm = m$$
$$fn = m$$
$$\times$$

The tree is closed and the inference valid. Notice that in the fourth line **EI** was applied to the third line, introducing a new letter "n". Also notice that both the names fn and m have been substituted in for the variable x in the first line.

Exercises

D1. Symbolize each of the following, using "f" as an operation symbol for "the father of" and "g" as an operation symbol for "the mother of":

(i) m is n's paternal grandmother

(ii) m is a father

(iii) m is n's full sibling

(iv) m is n's grandmother

(v) m is a grandfather

(vi) m is n's first cousin.

D2. Let f and g be operation symbols for the functions "the **father** of" and "the **mother** of", and read xPy as "x is the **parent** of y" and Mx as "x is **male**". For each of the statements below, explain the precise relationship asserted between m and n as concisely as you can in English.

(a) $m \neq n \wedge fm = fn \wedge gm = gn$ (b) mPfn (c) $m \neq n \wedge$ fmPn $\wedge$ gmPn $\wedge$ Mn

(d) $(fm = fn \leftrightarrow gm \neq gn) \wedge \neg$Mn (e) Mn $\wedge\ \exists$x(xPm $\wedge$ fn = fx $\wedge$ gn = gx $\wedge$ n = fm)

(f) $\neg$Mm $\wedge\ \exists$x(xPm $\wedge$ fnPx $\wedge$ gnPx $\wedge\ \neg$nPm.

7. Natural Deduction for Quantificational Logic

Natural deduction is easily extended to quantificational logic. All the rules of deduction in propositional logic continue to hold in predicate logic, but to deal with quantifiers, some new rules need to be added. The system **Q** is obtained from **P** by adding the following rules

Existential Generalization (EG).

$$\frac{p(n)}{\exists x p(x)}$$

Here n *may be any name.*

Existential Instantiation (EI)

$$\frac{\exists x p(x)}{p(n)}$$

Here n must be a name new to the deduction and must not appear in the conclusion.

Universal Instantiation (UI)

$$\frac{\forall x p(x)}{p(n)}$$

Here n *may be any name.*

Universal Generalization (UG)

$$\frac{p(n)}{\forall x p(x)}$$

Here (1) n *must not occur in* $\forall x p(x)$

(2) n *must not occur in the assumptions or the conclusion of the derivation*

(3) n *must not have been introduced by EI.*

Quantifier Interchange (QI)

$$\neg\forall x p(x) \leftrightarrows \exists x \neg p(x)$$
$$\neg\exists x p(x) \leftrightarrows \forall x \neg p(x)$$

Here are a few examples of derivations in **Q.**

1. $\forall x(Fx \rightarrow Gx)$ *premise*
2. $\forall xFx$ *premise*
3. Fa *2, UI*
4. $Fa \rightarrow Ga$ 1, UI
5. Ca *3, 4, MP*
6. $\therefore \forall xGx$ *5, UG*

In the next two, we assume that p contains no occurrence of x.

1. $\exists x(Fx \wedge p)$ *premise*
2. $Fm \wedge p$ *1, EI*
3. Fm *2, Conj*
4. p *2, Conj*
5. $\exists xFx$ *3, EG*
6. $\therefore \exists xFx \wedge p$ *5, 4, Conj*

Here is in an indirect deduction in **Q** of the inference

$\forall$x (Fx $\vee$ p)
$\forall$x Fx $\vee$ p

1. $\forall$x (Fx $\vee$ p) *premise*
2. $\neg$($\forall$x Fx $\vee$ p) *AID*
3. $\neg\forall$x Fx $\wedge\neg$ p *2, DM*
4. $\neg\forall$x Fx *3, Conj*
5. $\exists$x $\neg$Fx *4, QI*
6. $\neg$Fm *5, EI*
7. $\neg$p *3, Conj*
8. $\neg$Fm $\wedge\neg$p *6,7, Add*
9. $\neg$(Fm $\vee$ p) *8, DM*
10. Fm $\vee$ p *1, UI*
11. (Fm $\vee$ p) $\wedge$ $\neg$(Fm $\vee$ p) *9, 10, Conj* Contradiction

Exercises

E1. Construct derivations in **Q** for the following valid inferences:

(i) $\forall$x(Hx $\rightarrow$ Gx); $\exists$x($\neg$Gx $\wedge$ Fx); $\therefore$ $\exists$x(Fx $\wedge$ $\neg$Hx)

(ii) $\forall$x (Gx $\rightarrow$ Hx); $\exists$x(Gx $\wedge$ Fx); $\therefore$ $\exists$x(Fx $\wedge$ Hx)

(iii) $\forall$x(Fx $\vee$ $\neg$Gx); $\forall$x(Fx $\rightarrow$Hx); $\forall$x($\neg$Gx $\rightarrow$ Jx); $\exists$x$\neg$Jx; $\therefore$ $\exists$xHx

(iv) $\forall$x(Fx $\wedge$ Gx); $\therefore$ $\forall$xFx $\wedge$ $\forall$xGx).

(v) $\forall$xFx $\wedge$ $\forall$xGx); $\therefore$ $\forall$xFx $\wedge$ $\forall$xGx.

(vi) $\exists$x(Fx $\vee$ Gx); $\therefore$ $\exists$xFx $\vee$ $\exists$xGx.

(vii) $\exists$xFx $\vee$ $\exists$xGx;. $\therefore$ $\exists$x(Fx $\vee$ Gx).

(viii) $\forall$x (Fx $\rightarrow$ Hx); $\forall$x (Gx $\rightarrow$ Hx); $\therefore$ $\forall$x ((Fx $\vee$ Gx) $\rightarrow$ Hx).

E2. Construct indirect derivations in **Q** for the following inferences: **(i)**

$\exists$x$\forall$yLxy; $\therefore$ $\forall$y$\exists$xLxy.

(ii) $\forall x(Px \vee Qx); \therefore \forall xPx \vee \exists xQx$.

8. Second-Order Logic.

Quantificational logic is often known as *first-order* logic, because in forming its sentences quantification is restricted to individuals, that is, *first-order* entities. *Second-order* logic is an extension of first-order logic which allows existential and universal quantification of *second-order* entities such as predicates, relations, and operations. As examples of second-order statements we have:

$$(1)\ \forall x\forall y[x = y \leftrightarrow \forall P(Px \rightarrow Py)] \qquad (2)\ \forall x\forall y\exists R\, Rxy.$$

The first of these is *Leibniz's Principle of the Identity of Indiscernibles,* namely that individuals are identical just when one possesses every property the other does. The second asserts that any two individuals are related in some way or other. In these sentences the letter "*P*" is used as a unary predicate variable, for properties of individuals, and the letter "*R*" is used as a binary relation variable, for relations between individuals.

The formation and interpretation rules for statements of second-order logic are straightforward extensions of the corresponding first-order rules. One needs to note only that, in the second-order case, a *name* can now be a predicate, relation, or operation symbol, or an expression that, as in the examples above, can be construed as one. An interpretation of such a name is then a predicate, relation (of the appropriate number of argument places) or operation on the domain of the interpretation. The notions of *validity,* are *consistency* are thus automatically extended to second-order statements.

The *tree method* can be applied to reasoning involving second-order statements — *second -order reasoning* —in essentially the same way as for first-order reasoning. For instance, let us apply the tree test for validity to the statement (1) above. We get the tree below, in which the vertical lines have been omitted.

$$\neg \forall x \forall y[x = y \leftrightarrow \forall P(Px \rightarrow Py)]$$

$$\vdots$$

$$\neg[m = n \leftrightarrow \forall P(Pm \rightarrow Pn)]$$

Left branch:

$$m = n$$
$$\neg\forall P(Pm \rightarrow Pn)$$
$$\exists P\neg(Pm \rightarrow Pn)$$
$$(^{**}) \quad \neg(Km \rightarrow Kn)$$
$$Km$$
$$\neg Kn$$
$$Kn$$
$$\times$$

Right branch:

$$m \neq n$$
$$\forall P(Pm \rightarrow Pn)$$
$$m = m \rightarrow m = n \quad (^{*})$$

which branches into

$$m \neq m \qquad\qquad m = n$$
$$\times \qquad\qquad\quad \times$$

To obtain line (*) we applied **UI** (extended in the obvious way to second-order sentences) to the statement $\forall P(Pm \rightarrow Pn)$ immediately above by choosing the instance of the predicate variable "P" to be the expression "m = ", that is, the property of being m. To obtain line (**) we applied **EI** (extended in the obvious way to second-order sentences) to the statement $\exists P\neg(Pm \rightarrow Pn)$ immediately above by introducing a new predicate name K and substituting it for P. The tree is then seen to close, so that the sentence in question is a logical truth. This means, in effect, that in second-order logic identity $x = y$ can be defined as $\forall P(Px \rightarrow Py)$.

Similarly, in the case of sentence (2), we get the tree

$$\neg \exists R\, Rmn$$
$$|$$
$$\forall R\, \neg Rmn$$
$$|$$

$$\neg(m = n \vee m \neq n)$$

:

:

×

where the last line is obtained from second-order **UI** by choosing for R the relation x = y ∨ x ≠ y.

Exercises

F1. Using the tree method , show that the following arguments are valid: **(i)** ∀P(Pm → Pn), ∀P(Pn → Po). ∴ ∀P(Pm → Po), **(ii)** ∀P(Pm → Pn) ∴ ∀P(Pn → Pm).

VII. CONTEXTAL (MODAL) LOGIC

I. Contextual Statements.

In classical propositional logic statements are simply true or false. The truth value of a statement is taken to be entirely independent of the context or situation in which it is asserted. *Contextual logic,* by contrast, gives expression to the fact the truth or falsity of statements *can depend on the context of assertion.* For instance,

snow is falling here

is a context-dependent statement of the kind we have in mind: its truth depends on the exact *location* of "here", which, accordingly, plays the role of the context.

Another example of a context-dependent statement is

snow ids falling throughout a 10 square mile circular region centred 50 miles north of here.

There is a connection between these two types of statement. Let us write simply p for the partial statement *snow is falling.* Then the statement *snow is falling at (location) x* is equivalent to the statement

p *is true at x*.

Given a location *x,* let us call any location *y* within a 10 square mile circular region centred 50 miles north of *x* a *location of relevance to x* and the set of such locations the *region of relevance determined by x*: this will of course vary with *x*. Let us write □p (read "box p") for the partial statement *snow is falling at every location of relevance*, or, equivalently, *snow is falling throughout the region of relevance.* In that case, the statement

snow is falling throughout a 10 square mile circular region centred 50 miles north of x

may be abbreviated to the statement

$\Box p$ *is true at x*.

Similarly, if we write $\Diamond p$ (read "diamond p") for the partial statement

snow is falling at some location of relevance,

then the statement

snow is falling at some location within a 10 square mile circular region centred 50 miles north of x

may be written

$\Diamond p$ *is true at x*.

We note the equivalences

(E) $\neg\Diamond p \Leftrightarrow \Box\neg p$ and $\neg\Box p \Leftrightarrow \Diamond\neg p$.

For $\neg\Diamond p$ is true at x $\Leftrightarrow$ $\Diamond p$ is false at x $\Leftrightarrow$ p is false at every location relevant to x $\Leftrightarrow$ $\neg p$ is true at every location relevant to x $\Leftrightarrow$ $\Box\neg p$ is true at x. Similarly for the other equivalence.

The symbols $\Box$ and $\Diamond$ are *operators*, which, like $\neg$, when applied to a propositional statement (such as "snow is falling") yield new propositional statements. They are called *contextual* or *modal* operators. Thus the class of *contextual* (*propositional*) *statements* is defined by adding to the formation rules for propositional statements the clause:

if p *is a statement, so are* $\Box$p *and* $\Diamond$p.

As indicated by the example above, we think of the truth values of contextual statements as being implicitly determined by *contexts*. This idea leads us to adopt the following *tree rules* for contextual statements. First, the

$\Diamond$ *rule:*

$$\frac{\Diamond p}{p}$$

This may be read: *if* $\Diamond$p *occurs in a tree, a new context may be introduced immediately below and* p *asserted there.*

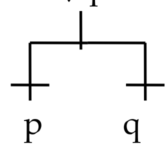
Here we have incorporated a new device into our trees, namely that of introducing or moving to a *new context*. *We will indicate a change of context by means of a solid horizontal line*: thus two statements in a given path not separated by a horizontal line are said to be in the *same context:* it follows that a tree containing n horizontal lines contains $n + 1$ contexts. It is important to understand that each application of the $\Diamond$ rule to statements appearing in the same context *requires the introduction of a separate and independent new context.* This is illustrated by the following example

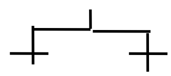
:

:

◊p

◊q



Here it is important to note that *the fork*  *does* not *indicate the splitting of the given path into two branches*; it signifies merely the introduction of two independent contexts *within a single path*.

Our second new rule is the

□ rule:

□p

....

p

This may be read: *if* □ p *occurs in a given context, and a new context is introduced just below that context, then* p *may be asserted there.*

Note that in presenting these rules a solid line will indicate that it is permissible to introduce a new context, while a broken line means only that if *some other rule* allows us to introduce a new context, then the rule in question allows us to assert something in it.

We also adopt the

Interchange rules

$\neg\Box p$	$\neg\Diamond p$
|	|
$\Diamond\neg p$	$\Box\neg p$

These are the tree rules corresponding to the equivalences **(E)** stated above.

Finally, we declare a path to be *closed* only when it contains some statement and its negation *not separated by a horizontal line*, that is, within the same context. As usual, a tree is said to be *closed* if all its paths are closed.

We write ■ for the system of tree rules consisting of the rules for the propositional operators, the $\Box$- and $\Diamond$-rules, the interchange rules, and the new rule for closing a path. This system is called *basic contextual logic.* A tree constructed in accordance with the ■-rules is called a ■-*tree* A statement p is ■-*valid* if there is a closed ■-tree with initial statement ¬p[15].

Let us use these new tree rules to establish some simple properties of the system. First, we show that the statement

$$\Box p \to q) \to (\Box p \to \Box q)$$

is ■-valid.

$$\neg[\Box (p \to q) \to (\Box p \to \Box q)]$$

|

15 In general, if **R** is a collection of tree rules, a **R**-tree is a tree constructed in accordance with the rules of **R,** and a statement *p* is **R**-*valid* if there is a closed **R**-tree with initial statement ¬*p*. This should be borne in mind in connection with the logical systems to be presented in the sequel.

$$\Box(p \to q) \quad (\#)$$
$$\neg(\Box p \to \Box q)$$
$$|$$
$$\Box p \quad (*)$$
$$\neg\Box q$$
$$|$$
$$\Diamond\neg q \quad (+)$$
$$\overline{\qquad\qquad}$$
$$\neg q \quad (++)$$
$$|$$
$$p \quad (**)$$
$$|$$
$$p \to q \quad (\#\#)$$
$$\neg p \qquad\qquad q$$
$$\times \qquad\qquad \times$$

Here (++) is derived from (+) by the ◊-rule, and (**) from (*), as well as (##) from (#) by the □-rule.

Next, we show that, if p is ■-valid, then so is □p, and conversely. For the ■-validity of p means that there is a closed ■-tree with initial statement ¬p:

(*)

$$\neg p$$
$$:$$
$$\times$$

This yields a closed tree

$$\neg\Box p$$
$$|$$
$$\Diamond\neg p$$
$$\overline{\qquad\qquad}$$
$$\neg p$$
$$: \quad (*)$$
$$\times$$

Here the nodes below the horizontal line reproduce the tree (*). So p is valid.

Conversely, suppose that $\Box p$ is ■-valid. Then there is a closed ■-tree

(**)

$$\begin{array}{c} \neg\Box p \\ \vdots \\ \times \end{array}$$

Now the only ■-rule applicable to $\neg\Box p$ is the appropriate interchange rule, so that the tree (**) must begin:

$$\begin{array}{c} \neg\Box p \\ | \\ \Diamond\neg p \\ \vdots \end{array}$$

Similarly, the only rule applicable to $\Diamond\neg p$ is the $\Diamond$-rule, so (**) must look like

$$\begin{array}{c} \neg\Box p \\ | \\ \Diamond\neg p \\ \hline \neg p \\ \vdots \\ \times \end{array}$$

But then the portion of this tree below the horizontal line is a closed tree with initial statement $\neg p$. Accordingly p is ■-valid .

Exercises

A1. By constructing closed trees, establish the ■-validity of the statements $\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$, $\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$, $\Box p \vee \Box q \rightarrow \Box(p \vee q)$, $(\Box p \rightarrow \Diamond q) \rightarrow \Diamond(p \rightarrow q)$, and $(\Box p \rightarrow \Diamond q) \rightarrow \neg\Box\mathbf{f}$ (here **f** is any contradiction, e.g. $A \wedge \neg A$).

A2. Show that ■ satisfies the *disjunction principle*: if $\Box p \vee \Box q$ is ■-valid, then at least one of p, q is ■-valid. (Hint: consider a closed tree with initial statement $\neg(\Box p \vee \Box q)$ and apply the same sort of analysis as was applied to the tree (**) on the previous page.)

2. Interpretations of Contextual Statements

As might be expected, the formal definition of an *interpretation* of contextual statements involves both the abstract notion of a *context* and the relation of *relevance* among concepts. Thus we define a *contextual structure* to be an ordered pair $(C, R) = \mathbf{C}$ in which C is a nonempty set and R is a binary relation on C. The members of C are called *contexts* of **C** and R the *relevance relation* of **C**. We use symbols a, b, c to indicate members of C; aRb is read "b is *relevant* to a."

An *interpretation* of contextual statements in a contextual structure **C** is an assignment, to each pair (p, a) consisting of a contextual statement p and a context a, of an element $\boldsymbol{I}(p, a)$ of the set of truth values {t, f} in such a way that

(1) $\mathbf{I}(\neg p, a) = t \;\Leftrightarrow\; \mathbf{I}(p, a) = f$

(2) $\mathbf{I}(p \wedge q, a) = t \;\Leftrightarrow\; \mathbf{I}(p, a) = \mathbf{I}(q, a) = t$

(3) $\mathbf{I}(p \vee q, a) = t \;\Leftrightarrow\; \mathbf{I}(p, a) = t$ or $\mathbf{I}(q, a) = t$

(4) $\mathbf{I}(p \rightarrow q, a) = t \;\Leftrightarrow\; \mathbf{I}(p, a) = f$ or $\mathbf{I}(q, a) = t$

(5) $\mathbf{I}(p \leftrightarrow q, a) = t \;\Leftrightarrow\; \mathbf{I}(p, a) = \mathbf{I}(q, a)$

(6) $\mathbf{I}(\Box p, a) = t \;\Leftrightarrow\; \mathbf{I}(p, b) = t$ for every b in C such that aRb

(7) $\mathbf{I}(\Diamond p, a) = t \;\Leftrightarrow\; \mathbf{I}(p, b) = t$ for some b in C such that aRb.

We think of $\boldsymbol{I}(p, a) = t$ as asserting that p *is true* (*under* **I**) *in the context a*.

Thus clause **(6)** may be construed as saying that

□p *is true in a context just when* p *is true in all contexts relevant to the given one,*

and clause **(7)** reads

◊p *is true in a context just when* p *is true in at least one context relevant to the given one*

It follows immediately that, if *a* is a context with *no* contexts relevant to it, then, for any statement p, □p is true in *a* and ◊p is false in *a*.

It will be convenient to write

$$a \Vdash_I \mathrm{p} \;\text{ or }\; a \Vdash \mathrm{p}$$

for ***I***(p ,*a*) = t. The assertion $a \Vdash \mathrm{p}$ is read "*a forces* (the truth of)p". We also write

$$a \nVdash \mathrm{p}$$

for the negation of $a \Vdash \mathrm{p}$.

Using this new symbol ⊩ the conditions just given for contextual interpretations take the following form:

(1) $a \Vdash \neg\mathrm{p} \;\Leftrightarrow\; a \nVdash \mathrm{p}$.

(2) $a \Vdash \mathrm{p} \wedge \mathrm{q} \;\Leftrightarrow\; a \Vdash \mathrm{p}$ and $a \Vdash \mathrm{q}$

(3) $a \Vdash \mathrm{p} \vee \mathrm{q} \;\Leftrightarrow\; a \Vdash \mathrm{p}$ or $a \Vdash \mathrm{q}$

(4) $a \Vdash \mathrm{p} \rightarrow \mathrm{q} \;\Leftrightarrow\; (a \Vdash \mathrm{p} \Rightarrow a \Vdash \mathrm{q})$

(5) $a \Vdash p \leftrightarrow q \Leftrightarrow (a \Vdash p \Leftrightarrow a \Vdash q)$

(6) $a \Vdash \Box p \quad \Leftrightarrow \quad b \Vdash p \quad$ for every b in C such that aRb

(7) $a \Vdash \Diamond p \quad \Leftrightarrow \quad b \Vdash p \quad$ for some b in C such that aRb.

Exercise. Show that $a \Vdash \Diamond p$ if and only if $a \Vdash \neg\Box\neg p$.

Clearly any assignment of truth values to pairs (A, a), where A is a statement letter and a a context of **C** generates a unique interpretation determined by clauses (1) - (7). So in specifying an interpretation in a given contextual structure we need only specify the truth values it assigns to pairs of that form.

We say that a contextual statement p is *true* under an interpretation ***I*** in a contextual structure **C** = (C, R) if $a \Vdash_I p$ for all a in C, that is, if p holds under ***I*** in every context of **C**. We say that p is *satisfiable* if $a \Vdash_I p$ for some interpretation ***I*** and some context a.

We now show that ■*-valid contextual statements are true under every interpretation*. (Recall that "■-valid" means "negation generating a closed ■-tree.") This is proved in the same way as the inference correctness for propositional trees. We first specify what it means for a tree rule to be *correct.* If the rule is an old (propositional) rule or either of the interchange rules, we say that it is correct if whenever its premise holds under a given interpretation in a given context, all the statements in at least one of its lists of conclusions hold under that interpretation. As for the □- and ◊-rules, we say that either is correct if whenever its premise is true in a given context under a given interpretation, its conclusion holds under the same interpretation *in some context relevant to the given one.* It is readily shown that all tree rules are correct in this sense. Thus, starting with a satisfiable statement q, there is an interpretation under which, and a context in which, q is true. Since each tree rule is correct, it readily follows that any tree with initial statement q will contain at least one complete open path. If p is valid, then the finished tree with initial statement ¬p is closed, and so ¬p cannot be satisfiable; in other words p is true under every interpretation.

As in the case of ordinary propositional logic, the tree method for contextual logic can be used to generate *counterexamples* or *countermodels*, that is, interpretations in which invalid statements are *false*. We give a couple of examples which will serve to indicate the general procedure.

1. *A countermodel for* $\Box A \rightarrow \Box\Box A$. Here we generate the following finished ■-tree:

$\neg(\Box A \rightarrow \Box A)$
|
$\Box A$
$\neg \Box A$ 1
|
$\Diamond\neg \Box A$

$\neg \Box A$
|
A 2
|
$\Diamond\neg A$

$\neg A$ 3

Since this tree contains *two* horizontal lines, it contains *three* contexts which we label 1, 2, 3, and each context is relevant to the one immediately below it. This may be represented by a "relevance diagram"

. → . → .
1 2 3

in which the nodes represent contexts and each arrow goes from a context to one relevant to it. The diagram determines a contextual structure C = (*C*, *R*) in which *C* = {1,2,3} and *R* = {(1, 2), (2, 3)}. The countermodel will be an interpretation ***I*** in C making the initial statement of the tree false in the context (1) in which it appears. As with propositional trees we allow the truth values assigned to statement letters to be determined by whether they occur positively or negatively, only now this assignment will also depend on the *context* in which they occur. We see that, in the tree above, A occurs positively in context 2 and negatively in context 3. So our interpretation ***I*** we should define ***I***(A, 2) = *t*, ***I***(A, 3) = *f*. (The value of ***I***(A, 1) is irrelevant.) Our interpretation may then be displayed by the diagram

$$\begin{array}{ccccc} & & t & & f \\ \cdot & \rightarrow & \cdot & \rightarrow & \cdot \\ 1 & & 2 & & 3 \end{array}$$

Let us verify that □A → □A is false under **I** in context 1. Since 2 is the only context relevant to 1, and 2 ⊩ A, it follows that 1 ⊩ □A. On the other hand, since 3 is the only context relevant to 2, and 3 ⊮ A, it follows that 2 ⊮ □A, But this means that 1 ⊮□A. Therefore 1 ⊮ □A → □A, as claimed. So **I** is a countermodel for □A → □A.

2. *A countermodel for* □(A ∨ B) → □A ∨ □B. In this case we generate the following finished tree:

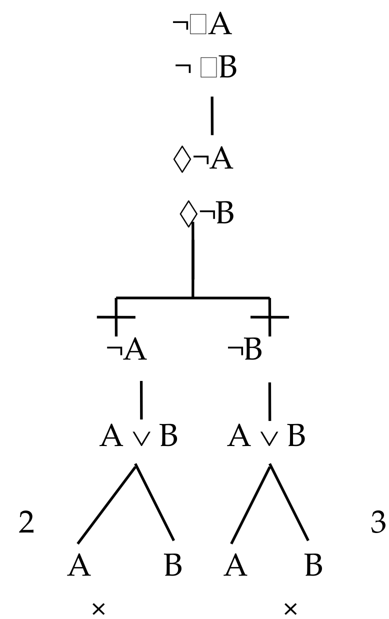
¬[□(A ∨ B) → □A ∨ □B]

|

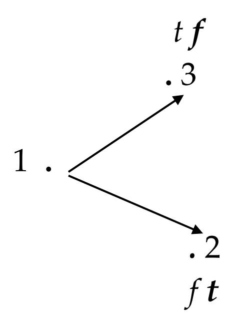
□(A ∨ B)

¬(□A ∨ □B)   1

|



In this case the diagram of the interpretation ***I*** determined by the tree is, writing *t, f* and ***t, f*** for the truth values assigned to *A,B* respectively,



Let us verify that $\Box(A \vee B) \rightarrow \Box A \vee \Box B$ is false under **I** in context 1. To begin with, since **I**$(A \vee B, 2)$ = **I**$(A \vee B, 3)$ = t, and 2,3 are the only contexts relevant to 1, it follows that 1 $\Vdash \Box(A \vee B)$. On the other hand, since **I**$(A, 2)$ = **I**$(B, 3)$ = f, it follows that **I**$(\Box A, 1)$ = **I**$(\Box B, 1)$ = f. Therefore 1 $\nVdash \Box A \vee \Box B$, and so 1 $\nVdash \Box(A \vee B) \rightarrow (\Box A \vee \Box B)$ as claimed.

Although we shall not prove it here, it can be shown that this method of construction works in general, that is, each ■-invalid contextual statement is false under some interpretation. Equivalently, any contextual statement true under every interpretation is valid.

Exercises

B1. Construct countermodels for the following statements: **(i)** $\Box A \to A$, **(ii)** $A \to \Box A$, **(iii)** $\Box A \to \Diamond A$, **(iv)** $\Diamond\Box A \to A$, **(v)** $\Box(\Box A \to A) \to \Box A$. Show that $\Box A \to \Box A$ is true in context 1 of the countermodel for **(i).**

3. Other Systems of Contextual Logic

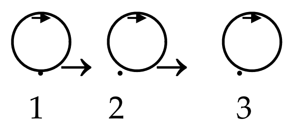
So far we have imposed no conditions whatsoever on relevance relations. One possible and, indeed, natural condition to consider is that of *reflexivity: aRa* for any $a \in C$. This condition means that each context is *self-relevant*. A contextual structure whose relevance relation R satisfies this condition is called *reflexive*. Truth in reflexive contextual structures is captured by adding the following $\Box$-*elimination rule* to our system $\blacksquare$ of tree rules:

$\Box$-elimination

$\Box$p
|
p

We write $\blacksquare$**1** for the resulting system of tree rules.

Clearly $\Box A \to A$ becomes a $\blacksquare$**1-**valid statement. On the other hand, the statement $\Box A \to \Box A$ remains $\blacksquare$**1-**invalid. This can be seen by returning to the tree on p. 43 which generates a countermodel for □A → □A. This tree can be finished in accordance with the rules of ■**1** by adding a node with "A" on it in context 1. Since we want the relevance relation of our interpretation to be reflexive, the original relevance diagram must now have loops attached to each node, as in



And in addition to assigning the value "*t*" to *A* in context 2, and "*f*" in context 3, the interpretation must assign "*t*" to *A* in context *1*. Then, as before, □*A* → □*A* is false in context 1 under this interpretation.

Another natural condition that can be imposed on a relevance relation is that it be *transitive*: $(aRb \wedge bRc) \Rightarrow aRc$. A contextual structure whose relevance relation satisfies this condition is called *transitive*. Truth in transitive contextual structures is captured by adding to the following rule to the system ■:

□***-repeat***
□p
....
□p

(Recall that the broken line indicates that if a new context is introduced, something may be asserted in it.) The resulting system is denoted by ■**2**.

The closed tree below shows that □p → □p is ■**2-**valid.

$$\neg(\Box p \to \Box p)$$
$$|$$
$$\Box p$$
$$\neg \Box\Box p$$
$$|$$
$$\Diamond\neg\Box p$$
$$\overline{\neg\Box p}$$
$$|$$
$$\Box p$$
$$\times$$

Here the last line is derived from the second by the □-repeat rule.

Invalidity in **■2** is established, as before, by using trees to generate countermodels, only now the contextual structure in each countermodel must be transitive. For instance, it will be found that this is the case for the countermodels for $\Diamond A \to A$ and $\Diamond\Box A \to A$ in the exercise **B1** below. So neither of these two statements are **■2-**valid.

The system **■3** is obtained by amalgamating **■1** and **■2**, in other words, by adding both the □-elimination and □-repeat rules to **■. ■3** captures truth in *ranked* contextual structures, those whose relevance relations are *transitive* and *reflexive*, that is, are *rankings.*

Another possible condition on a relevance relation is the extreme (but not unnatural) one that *all contexts are relevant to one another*. A contextual structure (C, R) satisfying this condition will then have $R = C \times C$—we shall call such contextual structures *full*—and the clause for the truth of p under an interpretation ***I*** in such a structure becomes

$$\boldsymbol{I}(\Box p, a) = t \Leftrightarrow \boldsymbol{I}(p, b) = t \text{ for every } b \in C.$$

That is, □p is true in a particular context if and only if p is true in all contexts. Similarly, ◊p is true in a particular context if and only if p is true in some context. In this event, it is natural to say that the truth of □p means the necessary truth of p— truth in every conceivable context—and the truth of ◊p means the possible truth—truth in some conceivable context. Construed in this way, □and ◊ are the so-called modal operators of necessity and possibility. For this reason, what we have called contextual logic is usually known as modal logic.

Let **■4** be the system obtained from **■1** by adding the additional "closure" rule:

(**■4closure**) If either of the pairs of statements (□p, ¬p), or (□¬p, p) occur in a path, close the path.

■4 captures truth in full contextual structures.

As an illustration of how this new rule works, let us show that the statement ◊□p → □p is **■4-**valid. Here is the relevant tree:

¬(◊□p → □p)

◊□p

¬□p

◊¬p

□p ¬p

×

Here $\Box p$ and $\neg p$ appear on different tines of the fork, but nevertheless in the tree's single path, which therefore closes.

Exercises

C1.Show that $p \rightarrow \Diamond p$ is **■1-**valid, and construct countermodels to show that the statements $\Diamond\Box A \rightarrow A$ and $\Box(\Box A \rightarrow A) \rightarrow \Box A$ are both **■1-**invalid.

C2. Show that **■1** satisfies the disjunction principle: for any statements p, q, if $\Box p \vee \Box q$ is **■1-**valid, then at least one of p, q is **■1-**valid.

C3. Show that the statement $\Diamond\Diamond p \rightarrow \Diamond p$ is **■2**-valid.

C4. Show that **■2** satisfies the disjunction principle: for any statements p, q, if $\Box p \vee \Box q$ is **■2-**valid, then at least one of p, q is **■2 -**valid.

C5. Show that the following statements are **■3-**valid. **(i)** $\Diamond\Diamond p \leftrightarrow \Diamond p$; **(ii)** $\Box\Diamond\Box\Diamond p \leftrightarrow \Box\Diamond p$; **(iii)** $\Box\Diamond(\Diamond\Box p \rightarrow \Box p)$

C6. Show that the following statements are **■3**-invalid: **(i)** $\Box\Diamond A \rightarrow A$ **(ii)** $\Diamond\Box A \rightarrow A$; **(iii)** $\neg\Box A \rightarrow \Box\neg\Box A$.

C7. Show that **■3** satisfies the disjunction principle: for any statements p, q, if $\Box p \vee \Box q$ is **■3-**valid, then at least one of p, q is **■3-**valid.

C8. (i) Establish the **■4**-validity of the following statements: **(i)** $\Box p \rightarrow p$, **(ii)** $p \rightarrow \Diamond p$, **(iii)** $\Box p \leftrightarrow \Box\Box p$**, (iv)** $\Diamond\Diamond p \leftrightarrow \Diamond p$, **(v)** $\Box\Diamond\Box\Diamond p \leftrightarrow \Box\Diamond p$, **(vi)** $\Box p \leftrightarrow \Diamond\Box p$, **(vii)**

$\Diamond p \leftrightarrow \Box\Diamond p$, **(viii)** $\Diamond\Box p \rightarrow p$. Deduce that in ■**4** any statement of the form ______…A, where each "_" is either □ or ◊, is either equivalent to $\Diamond A$, or to $\Box A$. Deduce from **(vii)** that ■**4** does not satisfy the disjunction principle.

C9. By constructing countermodels, show that the statements $\Box\Diamond A \rightarrow A$, $A \rightarrow \Box A$ and $\Box(\Box A \rightarrow A) \rightarrow \Box A$ are ■**4-**invalid. (Remember that a ■**4** countermodel has to be a full contextual structure.)

C10. (i) Show that the tree rules

$$\begin{array}{c} p \\ | \\ \Box p \end{array} \qquad\qquad \begin{array}{c} \Box p \\ | \\ p \end{array}$$

are correct for interpretations in contextual structures in which the relevance relation is the *identity relation,* that is, in which each context is *relevant only to itself*.

(ii) Show that the "premiseless" tree rule

$$\begin{array}{c} | \\ \Box p \end{array}$$

is correct for interpretations in contextual structures in which the relevance relation is *empty*, that is, in which *no contexts are relevant to one another.*

4. Other Interpretations of □ and ◊

We have seen that, in addition to its "contextual" interpretation, one possible meaning that can be assigned to □ is "it is *necessarily true* that". This is known as the *alethic* interpretation (from Greek *aletheia,* "truth"). There are several others, for example: "it is *known* that", the *epistemic* interpretation (from Greek *episteme*, "knowledge"); "it is *believed* that", the *doxastic* interpretation (from Greek *doxa*, "opinion"); "it is obligatory that", the *deontic* interpretation (from Greek *deon*, "duty"); "it is *demonstrable* that", the *apodeictic* interpretation (from Greek *apodeiknunai,* "demonstrate").

As we have remarked, the system ■**4** is an appropriate set of rules for the alethic interpretation. The system ■**3** provides a reasonably faithful set of rules for the epistemic interpretation, and ■**2** for the doxastic interpretation. As for the deontic interpretation, the only rule which would seem to be correct (in addition to those of the basic system ■) is

□p
|
◊p

"whatever is obligatory is permissible." The system obtained by adding this rule to ■ is denoted by ■**½.**

Exercises

D1. Show that the above rule for the deontic interpretation is equivalent to the following "closure" rule:

□p
:
:
□¬p
×

D2. Why isn't ■**1** suitable for the doxastic interpretation? What about ■**½?**

VIII. EVIDENTIAL (INTUITIONISTIC) LOGIC

1. Evidential Logic and Evidential Tree Rules.

The idea behind *evidential logic* is that statements are only asserted to be true when one is in possession of, or has the means of producing, *evidence* for the fact. Evidential logic is usually called *intuitionistic* logic because of its connection with a movement in the philosophy of mathematics called *intuitionism.* The basic tenet of intuitionism is that mathematical concepts and arguments are admissible only if they are adequately grounded in *mental intuition* - in other words, that the mind can produce evidence for their truth.

Confining our attention to statements of propositional logic, what should count as evidence for the truth of statements? In the case of elementary declarative statements, evidence could take the form of direct verification. For example, the experience of seeing snow falling is evidence for the truth of the statement *snow is falling*, and performing a computation is evidence for the truth of the statement $73 \times 137 = 10001$. In the legal sphere the issue of what constitutes evidence is more complex. Jurists recognize two types of evidence: *circumstantial* evidence and *direct* evidence. Circumstantial evidence relies on an inference to connect it to a conclusion of fact, such as a fingerprint at the scene of a crime. Direct evidence, on the other hand, supports the truth of an assertion without the use of any intermediate inference , such as a (truthful) eyewitness account of a crime actually being committed.

What should constitute evidence for the truth of *compound* statements ? Whatever is taken to be evidence for the truth of elementary statements, for compound statements it seems natural to characterize the idea of evidence in the following way. (Here we abbreviate "evidence for the truth of p" to "evidence for p", and "we have evidence" is to be understood as "in principle we can produce evidence".)

(1)*We have evidence for* $p \wedge q$ *provided we have evidence for* p *and evidence for* q

(2) We have evidence for $p \vee q$ *provided we have evidence* for p *or we have evidence for* q[16]

(3) We have evidence for $p \rightarrow q$ *provided that, whenever we have evidence for* p, *we can produce evidence for* q - *that is, evidence for* p *can always be converted into evidence for* q

(4) We have evidence for $\neg p$ *provided that we have evidence for the impossibility of ever producing evidence for* p.

---

[16] This clause is well illustrated by the handing down of verdicts in courts of law. At the end of a trial, the judge will ask the jury foreman, *is the defendant guilty or not guilty*? The foreman would risk being ejected from the court if he facectiously gave the answer `yes``, even though, strictly speaking, this answer is correct according to the rules of classical logic. What is required of the jury is an examination of evidence for the truth of the disjuncts `guilty`` `/``not guilty``, and a declaration that just *one* is true.

We shall take the idea of evidence to be of sufficient clarity that, whenever we are given a statement p, either we are certain that we *have* evidence for p, or we are certain that we *do not have* evidence for p. In other words, we shall assume that the *principle of bivalence* holds for the possession of evidence. That being the case, we define the *evidential status* of a statement p to be the truth value t if we have evidence for p, and the truth value f if we do not. We say that p is *evidentially true* or simply *e- true* if we have evidence for p. A statement is *e- valid* if it is e-true independently of the evidential status of the statement letters from which it is constructed. Thus e-validity of a statement means that it can *always be provided with evidence for its truth.* This is analogous to validity in classical logic, where a valid statement is one that is always true.

Many valid statements of classical logic are also e-valid. For example, consider the classically valid statement $p \wedge q \rightarrow p$. Why is this statement e-valid?. Because, by (3), we have evidence for $p \wedge q \rightarrow p$ provided that, whenever we have evidence for $p \wedge q$, we can produce evidence for p. But if we have evidence for $p \wedge q$, then by (1) we also have evidence for p. So, whenever we have evidence for $p \wedge q$, we can produce evidence for p. Thus we have evidence for $p \wedge q \rightarrow p$ and so it is e-valid.

But *certain classically valid statements are not e-valid.* For example, consider the law of excluded middle $A \vee \neg A$. For this to be e-valid it must be the case that, for any statement A, either A is e-true or ¬A is e-true. In other words, e-truth would have to satisfy the principle of bivalence. But that would mean that we must always either have evidence for A or evidence for $\neg A$. That is, we must always have evidence for A or evidence of the impossibility of ever providing evidence for A. Now clearly, there are many statements A for which we have no such evidence. For instance, take A to be the statement *Betelgeuse has planets.* We have no clue (at the present time at least) whether Betelgeuse has planets, let alone any demonstration of the impossibility of its having any. That is, we cannot affirm the e-truth of A, or the e-truth of $\neg A$. So the principle of bivalence fails for e-truth[17], and so the law of excluded middle is not e-valid. (Of course, it might happen that future astronomers actually determine whether Betelgeuse has planets or not, in which case one of the disjuncts A, $\neg A$ will become e-true. The point is that we do not always have evidence for $A \vee \neg A$.)

As another example, consider the law of double negation $\neg\neg A \rightarrow A$. For this to be e-valid, it is required that whenever we have evidence for $\neg\neg A$, we can produce evidence for A. Now, by (4), evidence for $\neg\neg A$ amounts to demonstrating the impossibility of providing evidence for $\neg A$, thus that we can show the impossibility of the impossibility of providing evidence for A. But asserting the impossibility of the impossibility of providing evidence for an assertion is much weaker than actually producing evidence for it. Again, consider the statement Betelgeuse has planets. Genuine evidence that Betelgeuse has planets would consist of actually observing them, or indirectly inferring their presence by the observed gravitational effects on other astronomical objects.[18] The double impossibility of producing such evidence hardly counts as genuine evidence in this sense.

Exercise. Show that $\neg\neg A \rightarrow A$ is *e*-valid.

Once evidence has been provided for the truth of a statement, we can say that the statement is *known* to be true. Thus evidential logic (so defined) is a kind of *epistemic* (or apodeictic) logic. The principal difference between them is that, in epistemic logic, as in any contextual logic, statement operators $\Box$, $\Diamond$ are introduced as explicit devices to represent knowledge or evidence, thereby enlarging the class of statements, while the

[17] Note that e-truth principle of bivalence fails for e-truth in general despite the fact that the principle of bivalence has been assumed to hold for the possession of evidence.

[18] Indirect evidence of this sort is the counterpart, in science, of circumstantial evidence in law.

meaning of—and thus the logical principles governing—the statements on which they operate *remains the same*: simple truth or falsity. In evidential logic, on the other hand, to assert a sentence is implicitly to assert that *we possess evidence for the truth of that sentence.* Thus the criterion of evidence is, so to speak, *injected into the meaning of the statements themselves*. As we have seen above, this results in *a change in the rules of reasoning.*

We shall conceive of evidence for statements as coming in *stages,* each stage consisting of a *body of evidence.* These stages will be conceived of as being structured by the relation of *enlargement.* A stage of evidence *b* is an *enlargement* of a stage of evidence *a* if the body of evidence at stage *a* is contained in the body of evidence at stage *b.* (On occasion we shall call an enlargement of a stage a *further* or *later* stage.) Clearly the enlargement relation is both reflexive and transitive, that is a *ranking* in the sense of Chapter V.

To illustrate the idea of stages of evidence, suppose that we are looking through a collection of photographs of various people, noting each one`s hair colour. If the sequence of people generated by flipping through the photographs is *A, B, C, ...*then we obtain a corresponding sequence of stages of evidence of the form: {*A has red hair*}, {*A has red hair, B has brown hair*}, {*A has red hair, B has brown hair, C has black hair, ...*}. For each of these stages of evidence is an enlargement of any of its predecessors, and each stage (apart from the last) has a unique sequence of further enlargements. Finally note that if, example, Jim`s photograph is included in the collection and he happens to have blond hair, then not until the stage of evidence {*A has red hair, B has brown hair, C has black hair, ..., Jim has blond hair*} has been reached will we actually possess evidence for the blondness of Jim`s hair.

As another illustration, consider someone flipping a coin, and noting whether the result is a head or tail. Here the corresponding stages of evidence take the form {*the first flip is a head (or a tail)*}, {*the first flip is a head (or a tail*), *the second flip is a head (or a tail*) }, {*the first flip is a head (or a tail*), *the second flip is a head (or a tail*) *the third flip is a head (or a* *tail*) }, etc. This case differs from the one above in that the stages of evidence are not fixed in advance. It is perfectly possible, for example, for the first and second flips both to be heads, and for the first to be a head and the second to be a tail. This gives rise to *two* possible ways of enlarging the stage {*the first flip is a head*}, namely

{*the first flip is a head, the second flip is a head*}

and

{*the first flip is a head, the second flip is a tail*}.

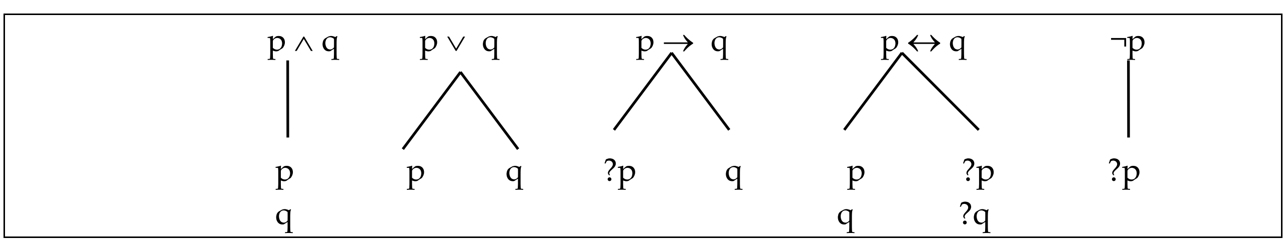
Note that *neither of these two stages of evidence is an enlargement of the other*. It follows that, while the enlargement relation is always reflexive and transitive, it is *not necessarily total.*

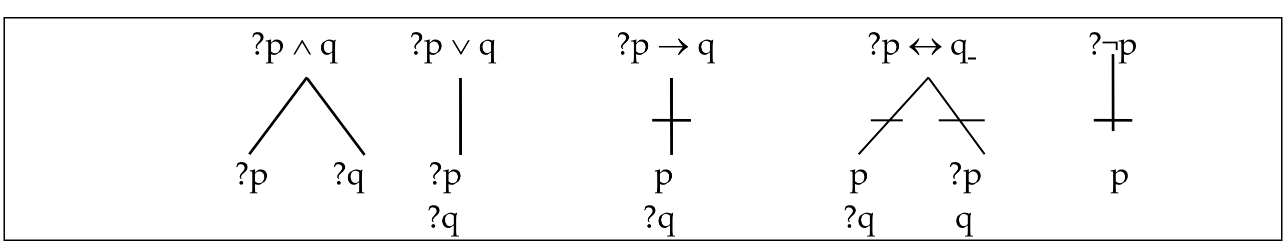
As a kind of epistemic logic, evidential logic is related to the system ■**3** of contextual logic considered in the previous chapter. Recall that this latter captures truth in *ranked* contextual structures, that is, those in which the relevance relation is *reflexive* and *transitive.* These structures also furnish natural interpretations of statements of evidential logic. When playing that role, *contexts* in such structures should be thought of *stages of evidence*, and relevance relations *R* as relations of *enlargement of evidence.* That is, if *a* and *b* are stages of evidence, *aRb* is understood to mean that the body of evidence at stage *b* is an enlargement of the body of evidence at stage *a.* In particular, evidence at stage *a* continues to count as evidence at stage *b* (so evidence does not `decay`). When *R* is thought of in this way, it is quite natural to require it to be both reflexive and transitive. The ideas of stage of evidence and enlargement relation will be used in the next section to provide a rigorous notion of interpretation for evidential logic.

Although in presenting evidential logic we have not introduced new propositional operators in the usual sense, the *tree rules* we shall formulate for it will involve a new piece of notation—the *interrogative sign ?*—which will allow us to express in a purely formal way the assertion that we *do not* possess evidence for the truth of a given statement, in other words, that the statement is not e-true. Thus, for any statement p, we will be able to write ?p and think of it as asserting that, *we do not possess evidence for* p , or p *is not e-true,* or, put epistemically, p *is not known to be true* . Clearly ?p does not entail ¬p, that is, it does not entail that we will *never* possess evidence for p. For example, we do not at the present time possess evidence for *Betelgeuse has planets,* but that by no means precludes the possibility that such evidence will emerge in the future. As we have said, we do *not* regard *?* as a new logical operator, nor do we regard *?p* as a new statement of our logical system. Rather, expressions of the form ?p are to be viewed as *purely formal constituents of trees*. The "?" sign is only allowed to be placed *at the front* of any statement. Thus no brackets are needed for writing interrogatives: for example in ?p $\wedge$ q the *?* sign must apply to the *whole* statement p $\wedge$ q, and not just to p.

We call ?p the *questioning* of p.

We now state our tree rules for evidential propositional logic. They fall into the following three groups:





Transport rule. We are allowed to carry any statement not marked by "?" across any horizontal line introduced by the ?→ , ?↔ , and ?¬ rules.

Closure rule. A path is closed when (and only when), both p and ?p occur on it not separated by a horizontal line. When this is the case, the path is marked, as before, by "×". (And, as usual, a tree is closed if all its paths are.)

Note that all these rules—apart from those for ¬, ?→ ,?↔ , ?¬ and the transport rule—are essentially the corresponding classical rules with "?" in place of "¬". The ¬ rule (which is obviously correct) is a device for converting ¬p into an expression we can work with, since our rules are formulated in terms of "?" rather than "¬". Clearly the negation rule allows us to close a path if both p and ¬p occur in it not separated by a horizontal line.

The → rule deserves a detailed explanation. The key point is that—in contrast with classical logic—implication in evidential logic is not material implication: that is, p → q is not equivalent to ¬p ∨ q. In classical logic, the justification for identifying p ∨ q with ¬p ∨ q rests on the principle of bivalence, that either p is true or ¬p is true. Then, if p is true, so is q by the entailment of q by p. It follows that either ¬p is true or q is true, On the other hand evidential logic does not (as we have seen) satisfy the principle of bivalence, this justification breaks down. What rule should we then adopt for implication? Let us assume that p implies q is *e*-true. This means that, if evidence for p comes to light, then so will evidence for q. Now recall that we have adopted the principle of bivalence for the possession of evidence. So either we have evidence for p or we do not have evidence for p. In the former case, we will also possess evidence for q, that is, q is *e*-true. It follows that either we don't have evidence for p, that is, ?p , or q is e-true. This the content of the → rule.

Next, consider the *?*$\rightarrow$ rule. Recall that in contextual logic the horizontal line indicated passage to a new context. Here the horizontal line may be taken to signalize *advancing to a new (enlarged) stage of evidence*. Why is it needed? Because, if at some stage, we have no evidence for $p \rightarrow q$, it *could* turn out that, at some further stage, evidence for *p* comes to light thereby yielding evidence for q . This is the content of the *?*$\rightarrow$ rule. Similarly for the *?*$\leftrightarrow$ rule.

Lastly, consider the *?*$\neg$ rule. To assert $?\neg p$ at a given stage is to say that we have no evidence for $\neg p$, in other words that we have no evidence that it is impossible to produce evidence for p. But in that case it is possible that evidence for p may come to light, so that p becomes e-true at some further stage.

Two statements on the same path not separated by a horizontal line may be said to occur *at the same stage of evidence*. The closure rule expresses the obvious fact that at a given stage of evidence, it cannot be the case that we both possess and do not possess evidence for a given statement.

The transport rule is designed to reflect the idea that evidence for truth does not "decay" we have evidence for a statement p at some stage, then that evidence will persist at all further stages. This property is called the *persistence* of *e*-truth.

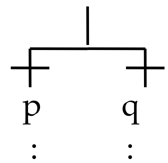
It is important to note that in applying the *?*$\rightarrow$ or *?*$\neg$ rules to statements on a single path *occurring at the same stage of evidence*—that is, not separated by a horizontal line—it is necessary to introduce a separate and independent horizontal line for each such application. This is illustrated by

$$?\neg p \vee \neg q$$

$$|$$

$$?\neg p$$

$$?\neg q \qquad (*)$$



in which the fork is obtained by independent applications of the ?¬ rule to ?¬p and ?¬q *within a single path.*

The tree test for validity is applied to evidential statements in just the same way as for classical statements, except that ? replaces ¬. That is, to determine whether a statement p is evidentially valid (according to the tree rules), start a tree with ?p: if the tree closes in the new sense, p is valid. (If it doesn't, then p is invalid and a countermodel can be read off from the tree, as we shall see in the next section.)

Here are some examples of evidentially valid statements. In each case, the tree closes through straightforward application of the above rules.

1. $p \rightarrow p$

$?p \rightarrow p$

---

p

$?p$

×

2. $p \rightarrow (q \rightarrow p)$.

$?p \rightarrow (q \rightarrow p)$

---

p

$?q \rightarrow p$

---

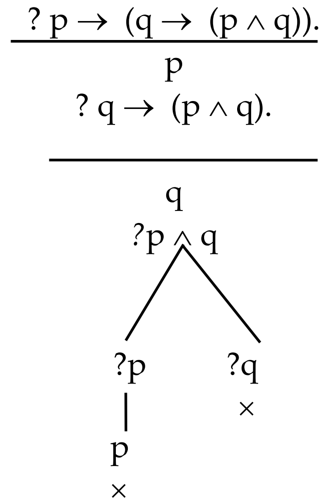
q

$?p$

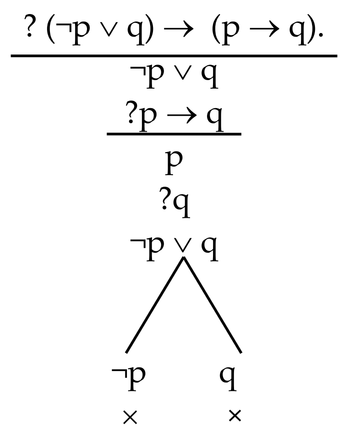
p

×

3. $p \rightarrow (q \rightarrow (p \wedge q))$.



4. $(\neg p \vee q) \rightarrow (p \rightarrow q)$.



In exercise **A4** below it is shown that each classically valid statement is evidentially valid. From this it follows immediately that, if , if p is classically valid, then $\neg\neg p$ is evidentially valid. In fact, (although we shall not prove this here), the converse also holds, so that. for any statement p, p *is classically valid if and only if* $\neg\neg p$ *is evidentially valid.* In exercise **A1** below it is shown that, for, any statement p, $\neg p$ and $\neg\neg\neg p$ are evidentially equivalent. From we infer that, *for negated statements, classical validity coincides with evidential validity.* For $\neg p$ is classically valid if and only if $\neg\neg\neg p$ is evidentially valid and $\neg\neg\neg p$ is evidentially equivalent to $\neg p$ **.** It follows that $\neg p$ is classically valid if and only if $\neg p$ is evidentially valid. What this shows is that *evidence can never be provided for a classical contradiction.*

Exercises

A1. By constructing closed trees, show that each of the following statements is evidentially valid. **(i)** $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$. **(ii)** $(p \wedge q) \to p$. **(iii)** $p \to (p \vee q)$. **(iv)** $(p \to r) \to ((q \to r) \to (p \vee q \to r))$. **(v)** $(p \to q) \to ((p \to \neg q) \to \neg p)$. **(vi)** $\neg p \to (p \to q)$. **(vii)** $(p \wedge (p \to q)) \to q$. **(viii)** $[(p \vee \neg p) \to ((p \to q) \to (\neg p \vee q))]$. **(ix)** $p \to \neg\neg p$. **(x)** $\neg\neg\neg p \leftrightarrow \neg p$. **(xi)** $(p \vee \neg p) \to (\neg\neg p \to p)$. **(xii)** $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$. **(xiii)** $p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$. **(xiv)** $\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$. **(xv)** $(\neg p \vee \neg q) \to \neg(p \wedge q)$. **(xvi)** $\neg p \leftrightarrow (p \to \mathbf{f})$, where **f** is any contradiction. **(xvii)** $\neg\neg(p \wedge q) \leftrightarrow \neg\neg p \wedge \neg\neg q$. **(xviii)** $\neg\neg(p \vee \neg p)$. **(xix)** $\neg(p \wedge \neg p)$.

A2. Call a statement p *evidentially contradictory* if there is a closed evidential tree with initial statement p. Show that *p* is evidentially contradictory if and only if $\neg p$ is evidentially valid.

A3. Show that evidential logic has the *weak disjunction property*: if $\neg p \vee \neg q$ is evidentially valid, then so is at least one of $\neg p$, $\neg q$. (Evidential logic actually has the full disjunction property.)

A4. Observe that replacing "?" by "$\neg$" and erasing the horizontal lines in each evidential tree rule transforms it into the corresponding classical tree rule. Deduce that any evidentially valid statement is classically valid.

2. Interpretations of evidential statements.

We now give a precise definition of the idea of an *interpretation* of statements of evidential logic, similar to that given for contextual statements. The basis for our interpretation will be the idea of *stages of evidence,* structured by the relation of *possible enlargement of evidence.* As remarked above, it is quite natural to require this relation to be both reflexive and transitive, in other words, a *ranking*. That being the case, we shall base our interpretation of evidential statements on *ranked contextual structures.*

For simplicity let us call a ranked contextual structure a *frame*. If **C** = (C, R) is a frame, the elements of C will be called *stages of evidence*. Instead of aRb, we will write $a \leq b$ (or $b \geq a$) and read this "b is *later* than (or the same as) a." An (evidential) *interpretation* of statements in a frame **C** is a function $\boldsymbol{I}$ which assigns, to each pair (p, a) consisting of a sentence p and a stage of evidence a, an element $\boldsymbol{I}(p, a)$ of $\{t, f\}$ in such a way that:

(i) For any statement letter A, if $\boldsymbol{I}(\mathrm{A}, a) = \mathrm{t}$ and $a \leq b$, then $\boldsymbol{I}(\mathrm{A}, b) = \mathrm{t}$.
(ii) $\boldsymbol{I}(\mathrm{p} \wedge \mathrm{q}, a) = \mathrm{t} \quad \Leftrightarrow \quad \boldsymbol{I}(\mathrm{p}, a) = \boldsymbol{I}(\mathrm{q}, a) = \mathrm{t}$.
(iii) $\boldsymbol{I}(\mathrm{p} \vee \mathrm{q}, a) = \mathrm{t} \quad \Leftrightarrow \quad \boldsymbol{I}(\mathrm{p}, a) = \mathrm{t}$ or $\boldsymbol{I}(\mathrm{q}, a) = \mathrm{t}$.
(iv) $\boldsymbol{I}(\neg\mathrm{p}, a) = \mathrm{t} \quad \Leftrightarrow \quad \boldsymbol{I}(\mathrm{p}, b) = \mathrm{f}$ for all $b \geq a$.
(v) $\boldsymbol{I}(\mathrm{p} \rightarrow \mathrm{q}, a) = \mathrm{t} \quad \Leftrightarrow \quad (\boldsymbol{I}(\mathrm{p}, b) = \mathrm{t} \Rightarrow \boldsymbol{I}(\mathrm{q}, b) = \mathrm{t})$ for all $b \geq a$.
(vi) $\boldsymbol{I}(\mathrm{p} \leftrightarrow \mathrm{q}, a) = \mathrm{t} \quad \Leftrightarrow \quad \boldsymbol{I}(\mathrm{p}, b) = \boldsymbol{I}(\mathrm{q}, b)$ for *all* $b \geq a$.

Clearly any assignment of truth values satisfying condition **(i)** to all pairs (A, a) where A is a statement letter generates a unique interpretation determined by clauses **(ii)** - **(vi)**. So in specifying an interpretation in a given frame we need only specify the truth values it assigns to these pairs, at the same time ensuring that condition **(i)** is satisfied.

If $\boldsymbol{I}(\mathrm{p}, a) = t$, we say that p is *true* (under $\boldsymbol{I}$) at *stage a.* Thus clause **(i)** stipulates that, if a statement letter is true at some stage, it remains true at all later stages: its truth is, in short, *persistent.* It can be shown that then the truth of any statement is persistent in this sense. Notice also that, according to clauses (ii) and (iii), the truth of p $\wedge$ q and p $\vee$ q at a given stage is completely determined by the truth of p and the truth of q at that stage. However, this is not the case for ¬p, p → q, or p ↔ q. For example, according to clause **(iv),** ¬p is true at a given stage if and only if p is false *at all later stages* (recall that "later" includes the given stage). And according to clause **(v),** p → q is true at a given stage if and only if the truth of p implies that of q *at all later stages.*

If p is true at *every* stage under ***I***, we shall say simply that p is *valid* under ***I***.

We shall usually write $a \Vdash_I$ p or $a \Vdash$ p for ***I***(p, *a*) = t, and $a \nVdash$ p for ***I***(*p*, a) = f. $a \Vdash$ p may be read *a forces* p. Using these new symbols the rules for interpretation of evidential statements in a frame now take the form

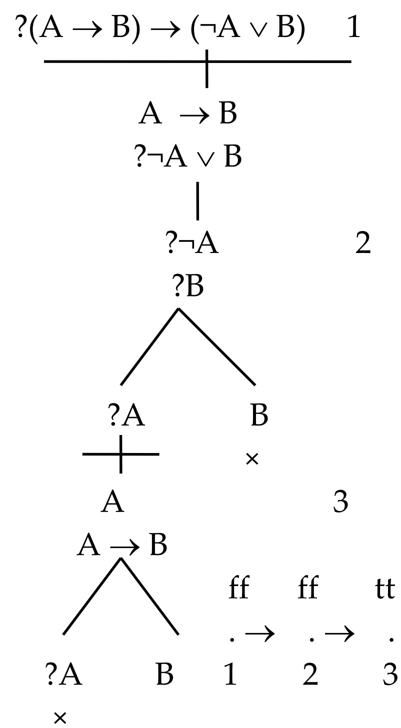
(i) For any statement letter A, if $a \Vdash$ A and $a \leq b$, then $b \Vdash$ A.
(ii) $a \Vdash$ p ∧ **q** ⇔ $a \Vdash$ p and $a \Vdash$ **q**
(iii) $a \Vdash$ p ∨ **q** ⇔ $a \Vdash$ p or $a \Vdash$ **q**
(iv) $a \Vdash$ ¬p ⇔ $b \nVdash$ p for all $b \geq a$.
(v) $a \Vdash$ p → q ⇔ ($b \Vdash$ p ⇒ $b \Vdash$ q) for all $b \geq a$.
(vi) $a \Vdash$ p ↔ q ⇔ ($b \Vdash$ p ⇔ $b \Vdash$ q) for all $b \geq a$).

The *persistence property* for statements may then be expressed as:

if $a \Vdash$ p *and* $a \leq b$*, then* $b \Vdash$ p.

It can be shown without much difficulty (see section 3) that that the rules we have given are correct (in the usual sense) for evidential statements provided we take the truth of ?p at any stage as meaning the *falsity of* p *at that stage*, with no reference to future stages—that is, $a \Vdash$?p is taken to mean $a \nVdash$ p. Accordingly, in using the tree method in the familiar way (that is, as for statements of contextual logic) to generate countermodels for evidentially invalid statements, statement letters will be assigned the value f at stages where they occur preceded by ?. This will result in the truth values of sentences changing from f to t as evidence "expands". While perhaps a touch counterintuitive, it is the price that must be paid for employing just the two truth values t and f in evidential interpretations. It is, nevertheless, perfectly consistent, since, while *truth* is required to persist, the same is not demanded of falsity.

By way of illustration, we now construct a countermodel for (A → B) → (¬A ∨ B).



The tree is finished and has one open path. That path contains three stages of knowledge which we label 1, 2, 3. At stage 2, both ?A and ?B appear, so our interpretation will assign f to A and B there. Similarly, at stage 3, it assigns t to both A and B. Thus the countermodel **I** will be an interpretation in the frame (C, R) where C = {1, 2, 3}, R is the usual "equal to or less than" relation on {1, 2, 3}, and **I**(A, 2) = **I**(B, 2) = f, **I**(A, 3) = **I**(B, 3) = t. (To respect persistence we must also take **I**(A, 1) = **I**(B, 1) = f, but this fact will not figure in our calculations.) Clearly 2 $\Vdash$A $\to$B. On the other hand, since 3 $\Vdash$ A , we have 2 $\nVdash$¬A, so that 2 $\nVdash$ ¬A ∨ B Therefore 2 $\nVdash$ (A → B) → (¬A ∨ B), so **I** is a countermodel for (A → B) → (¬A ∨ B).

Exercises

B1. Prove that a $\Vdash$ ¬¬p ⇔ for all b ≥ a there is c ≥ b such that c $\Vdash$ p.

B2. Using the tree method as above, construct evidential countermodels for the following classically valid statements. **(i)** ¬¬A → A. **(ii)** A ∨ ¬A . **(iii)** (A → B) ∨ (B → A). **(iv)** (¬A → ¬B) ∨ (¬B → ¬A). **(v)** ¬A ∨ ¬¬A. **(vi)** ¬(A ∧ B) → (¬A ∨ ¬B).

3. Correctness of the evidential tree rules.

Let us go into the question of the correctness of the evidential tree rules in more detail. First, what is actually meant by correctness? We say that a tree rule is *correct* if either

(i) it is a rule that introduces no horizontal lines and whenever its premise is true at a given stage under a given interpretation, at least one of its lists of conclusions is true at that stage under that interpretation; or

(ii) it is a rule that introduces new horizontal lines and whenever its premise is true at a given stage under a given interpretation, at least one of its lists of conclusions is true under the same interpretation at a stage later than (or the same as) the given one.

Correctness of all rules not introducing new horizontal lines is then clear. The correctness of (most of) the remaining rules may be indicated as follows.

$$\begin{array}{c} ?p \rightarrow q \\ \hline p \\ ?q \end{array}$$

if $a \nVdash p \rightarrow q$, then there is $b \geq a$ such that $b \Vdash p$ and $b \nVdash q$

$$\begin{array}{c} ?\neg p \\ \hline p \end{array}$$

if $a \nVdash \neg p$, then there is $b \geq a$ s.t. not $b \nVdash p$, i.e. $b \Vdash p$

Now suppose we start a tree T with a satisfiable statement q, i.e., one for which there is an interpretation **I** and some stage at which it is true. Then each application of a tree rule will yield a stage at which at least one of its lists of conclusions is true under **I,** so that T will always contain an open path.

If p is (tree) valid, then the finished tree starting with ?p is closed, so that, by the above, ?p cannot be satisfiable, in other words, p is valid under every interpretation.

Conversely, if the finished tree starting with ?p contains an open path, this will generate an interpretation in which p is false at its first stage, as presented in the example above.

4. Evidential Quantificational Logic.

Evidential propositional logic can be extended to quantificational logic. To do this we must explain the meaning of quantifiers from the evidential point of view.

Suppose that we are given an interpretation ***I*** with universe *A* of a vocabulary for quantificational logic. Consider first a universal statement ∀xp(x). What should constitute evidence for the truth of ∀xp(x) under **I**? A simple requirement would be:

- evidence for the truth of ∀xp(x) under **I** is the provision, for each $a \in A$, of evidence for the truth of p(a) under **I.**

For existential statements the idea of evidence is somewhat more involved, since evidence for the existence of something involves the presentation or description of that something. We shall use this formulation:

- evidence for the truth of ∃xp(x) under **I** is the presentation of a definite element *d* of *A* and the provision of evidence of the truth of p(*d*) under **I.**

These two clauses extend the concept of e-truth and e-validity to quantificational statements.

The tree rules for quantifiers can also be suitably formulated for evidential statements, yielding the appropriate trees, and so also the corresponding notion of tree validity for evidential statements.

Many classically valid quantificational statements are evidentially valid. For example, consider the classically valid statement ∃x¬p(x) → ¬∀xp(x). To see that this is *e*-valid, suppose we have evidence for the truth of ∃x¬p(x) under an interpretation ***I.*** This means that we have a definite element *d* of the domain A of ***I*** together with evidence of the truth of ¬p(*d*) under ***I.*** This in turn means that

(1) we have evidence for the impossibility of ever obtaining evidence for the truth of p(d) under **I.**

Now evidence for the truth of $\forall xp(x)$ under **I** provides evidence, in particular, for the truth of p(d) under **I.** It follows that

(2) evidence for the impossibility of ever obtaining evidence for the truth of p(d) under **I** provides evidence for the impossibility of ever obtaining evidence for the truth of $\forall x$p(x) under **I**, that is, evidence for the truth of $\neg\forall x$p(x) under **I.**

From (1) and (2) it follows that we have evidence for the impossibility of ever obtaining evidence for the truth of $\forall x$p(x) under **I.** Thus evidence for the truth of $\exists x\neg$p(x) under **I** yields evidence for the truth of $\neg\forall$xp(x) under **I.** Thus $\exists$x$\neg$p(x) $\rightarrow$ $\neg\forall$xp(x) is *e*-valid.

In a similar way, the following statements can be shown to be e-valid.

(A) $\neg\exists$xp(x) $\rightarrow$ $\forall$x$\neg$p(x) $\exists$x (p $\wedge$ q(x)) $\rightarrow$ p $\wedge$ $\exists$x q(x) p $\vee$ $\forall$xq(x) $\rightarrow$ $\forall$x(p $\vee$ q(x) $\forall$x(p(x) $\rightarrow$ q) $\leftrightarrow$ ($\exists$xp(x) $\rightarrow$ q) $\exists$x(p $\rightarrow$ q(x)) $\rightarrow$ (p $\rightarrow$ $\exists$xq(x) $\exists$x(p(x) $\rightarrow$ q) $\rightarrow$ ($\forall$xp(x) $\rightarrow$ q)

But not all classically valid quantificational statements are e-valid. For example, consider the classically valid statement $\neg\forall$xp(x) $\rightarrow$ $\exists$x$\neg$p(x). Suppose we have evidence for the truth of $\neg\forall$xp(x) under an interpretation **I.** This means that we have evidence for the impossibility of ever obtaining evidence for the truth of $\forall$xp(x) under **I.** This means in turn that

(3) we have evidence for the impossibility of providing , for each $a \in$ A, evidence for the truth of p(a) under **I.**

Now evidence for the truth of $\exists x \neg p(x)$ under ***I*** amounts to providing

(4) a definite element *d* of the domain *A* of **I** together with evidence for the impossibility of providing evidence for the truth of p(d) under **I.**

But the evidence provided under (3) is by no means sufficient to yield what is demanded by (4){ in particular, (3) provides no definite element of A at all. It follows that $\neg\forall x p(x) \to \exists x \neg p(x)$ is not *e*-valid.

In a similar way, the following classically valid statements can be shown to be e-invalid.

(B) $\forall x \neg\neg p(x) \to \neg\neg\forall x p(x)$ $\quad \forall x(p \vee q(x)) \to p \vee \forall x q(x)$

$(p \to \exists x q(x)) \to \exists x(p \to q(x))$ $\quad (\forall x p(x) \to q) \to \exists x(p(x) \to q)$

As with in the propositional case, it can be shown that, for any quantificational statement p, p *is classically valid if and only if* $\neg\neg$p *is evidentially valid.*

In mathematics, evidence for the truth of a statement is a mathematical *proof* of that statement. Here are some mathematical examples of evidentially invalid statements:

Let *A*(*n*) be the assertion: "the *n*th place in the decimal expansion of π is a 7 and is preceded by six 7s". Then for $\exists x A(x) \vee \neg\exists x A(x)$ to be evidentially valid we would require a proof yielding a number *n* such that *A*(*n*), or one showing that no such *n can* exist. But we do not have either of these. So $\exists x A(x) \vee \neg\exists x A(x)$ is not evidentially valid. On the other hand, $\neg\neg(\exists x A(x) \vee \neg\exists x A(x))$ is evidentially valid (since any statement of the form $\neg\neg(p \vee \neg p)$ is). Therefore the law of double negation fails evidentially for the statement

$\exists x A(x) \vee \neg\exists x A(x)$. Finally, $\neg\neg(\exists x A(x) \vee \neg\exists x A(x))$ is evidentially equivalent to $\neg(\exists x A(x) \wedge \neg\exists x A(x))$ (which is evidentially valid as an instance of the evidentially valid statement $\neg(p \wedge \neg p)$. But we have no evidence for $\neg\exists x A(x) \vee \neg\neg\exists x A(x)$ since we do not have a proof of the impossibility of the existence of a sequence of seven 7s in the decimal expansion of π; nor do we have a proof of the impossibility of the nonexistence of such a sequence.

Exercises

C1. Demonstrate the e-validity of the statements listed under **(A)**

C2. Demonstrate the e-invalidity of the statements listed under **(B).**

APPENDIX A

THE PROPOSITIONAL CALCULUS

In this Appendix we describe a formal system—the *propositional calculus*—for *proving* propositional statements and as a result obtain a purely syntactical characterization of valid propositional inferences and tautologies. To set up the system we choose certain tautologies as *axioms* and lay down a certain *rule of inference* which will enable us to construct *proofs*.

In what follows we shall omit the logical operator $\leftrightarrow$ *in forming statements, and the rules governing* $\leftrightarrow$ *in constructing trees*. (Of course, we can always define $\leftrightarrow$ by $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ if we wish.)

1. Axioms

The *propositional calculus* (**PC**) has as *axioms* all statements of the form (1)–(10) below.

(1) $p \rightarrow (q \rightarrow p)$
(2) $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$
(3) $(p \wedge q) \rightarrow p$
(4) $(p \wedge q) \rightarrow q$
(5) $p \rightarrow (q \rightarrow p \wedge q)$
(6) $p \rightarrow (p \vee q)$
(7) $q \rightarrow (p \vee q)$
(8) $(p \rightarrow r) \rightarrow [(q \rightarrow r) \rightarrow (p \vee q \rightarrow r)]$
(9) $(p \rightarrow q) \rightarrow [(p \rightarrow \neg q) \rightarrow \neg p]$
(10) $\neg\neg p \rightarrow p$.

The sole *rule of inference* for **PC** is called *modus ponens* (Latin: "affirming mood"):

MP $\dfrac{p,\ p \to q}{q}$

In words, *from* p, $p \to q$, *infer* q.

2. Proofs

Let S be a set of statements. A *proof from* S is a finite sequence $p_1,\ldots,p_n$ of statements such that, for any $i = 1,\ldots,n$, p_i is either (a) an axiom, (b) a member of S, or (c) inferrable using **MP** from earlier members of the sequence, i.e., there are numbers $j,k < i$ such that p_k is $p_j \to p_i$.

A proof from the *empty* set of statements is called simply a *proof*. A proof with last statement p is called a *prooof* of p. We write $S \vdash p$ to indicate that p is *provable* from S, i.e. that there is a proof of p from S. If S is empty, so that p is *provable*, i.e. there is a proof of p, we write just $\vdash p$, and call p a *theorem* of **PC**.

Example. $\vdash p \to p$.

The following is a proof of the statement $p \to p$.

1. $(p \to ((p \to p) \to p)) \to ((p \to (p \to p)) \to (p \to p))$ (Ax.2)
2. $p \to ((p \to p) \to p)$ (Ax.1)
3. $(p \to (p \to p)) \to (p \to p)$ (**MP** on 1,2)
4. $p \to (p \to p)$ (Ax.1)
5. $p \to p$ (**MP** on 3,4)

We now prove the important

Deduction Theorem *For any set* S *of statements and any statements* p,q:

$$S,p \vdash q \text{ if and only if } S \vdash p \to q.$$

Proof. First suppose that $S \vdash p \to q$. Then there is a proof D of $p \to q$ from S. Clearly, if we add the sequence p, q to D, the result is a proof of q from S,p. Therefore $S,p \vdash q$.

Conversely, suppose that S,p $\vdash$ q. Then there is a proof $r_1,\ldots,r_n$ of q from S,p (so that q is r_n). *We claim that* S $\vdash$ $p \to r_i$ *for any* i = 1,…,n.

Suppose that the claim were false. Then there is a *least* number k such that it is *not* the case that S $\vdash$ $p \to r_k$. There are then 4 possibilities: (1) r_k is an axiom; (2) r_k is in S; (3) r_k is p; (4) r_k is inferrable using **MP** from some r_i and r_j with i,j < k, where r_j is $r_i \to r_k$.

We show that in each of these four cases we have S $\vdash$ $p \to r_k$. This will contradict the assertion that the claim is false, and it must accordingly be true.

Case (1). r_k is an axiom. In this case the sequence of statements r_k, $r_k \to (p \to r_k)$, $p \to r_k$ is a proof of $p \to r_k$, so that S $\vdash$ $p \to r_k$.

Case (2). r_k is in S. In this case the same sequence of statements as in case (1) is a proof of $p \to r_k$ from S.

Case (3). r_k is p. Here we have $\vdash$ $p \to r_k$ by our Example above, so *a fortiori* S $\vdash$ $p \to r_k$.

Case (4). For some i,j < k r_j is $r_i \to r_k$. Since k was assumed to be the *least* number for which it is *not* the case that S $\vdash$ $p \to r_k$, and i,j < k, we must have S $\vdash$ $p \to r_i$ and

S $\vdash$ $p \to r_j$, i.e., S $\vdash$ $p \to (r_i \to r_k)$. By axiom 2,

$$(p \to (r_i \to r_k)) \to ((p \to r_i) \to (p \to r_k)).$$

Hence, applying **MP**,

$$S \vdash (p \to r_i) \to (p \to r_k)$$

and applying it once more,

$$S \vdash p \to r_k.$$

We have obtained a contradiction in each case, so the claim is true. In particular, taking i = n, we get S |- p → r_n, i.e. S |- p → q. This completes the proof.

3. Soundness

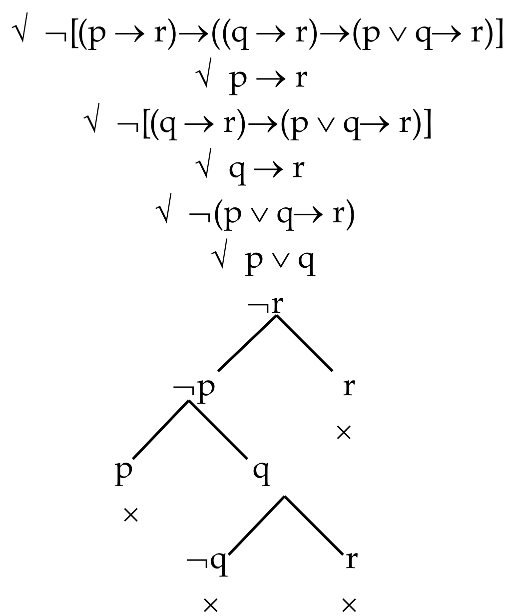
Our next result is the

Soundness Theorem for the Propositional Calculus.

Any theorem of **PC** *is a tautology.*

Proof. Note first that, if a valuation satisfies both p and p → q, then it satisfies q. Thus if both p and p → q are tautologies, so is q. In other words, **MP** leads from tautologies to tautologies.

It is also not hard to show that any axiom of **PC** is a tautology. For example, we may use the tree method to establish this for Axiom 8:



Hence any proof in **PC** consists entirely of tautologies, and the theorem follows.

As an immediate consequence of this, it follows that **PC** is *consistent* in the sense that for no statement p do we have both |- p and |- ¬p.

We are next going to establish a strengthened version of the Soundness Theorem by employing the Deduction Theorem.

Strengthened Soundness Theorem for PC. *If* S $\vdash$ p, *then* S $\models$ p.

Proof. Suppose S $\vdash$ p, where S = $\{s_1, s_2, \ldots, s_n\}$. The trick is simply to apply the Deduction Theorem to $s_1, s_2, \ldots, s_n \vdash p$ and carry each of the statements in the sequence $s_1, s_2, \ldots, s_n$ over to the right-hand side of the $\vdash$ sign so that the (unstrengthened) Soundness theorem, which we've already proved, can be invoked.

Thus, applying the Deduction Theorem n times in succession to S $\vdash$ p yields:

$$\vdash s_1 \rightarrow (s_2 \rightarrow (\ldots (s_{n-1} \rightarrow (s_n \rightarrow p))\ldots)) \qquad (*)$$

(For example, if n=3 the first application of the Deduction Theorem yields

$$s_1, s_2 \vdash s_3 \rightarrow p,$$

the next application yields

$$s_1 \vdash s_2 \rightarrow (s_3 \rightarrow p),$$

and the final application yields

$$\vdash s_1 \rightarrow (s_2 \rightarrow (s_3 \rightarrow p)).\ )$$

But by the (unstrengthened) Soundness Theorem, we can infer from (*) that:

$$\models s_1 \rightarrow (s_2 \rightarrow (\ldots (s_{n-1} \rightarrow (s_n \rightarrow p))\ldots)) \qquad (**)$$

(**) simply asserts that the nested conditional above that we have concocted by applying the Deduction Theorem is a tautology. But it is clear from the truth table for $\rightarrow$ that this could not be so unless there is never a case where all the statements in the set S = $\{s_1, s_2, \ldots, s_n\}$ are true and p is false. So (**) cannot be correct unless S $\models$ p, which is what we set out to prove.

Our final task will be to prove the converse of the Strengthened Soundness Theorem.

4. Completeness

Let us call a set S of statements *formally inconsistent* if $S \vdash p$ and $S \vdash \neg p$ for some statement p. We now establish the following facts:

<u>Fact A.</u> S *is formally inconsistent if and only if* $S \vdash q$ *for all statements* q.

<u>Fact B.</u> $S \vdash p$ *if and only if* $\{S, \neg p\}$ *is formally inconsistent*.

<u>Proof of A.</u> Clearly, if $S \vdash q$ for all statements q then S is formally inconsistent. To establish the converse, we begin by showing that $\vdash \neg p \to (p \to q)$. First, note that the following sequence qualifies as a deduction of q from $\neg p, p$:

$$\neg p$$
$$p$$
$$p \to (\neg q \to p)$$
$$\neg p \to (\neg q \to \neg p)$$
$$\neg q \to p$$
$$\neg q \to \neg p$$
$$(\neg q \to p) \to ((\neg q \to \neg p) \to \neg\neg q))$$
$$(\neg q \to \neg p) \to \neg\neg q$$
$$\neg\neg q$$
$$\neg\neg q \to q$$
$$q$$

It follows that $\neg p, p \vdash q$. Two applications of the Deduction Theorem now give $\vdash \neg p \to (p \to q)$ as claimed.

Now if S is formally inconsistent, we have $S \vdash p$ and $S \vdash \neg p$. Since

$$\vdash \neg p \to (p \to q),$$

two applications of **MP** yield S $\vdash$ q. This proves **A**.

<u>Proof of B.</u> If S $\vdash$ p, then S, ¬p $\vdash$ p and S, ¬p $\vdash$ ¬p, so S, ¬p is formally inconsistent.

Conversely suppose that S, ¬p is formally inconsistent. Then by Fact **A**, S,¬p $\vdash$ ¬¬p. So by the deduction theorem S $\vdash$ ¬p → ¬¬p. Now we have

$$p \to p, p \to \neg p \vdash \neg p$$

as the following proof shows:

$$p \to \neg p$$
$$(p \to \neg p) \to ((p \to \neg p) \to \neg p)$$
$$p \to p$$
$$(p \to \neg p) \to \neg p$$
$$\neg p$$

Since $\vdash$ p → p, it follows that p → ¬p $\vdash$ ¬p. So, substituting ¬p for p, we get ¬p → ¬¬p $\vdash$ ¬¬p. But ¬¬p → p is an axiom, so an application of **MP** yields ¬p → ¬¬p $\vdash$ p. But we have already observed that S $\vdash$ ¬p → ¬¬p, so another application of **MP** yields S $\vdash$ p as required. This proves **B**.

We now sketch a proof of the

<u>Theorem.</u> *The initial set of statements of any closed tree is formally inconsistent*.

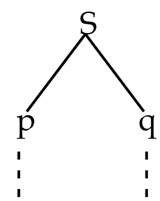
<u>Proof</u> (sketch). Let us define the *depth* of a tree to be the length of its longest path. Suppose that the assertion of the theorem is false. Then there is a closed tree with a *formally consistent* (i.e., not formally inconsistent) set of initial statements. Among these choose one, **T** say, of *least depth,* d say. Then **T** is a closed tree whose set S of initial statements is formally consistent. We shall derive a contradiction from this.

There are two cases to consider.

Case 1: d = 1. In this case **T** is identical with S. Since **T** is closed there must be some statement p for which both p and ¬p are in S. Clearly S is then formally inconsistent.

Case 2: d > 1. In this case, by assumption, the set of initial statements of *any* closed tree of depth < d is formally inconsistent. Now examine the statements at level 2 of **T**. We claim that however these statements were obtained, we can always conclude that S is formally inconsistent.

For example, suppose that the statements at level 2 of **T** arise by applying the ∨-rule to a statement in S of the form p ∨ q. Then **T** starts thus:



If in **T** we fuse S with p and expunge q as well as all nodes following it, we get a closed tree (recall that **T** was assumed closed) of depth < d with S,p as its set of initial statements. But then S,p is formally inconsistent. Similarly, S,q is formally inconsistent. Since p ∨ q is in S, it follows that S is formally inconsistent. For if r is any statement, we have S,p |– r and S,q |– r so that S |– p → r and S |– q → r. Two applications of **MP** and Axiom 8 now yield S |– p ∨ q → r; but since p ∨ q is in S, **MP** yields S |– r. Since this holds for any statement r, S is formally inconsistent.

Similar arguments work for the other rules; in all cases we are able to conclude that S is formally inconsistent.

We have shown that assuming the theorem false leads to a contradiction. So the theorem is proved.

As a consequence of this, we finally obtain the

<u>Completeness Theorem for PC.</u> *If* S |= p *then* S |– p.

<u>Proof.</u> If S |= p, then by inference adequacy any finished tree **T** associated with the inference of p from S is closed. It follows from the previous theorem that the set S, ¬p of initial statements of **T** is formally inconsistent. Hence, by fact **B** , S |– p.

Exercises

A1. The following is a purported proof in **PC** of p from $\neg\neg p$. Verify that it is or is not by identifying the origin of each statement in the sequence.

$\neg\neg p$, $(\neg p\rightarrow((\neg p\rightarrow\neg p)\rightarrow\neg p))$, $\neg p\rightarrow(\neg p\rightarrow\neg p)$,
$(\neg p\rightarrow((\neg p\rightarrow\neg p)\rightarrow\neg p))\rightarrow((\neg p\rightarrow(\neg p\rightarrow\neg p))\rightarrow(\neg p\rightarrow\neg p))$, $((\neg p\rightarrow(\neg p\rightarrow\neg p))\rightarrow(\neg p\rightarrow\neg p))$,
$(\neg p\rightarrow\neg p)$, $(\neg p\rightarrow\neg p)\rightarrow((\neg p\rightarrow\neg\neg p)\rightarrow\neg\neg p)$, $\neg\neg p\rightarrow p$, $(\neg p\rightarrow\neg\neg p)\rightarrow\neg\neg p$,
$\neg\neg p\rightarrow(\neg p\rightarrow\neg\neg p)$, $\neg p\rightarrow\neg\neg p$, $\neg\neg p$, p

Of course, there is a very simple deduction of p from $\neg\neg p$. What is it?

A2. The sequence 1.-14. below (see over) allegedly establishes that:

$$\neg p\rightarrow\neg\neg p,\ q \vdash q\wedge p$$

Check to see whether this is so by justifying each statement below with the words "in the initial set", "modus ponens", or "axiom # so-and-so" (filling in the relevant axiom number). If a particular statement *cannot* be justified, say so!

1. $[\neg p\rightarrow((\neg p\rightarrow\neg p)\rightarrow\neg p)]\rightarrow[(\neg p\rightarrow(\neg p\rightarrow\neg p))\rightarrow(\neg p\rightarrow\neg p)]$
2. $\neg p\rightarrow((\neg p\rightarrow\neg p)\rightarrow\neg p)$
3. q
4. $(\neg p\rightarrow(\neg p\rightarrow\neg p))\rightarrow(\neg p\rightarrow\neg p)$
5. $\neg p\rightarrow(\neg p\rightarrow\neg p)$
6. $q\rightarrow(p\rightarrow(q\wedge p))$
7. $\neg p\rightarrow\neg p$
8. $\neg p\rightarrow\neg\neg p$
9. $(\neg p\rightarrow\neg p)\rightarrow[(\neg p\rightarrow\neg\neg p)\rightarrow\neg\neg p]$
10. $(\neg p\rightarrow\neg\neg p)\rightarrow\neg\neg p$
11. $\neg\neg p$

12. $\neg\neg p \to p$

13. p

14. $q \wedge p$

A3. There are two sequences of statements below (set aside in two separate columns), each purporting to be a deduction from the set of statements $S = \{p, q \to r\}$. Identify the origin of each statement in each sequence, and thus discern whether or not these sequences really are deductions from S.

p	$q \to r$
$q \to r$	p
$p \to [(q \to r) \to (p \wedge (q \to r))]$	$p \to (r \to p)$
$(q \to r) \to (p \wedge (q \to r))$	$r \to p$
$p \wedge (q \to r)$	$q \to p$

A4. By the Completeness and Strengthened Soundness Theorems for the propositional calculus, each concept on the left below corresponds to one on the right and vice-versa. Match them up.

tautology	deduction
unsatisfiable	theorem
valid argument	proof
valid argument without premises	formally inconsistent

A5. State the theorems below first in symbols and then in your own words.

(i) (Strengthened) Soundness Theorem

(ii) Completeness Theorem

(iii) What is the point of introducing the propositional calculus and proving these theorems?

A6. Indicate whether each of the following statements is true or false.

(a) A statement deducible from its negation cannot be a theorem.

(b) Consistency of the propositional calculus follows from the completeness theorem.

(c) If $p \vdash (q \to \neg p)$, then the pair {p,q} is formally inconsistent.

(d) The conclusion of a proof cannot be formally inconsistent.

(e) A theorem cannot be proven from a formally inconsistent set of statements.

(f) Assuming every tautology is a theorem, completeness of the propositional calculus follows from the Deduction Theorem.

(g) $a \vdash c$ implies $a, \neg b \vdash b \to c$ for all statements a, b and c.

(h) The propositional calculus would not be sound unless it employed the modus ponens rule.

For the last two questions, call a set of statements maximally consistent if and only if it is formally consistent but not a subset of any other formally consistent set of statements.

(i) No maximally consistent set of statements can contain all theorems.

(j) Every maximally consistent set of statements must contain either p or $\neg p$, for any statement p.

A7. Why does $p \vdash r$ imply $p \vdash (q \to r)$ for any statements p, q and r?

A8. (a) Assuming the Completeness and Strengthened Soundness Theorems, prove the Deduction Theorem.

(b) Use the Deduction Theorem to show directly (i.e. without explicitly constructing a deduction sequence) that $\vdash (p \to p)$ and that $\neg\neg p \vdash p$.

A9. (i) For any statements p, q and set of statements S, S $\vdash$ p and p $\vdash$ q implies S $\vdash$ q. Why?

(ii) Let S_1, S_2, ...,S_n be n *sets* of statements, and let S be the set of statements $\{p_1,p_2,...,p_n\}$. Show that if $S_i \vdash p_i$ for all i=1 to n and S $\vdash$ p for some statement p, then $S_1 \cup S_2 \cup ... \cup S_n \vdash p$.

(iii) Show that if S $\vdash$ p and S $\vdash$ q, then S $\vdash$ $(p \wedge q)$. (You are not allowed to assume the Completeness Theorem!)

A10. (a) By relying on the Completeness and Strengthened Soundness Theorems, prove that a set S of statements is formally inconsistent if and only if S $\vdash$ p for all statements p.

(b) Without relying on Completeness and Soundness, show that S $\vdash$ $\neg$p implies that the set $T=S \cup \{p\}$ is formally inconsistent.

Appendix B

Logical Paradoxes

Here we give a brief account of some famous logical paradoxes.

1. The Liar Paradox.

The *liar paradox* **(*pseudomenos logos* - the "falsely named**") purports to show that common beliefs about *truth* and *falsity* actually lead to *contradiction.* This is done by formulating perfectly grammatical statements which cannot consistently be assigned a truth value.

In its simplest form, the liar paradox is the statement*: the statement I am now making is false,* or, *this statement is false.* Since this statement asserts its own falsehood, if it is true, it is false, and if it is false it is true. So it is both self-referential and self-contradictory.

This argument depends on the *rule of double negation,* namely, the assertion that, for any statement *p,* if *p is false* is false, then *p* is true. This in turn is a consequence of the *law of bivalence* or *the law of excluded middle*, namely, that for any statement *p, p* is either true or false (but not both).

In fact the rule of double negation is not needed to derive the contradiction in the liar paradox. For let ℓ denote the liar statement . Then ℓ asserts ℓ *is false.* So if ℓ is true, ℓ is false, and it follows that ℓ implies not ℓ. Hence ℓ implies (ℓ and not ℓ), so that ℓ implies a contradiction. It follows that ℓ is false, i.e. ℓ *is false* is true. But ℓ *is false* is just ℓ, so it follows that ℓ is true. So ℓ is both true and false, a contradiction.

The liar paradox can be traced back as far as ancient Greece. The earliest known example of a liar-type statement is the *Epimenides paradox (*c. 600 B.C.). Epimenides, a Cretan, is reported to have stated that "Cretans are always liars." However, Epimenides' statement is non-paradoxical - in fact false. For if `Cretans are always liars` is true, then Epimenides, a Cretan, has stated a truth, and therefore `Cretans are always liars`` must be false. This version of the paradox even appears in the Bible (Titus 1:12-13a): *It was one of them, their very own prophet, who said, "Cretans are always liars, vicious brutes, lazy gluttons." That testimony is true.*

It seems to have been the Greek philosopher Eubulides of Miletus (4th century B.C) who first stated the liar paradox in the form in which it has become familiar. Eubulides reportedly posed the question, "A man says that he is lying. Is what he says true or false?"

A version of the paradox occurs in *Don Quixote*, when Sancho Panza is stopped at a bridge and told by the guards that he can pass if he tells the truth and that he will be hanged if he lies; he says, "I shall be hanged." This leaves the guards in a dilemma. For if they let him pass, then he will have lied and so he should have been hanged; while if he is hanged then he will have told the truth and so he should have been allowed to pass.

The liar paradox can also be cast in forms involving more than one statement. For example, consider the following pair of sentences:

(1) the following statement is true

(2) the preceding statement is false.

Assume (1) is true. Then (2) is true. This would mean that (1) is false. Therefore (1) is both true and false.Assume (1) is false. Then (2) is false. This would mean that (1) is true. Thus (1) is both true and false. So in either case, (1) is both true and false.

There have been a number of attempts to resolve the liar paradox. Here are a few of them.

The great logician *Alfred Tarski* claimed that liar -type paradoxes could arise only in languages he termed semantically closed. By this he meant languages in which it is possible for one sentence to ascribe truth (or falsehood) of another sentence in the same language (or even of itself). To avoid contradiction when discussing truth values of sentences it is necessary, according to Tarski, to divide languages into strict semantic levels. Sentences of a language at a particular level can refer only to sentences in languages at a lower level. Accordingly, when a sentence refers to another sentence - in particular to the truth-value of that sentence - the first sentence is necessarily semantically higher than the second. This prevents sentences from being self-referential since a self-referential sentence would have to be semantically higher than itself. Accordingly, the liar sentence, being self-referential becomes meaningless in the sense that it cannot be legitimately formulated in any language.

The philosopher *Arthur Prior* dealt with the liar paradox by denying that there is anything paradoxical about the liar sentence. His analysis turns on the claim is that each statement implicitly asserts its own truth. Thus, for example, the statement "it is true that two plus two equals four", contains no more information than the statement "two plus two equals four", because the phrase "it is true that..." is implicitly present in any statement. Under this analysis the liar sentence this sentence is false now becomes it is true that this sentence is false, or this statement is both true and false. This last sentence is a straightforward contradiction of the form A and not A, and hence is false. No paradox now arises because the claim that A and not A is false, far from being a contradiction, is actually true.

The philosopher *Graham Priest* and others have suggested that the liar sentence should be considered to be *both true and false*. This is justified by a doctrine known as *dialetheism,* whose principal tenet is the claim that there can be *true contradiction*[19]*s.* Dialethism faces an immediate difficulty arising from the generally accepted law of logic *ex falso quodlibet*, "from a falsehood, anything follows", or *ex contradictione quodlibet*, "from a contradiction, anything follows". According to this law (also known as the *principle of explosion*) any proposition can be deduced from a contradiction. It would therefore follow that in dialethism *all* propositions would have to be true. In order to avoid this absurd conclusion dialetheists nearly always reject the explosion principle. Logical systems rejecting it are called *paraconsistent.*

In 1931 the great logician *Kurt Gödel* used a modified form of the liar paradox to prove his celebrated *Incompleteness Theorem*. This asserts that, in any sound and sufficiently rich mathematical theory *T* - arithmetic or set theory, for example - propositions can always be formulated which one can see to be true but whose truth cannot be proved within the theory. Gödel obtained such a proposition by replacing, in the liar sentence *this sentence is false*, the word "false" by the word "unprovable", producing the sentence *this sentence is unprovable*, This, the Gödel sentence *G*, thus asserts its own *unprovability*. It follows, assuming that *T* is sound, that *G is true but unprovable*. For suppose that *G* were false. Then, since *G* asserts its own unprovability, it would follow that the unprovability of *G* is false, i.e. *G* is provable. But then *G* would be both

[19] Nicely summed up by Priest`s variant on Hamlet`s famous utterance: *To be* and *not to be, that is the question.*

provable and false, contradicting the soundness of *T.* It follows that *G* must be true. Therefore, since *G* asserts its own unprovability, it must also be unprovable.

Gödel later strengthened the Incompleteness Theorem to show that, for any consistent and sufficiently rich mathematical theory *T,* the consistency of *T* cannot be proved in *T.* If we take *T* to be arithmetic, it follows that, if arithmetic is consistent, then the consistency of arithmetic cannot be proved in arithmetic itself. The great French mathematician André Weil expressed this in the saying *God exists, since mathematics is consistent, and the Devil exists, since we cannot prove it.*

2. The Liar, the Truth-Teller and the Dice-Man,

Related to the liar paradox is the following well-known puzzle (not a paradox, in fact). A person travelling to a certain town comes to a fork in the road and doesn't know which branch to take. Two brothers live next to the fork, one of whom always tells the truth and the other always lies. What single question should the traveller ask either one of the brothers in order to determine which branch to take? A suitable question is: *which branch would your brother tell me to take?* On hearing the answer to this question, the traveller then does the opposite: if the answer is *the right branch,* he takes the left branch, and if the answer is *the left branch,* he takes the right branch. This works because, if the traveller happens to put the question to the truth-teller, the latter will truthfully report what his lying brother's answer would be, i.e. a lie. Similarly, if the traveller happens to put the question to the liar, the latter will lie about the truthful answer his brother would give, again producing a lie.

A nice way of presenting this is to think of the truth teller (T) as a lens consisting of a simple piece of glass and the liar (L) as a lens in which images appear inverted.

Asking the truth teller what the liar would say is then analogous to looking through the combination of lenses in the order TL. T Asking the liar what the truth-teller would say is analogous to looking through the lens combination LT. In both cases the image is inverted, just as, in the original scenario, the answer is always a lie.

Now let's introduce into the scenario a third brother – call him the *dice man* -who *answers entirely at random.* Surprisingly, the traveller can extract the required information by asking just *two* questions. Let us call the truth teller and the liar *determinate.* The first question has then to be phrased in such a way that, *no matter to whom it is addressed,* the answer will enable the traveller to identify a *determinate* brother. Let us say that the truth teller (**T**) is *more truthful* than the dice man (**D**), who is in turn is *more truthful* than the liar (**L**). The traveller picks any of the brothers, *A* say, and labels the other two brothers *B* and *C* He then asks *A*

Is B more truthful than C?

If the answer is *yes,* then *C* is determinate. If the answer is *no*, then *B* is determinate. That this is the case can be seen from the following table.

A	*B*	*C*	Answer	Determinate brother
T	**L**	**D**	*No*	*B*
T	**D**	**L**	*Yes*	*C*
L	**T**	**D**	*No*	*B*
L	**D**	**T**	*Yes*	*C*
D	**L**	**T**	*No*	*B*
D	**L**	**T**	*Yes*	*C*

D	**T**	**L**	*No*	*B*
D	**T**	**L**	*Yes*	*C*

The essential point here is that the strategy works when *A* is determinate, and *it continues to work when A is the dice man.* This is the case because when *A* is the dice man, both *B* and *C* are determinate, so it actually doesn't matter which one is picked.

Once the traveller has identified a determinate brother, his second question, addressed to the latter, is

which branch would your determinate *brother tell me to take?*

On hearing the answer to this question, the traveller proceeds as in the two-brother case, that is, he does the opposite of what he is told. Thus he achieves his goal.

3. Curry's Paradox.

Like the liar paradox, Curry`s paradox involves self-reference.

Consider the sentence, call it *S:*

If the only sentence in this box is true, then God exists.

Then *S* is the only sentence in the above box, so that *S* is the same as the sentence *S*:*

if S is true, then God exists.

Thus if *S* is true, then *if* S *is true, then God exists* is true, so that

(1) if *S* is true, then God exists.

But then the sentence *S** above is true, and since *S** is *S*, it follows that *S* is true. It now follows from the truth of *S* and (1) that God exists.

Thus we seem to have proved that God exists. Now clearly if we replace the statement *God exists* by *any* given statement *A*, the same argument will prove that *A* is true. Accordingly we have shown that all statements are true. This absurd conclusion is Curry`s paradox.

4. The Grelling-Nelson Paradox

The *Grelling-Nelson paradox* was formulated in 1908 by the philosophers *Kurt Grelling* and *Leonard Nelson*. It appears to strike at the very foundations of language.

Define the adjectives *autological* and *heterological* by stipulating that an (English) adjective is *autological* if and only if it is applies to itself and *heterological* if and only if it does not apply to itself. More exactly, an adjective *A* is

- autological if and only if the word ``*A*`` has the property expressed by the adjective *A*,
- heterological if and only if the word ``*A*`` does *not* have the property expressed by the adjective *A*.

Clearly an adjective cannot be both autological and heterological.

For instance, the adjectives *polysyallabic, English* are autological since the word ``polysyllabic`` is polysyllabic and the word ``English`` is English On the other hand the adjectives *palindromic, French* are heterological since the word "palindromic" is not palindromic and the word "French" is not French.

The paradox arises upon asking whether the adjective *heterological* is hetrological.

We reason as follows:

heterological is heterological

if and only if the word "heterological" is heterological

if and only if *heterological* is autological.

This contradiction is the Grelling-Nelson paradox.

A deeper analysis dissolves the Grelling-Nelson paradox. To do this, we assume that we are given two collections: a collection of words called *adjectives* and a collection of *properties* of adjectives. If the adjective *a* has the property *P,* we say that *P applies to a.* We also suppose that each property *P* of adjectives is correlated with an unique adjective *a* called its *name:* in that case we say that *a names P.* Now consider the "heterological" property *H* of adjectives which is defined to apply to an adjective *a* just when *a* names a property which doesn't apply to *a.* We now show that *H applies to its own name.* To prove this we argue by contradiction. Write *h* for the name of *H* and suppose that *H* doesn't apply to *h.* Then *h* names a property, i.e. *H,* which doesn't apply to *h.* It now follows from the very definition of *H,* that *H* applies to *h.* We conclude that *H* does indeed apply to *h.* That being established, it follows from the definition of *H* that *h* must also name a property, call it *I*, *which doesn't apply to h.* Then the properties *H* and *I* have the same name (that is, *h*)*)* but they cannot themselves be the same, since *H* applies to its own name but *I* doesn't. We conclude therefore that *there must exist two different properties of adjectives with the same name.* One of these, as we have seen, is the "heterological" property *H.* This is a perfectly well-defined property which doesn't give rise to paradox. *But it must necessarily have the same name as a different adjective.*

Now consider the "autological" property *A* of adjectives which is defined to apply to an adjective *a* just when *a* names a property which *does* apply to *a.* As in the bibliographical case, nothing can be inferred about *A:* it may be autological or heterological; anf it may or may not have the same name as a different adjective.

SOLUTIONS TO SELECTED EXERCISES

CHAPTER I

A1. (a) <u>in</u>valid; (c), (e), (g), (i), (k), (m), (o) all valid

A2.

(a)	(c)	(e)	(g)	(i)
$K \vee F$	$F \to O$	$R \to W$	$\underline{(W \wedge Q) \to D}$	$F \to T$
$\underline{K}$	$\underline{O}$	$\underline{W \to \neg R}$	$\therefore (W \to D) \vee (Q \to D)$	$\underline{F \wedge \neg T}$
$\therefore \neg F$	$\therefore F$	$\therefore \neg R$		$\therefore L$
invalid	invalid	valid	valid	valid!

A3. With 'I'm right' = IR, 'You're a fool' = YF, and 'I'm a fool' = IF, the argument symbolized is:

$$
\begin{array}{c}
IR \to YF \\
IF \to \neg IR \\
\underline{YF \to IR} \\
IF \vee YF
\end{array}
$$

To check for counterexamples, and hence validity, we only need look at the cases in the truth table where the conclusion comes out false. That means we only look at cases where both IF and YF are false (otherwise IF ∨ YF comes out true!), and that can happen in two ways according to what the truth value of IR is. So the truncated truth table for this argument looks like:

IF	YF	IR	IR→YF	IF→¬IR	YF→IR	IF ∨ YF
f	f	t	f	t	t	f
f	f	f	t	t	t	f

The second row exhibits a case where the premises are all true but conclusion false, so the argument is *invalid*.

A5. (a) $Y \wedge [I \vee (\neg Y \wedge \neg I)]$, therefore $(Y \wedge I) \vee (\neg Y \wedge \neg I)$. Valid.

A6. (a) valid; (c) valid.

A7. NO. The argument's certainly valid, and its second premise is true. But supposing premise one is true (i.e. 'for its conclusion is false') leads to a contradiction: for since we would then have a sound argument, the conclusion would have to be true, contradicting premise one's truth (and also the argument's soundness!). So premise one can't be true! So in fact the conclusion *is* true, and therefore the argument is unsound (as its conclusion claims!).

A8. Both knights; both knaves.

A9. I = can't tell! He = taxpayer for sure.

B1. (a), (c) tautologous

B2. (a) correct; (c) <u>in</u>correct; (e) correct; (g) <u>in</u>correct;

(i), (k) correct; (m) <u>in</u>correct

B3. (a) Correct. If a statement is not contingent it is either a tautology or contradiction, therefore its negation is either a tautology or contradiction, hence its negation can't be contingent.

(c) Correct. For a conjunction to be a tautology, it must come out true under all possible valuations; and since it's a conjunction, that means each of its conjuncts has to come out true under all possible valuations, otherwise the entire conjunction will come out false under some valuation. Hence all the conjuncts must also be tautologies.

(e) Correct. If p→q is valid that means it's a tautology, which means there is never any case where it comes out false. By the truth-conditions for '→', that means there can never be any case where p comes out true and q false (otherwise p→q would be false). But if there is never any case where p is true and q false, the argument from p as premise to q as conclusion faces no counterexamples, and so must be valid. Conversely, suppose the argument from p to q is a valid one. Then there is never any case where p is true and q is false (otherwise we'd have a counterexample!). But since p→q is false only in such a case, there is never any case where p→q comes out false, which means that it is a tautology, i.e. valid.

B4. (a) (ii); (c) (iii)

B5. (a) If there *were* a case where a conjunct comes out false, the conjunction would have to come out false in that case too, and so couldn't be valid!

B6. (a) (iv); (b) (iv); (d) (iii)

B7. (a) false; (c) false; (e) false; (g) true

B8. (a) true; (c) true; (e) false;

(g) false - tautologies only imply tautologies!; (i) false - just add a contradiction!

B9. (a) satisfiable; (b) unsatisfiable; (d) satisfiable

B10. (a) false; (c) false; (e) false.

B11. (a) Valid.

B12. With 'The witness was not intimidated' = W, 'Flaherty committed suicide' = F, and 'A note was found' = N, the set of sentences is {W ∨ (F→N), W→¬F, N→F}. Fairly quickly one can see that W = true, F = false, and N = false is a satisfying truth valuation!

CHAPTER II

A1. (a), (c), (e) equivalent; (g) <u>in</u>equivalent; (i) equivalent

A2. (a) (i), (ii), (iv), (vi)-(viii) are all valid, the rest invalid.

(b) Just the tautologies, which are of course all equivalent to each other.

B1. (1) $\equiv ABC \vee \underline{A}BC \vee A\underline{B}C \vee \underline{ABC}$ (3) $\equiv \underline{A}BC \vee AB\underline{C} \vee \underline{A}B\underline{C} \vee \underline{ABC}$

B2. (c) Use the facts: $\{\wedge, \neg\}$ is expressively complete and $\neg A \equiv A \vee t$

B3. (a) Use the facts that $p \vee q \equiv \neg p \to q$ and $\{\neg, \vee\}$ are expressively complete.

B4. (a) To prove the hint: show that if p and q are statements in the letters A and B taking value t in at least one case where A and B have opposite truth values, then $p \to q$ has exactly the same property (i.e. it *too* takes value t in a case where A and B have opposite truth values). Using the hint, *any* statement using *just* A, B and $\to$ takes value t in a case where A and B have opposite values. But in such a case, $A \leftrightarrow B$ is false! So you could *never* express it using just A, B and $\to$.

(c) Use: if p(A, B), q(A, B) both are true in at least two cases, $p(A, B) \to q(A, B)$ has the same property. Then, since $A \wedge B$ only takes value t in one case, you're done!

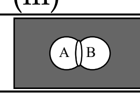
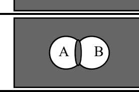
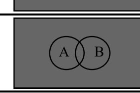
B5. f: $ABC \vee \underline{A}BC \vee \underline{ABC}$ h: $ABC \vee A\underline{BC} \vee \underline{A}B\underline{C} \vee \underline{ABC}$

B6. $h(A, A, A) \equiv \neg A$, $h(A, h(A, A, A), h(C, C, C)) \equiv A \vee C$

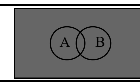
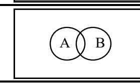
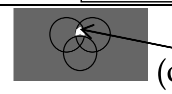
B7. (a) $A\underline{B}C \vee A\underline{BC} \vee \underline{A}BC \vee \underline{AB}C$

B8. (a) Since the set $\{\neg, \vee\}$ is expressively complete and those logical operators can be expressed in terms of $\{f, \to\}$ as: $\neg A \equiv A \to f$, $A \vee B \equiv (A \to B) \to B$

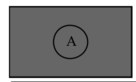
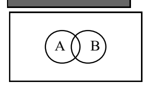
C2.

	(i)	(ii)	(iii)
(a)	$1 + A + B + A \cdot B$	contingent	
(b)	$1 + A + B$	contingent	
(c)	1	tautology	

C3.

	Binary	Venn
(a)	1	
(c)	0	
(e)	1 + pq + pqr	 (only the indicated space is empty)

C4.

	Binary	Venn
(a)	1	
(c)	0	
(e)	r+p+pr+pq+pqr note: this = 0 exactly when q=1 and r=0 so….	….shade everything except the region inside of q and outside of r!

C5. (a) $A\underline{B}$; A+AB; shade only what's inside circle A and outside circle B.

CHAPTER III

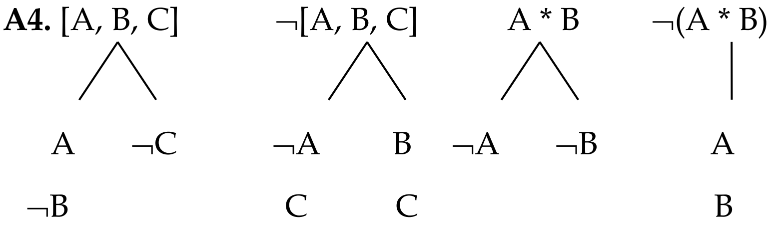
A1. (a) valid (c) invalid; counterexample: A=t, B=C=D=E=f

(e) $\neg E \vee A,\ R \rightarrow \neg A \quad \therefore\ \neg A \rightarrow (E \vee R)$ invalid, counterexample: A=E=R=f

A2. (a) valid; (c) valid

A3. (i) valid (iii) <u>in</u>valid c.examples: A true, all other statement letters false

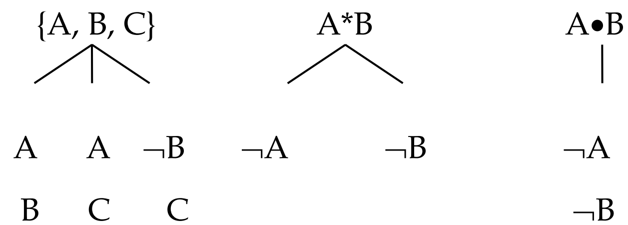
A4.



(i) valid

A5. (i) valid; (iii) invalid, c. examples: A=t, B=C=f

A6. Rules:



(i) valid; (iii) invalid

A7. (a) valid; (c) valid

A8. (a) valid; (c) valid

A9. (a) $K \rightarrow D$ Invalid, one counterexample

$K \rightarrow S$ [Note that the argument fails simply because Dumb could

$\underline{S \rightarrow \neg D}$ be so dumb that he doesn't even know it!]

$\neg D$

A10.

S = scientists don't know what they are talking about

B = the sun will eventually burn out

E = Earth will become dark and cold

M = Mars is teeming with life

H = the human race will migrate to other planets

D = the human race will die out

Argument has H=E=B=true, S=M=D=false for a counterexample

A11. Invalid, exactly 4 counterexamples

B1. (a) unsatisfiable; (c) satisfiable, A=t, B=f

B2. (a) tautology; (c) tautology

B3. (i) contradiction

B4. (i) P & J are both Knights or both Knaves.

B5. (i), (iii) equivalent; (v) inequivalent

B6. (a) contingent; (c) contingent

B7. (i) Both knaves; (iii) I=knave, He=can't tell! (this one's tricky)

B8. (a) contingent; (c) contradiction

B9. Lancelot is a knave, Arthur a knight and Merlin a knight.

B10. (a), (b) and (e) are the tautologies, the others aren't.

B11. They're all inconsistent!

B12. (a) Both knights.

B13.

1. No information derivable.

3. M is a saint if and only if J is a sinner.

5. M is a saint.

7. M is a saint and J a sinner.

9. J is a sinner.

11. M and J are both saints.

13. M is a sinner and J a saint.

15. Inconsistent.

17. M and J are both saints.

19. M is a sinner and J and his brother are not both saints.

21. C is a sinner and the others aren't both saints.

23. C and L are saints and M a sinner.

25. M is a saint and the others are sinners.

B14. (a) Satisfiable in 4 ways:

A	B	C	D
f	t	f	f
f	f	f	f
f	f	f	t
t	t	f	f

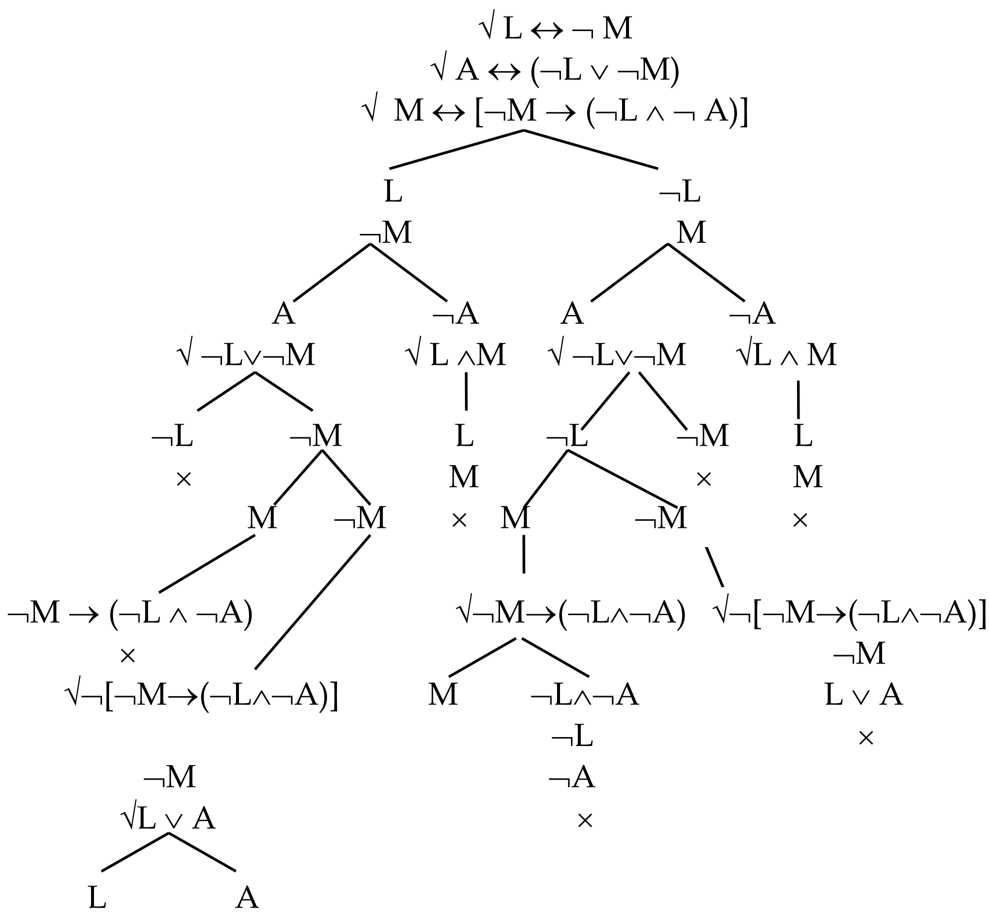
(c) Unsatisfiable

B15. (i) Non-tautologous; A=f, B=t only way to make false.

B16. (i) Satisfiable in one way:

A	B	C	D
f	f	f	t

B17. L=Lancelot a knight
A=Arthur a knight
M= Merlin a knight



From 3 open branches we see: A is true and M and L have opposite truth values.

B18. Guinevere *is* in Camelot today - Lancelot is *not* deceived!

B19. (a) Dean, Jerry, Stan are heroes, and Ollie is a scoundrel.

(c) One question you could ask is “Is it the case that you’re a knight if and only if there is no buried treasure on the island?” It is easy to verify (with the tree method) that if Dean answers Yes, there’s no gold on the island; and if he answers No, there is!

CHAPTER IV

A1. (i)

1.	$(F \rightarrow G) \wedge (H \rightarrow I)$	*premise*
2.	$J \rightarrow K$	*premise*
3.	$(F \vee J) \wedge (H \vee L)$	*premise*
4.	$F \rightarrow G$	1, *simp.*
5.	$(F \rightarrow G) \wedge (J \rightarrow K)$	4, 5, *conj.*
6.	$F \vee J$	3, *simp.*
7	$\therefore\ G \vee K$	5, 6, *CD.*

(vi)

1.	$A \rightarrow B$	*premise*
2.	$C \rightarrow D$	*premise*
3.	$A \vee C$	*premise*
4.	$A \rightarrow (A \wedge B)$	1, *exp.*
5.	$C \rightarrow (C \wedge D)$	2, *exp.*
6.	$[A \rightarrow (A \wedge B)] \wedge [C \rightarrow (C \wedge D)]$	4, 5, *conj.*
7.	$\therefore\ (A \wedge B) \vee (C \wedge D)$	3, 6, *CD*

A2. (i)

1.	$(D \wedge \neg E) \rightarrow F$	*premise*
2.	$\neg(E \vee F)$	*premise*
3.	$\neg E \wedge \neg F$	2, *De M*
4.	$\neg E$	3, *simp.*
5.	$\neg F$	3, *simp*
6.	$\neg F \rightarrow \neg(D \wedge \neg E)$	1, *cont*
7.	$\neg(D \wedge \neg E)$	5, 6, *MP*
8.	$\neg D \vee \neg\neg E$	7, *De M*
9.	$\therefore\ \neg D$	8, 4, *D.S.*

(viii)

1.	$(R \vee S) \rightarrow (T \wedge U)$	*premise*
2.	$\neg R \rightarrow (V \rightarrow \neg V)$	*premise*
3.	$\neg T$	*premise*

4. $\neg(T \wedge U) \rightarrow \neg(R \vee S)$ 1, *trans*
5. $(\neg T \vee \neg U) \rightarrow \neg(R \vee S)$ 4, *De M*
6. $\neg T \vee \neg U$ 3, *Add.*
7. $\neg(R \vee S)$ 5, 6, *MP*
8. $\neg R \wedge \neg S$ 8, *De M*
9. $\neg R$ 9. *simp*
10. $V \rightarrow \neg V$ 2, 9, *MP*
11. $\neg V \vee \neg V$ 10, *Impl*
12. $\therefore \neg V$ 11, *Taut*

CHAPTER V

A2. Let A be any set. Then for any x, $x \in \varnothing$ is a false statement, so $x \in \varnothing \Rightarrow x \in A$ is a true one. Hence $\varnothing \subseteq A$.

A4 (i) $x \in \mathbf{CC}A \Leftrightarrow \neg\neg x \in A \Leftrightarrow x \in A$.

(ii) $x \in A \Rightarrow x \in A$..

(iii) $x \in U \Leftrightarrow x \in A \vee x \in \mathbf{C}A$.

(iv) $x \in A \cap \mathbf{C}A \Leftrightarrow x \in A \wedge x \notin A \Leftrightarrow x \in \varnothing$.

(vi) $x \in (A \cup (B \cap C)) \Leftrightarrow x \in A \vee (x \in B \wedge x \in C) \Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \Leftrightarrow x \in A \cup B \wedge x \in A \cup C \Leftrightarrow x \in (A \cup B) \cap (A \cup C)$.

(vii) $x \in A \cap (B \cap C) \Leftrightarrow x \in A \wedge (x \in B \wedge x \in C) \Leftrightarrow (x \in A \wedge x \in B) \wedge x \in C \Leftrightarrow x \in (A \cap B) \cap C$.

(ix) $x \in \mathbf{C}(A \cap B) \Leftrightarrow \neg(x \in A \cap B) \Leftrightarrow \neg(x \in A \wedge x \in B) \Leftrightarrow x \notin A \vee x \notin B \Leftrightarrow x \in \mathbf{C}A \vee x \in \mathbf{C}B \Leftrightarrow x \in \mathbf{C}A \cup \mathbf{C}B$.

A5. (a) ⇔ (b). $A \subseteq B \Leftrightarrow \forall x\,(x \in A \Rightarrow x \in B) \Leftrightarrow \forall x\,(x \notin B \Rightarrow x \notin A) \Leftrightarrow \mathbf{C}B \subseteq \mathbf{C}A$.

(a) ⇔ (c) $A \subseteq B \Leftrightarrow \forall x\,(x \in A \Rightarrow x \in B) \Leftrightarrow \forall x\,((x \in A \vee x \in B) \Leftrightarrow x \in B) \Leftrightarrow A \cup B = B$.

(a) ⇔ (e) $A \cap \mathbf{C}B = \varnothing \Leftrightarrow \forall x \neg(x \in A \wedge x \in \mathbf{C}B) \Leftrightarrow \forall x\,(x \in A \Rightarrow x \notin \mathbf{C}B) \Leftrightarrow \forall x\,(x \in A \Rightarrow x \in B) \Leftrightarrow A \subseteq B$.

A6. (i) $x \in A - (A \cap B) \Leftrightarrow x \in A \wedge x \notin A \cap B \Leftrightarrow x \in A \wedge (x \notin A \vee x \notin B) \Leftrightarrow (x \in A \wedge x \notin A) \vee (x \in A \wedge x \notin B) \Leftrightarrow x \in A \vee x \notin B \Leftrightarrow x \in A - B$.

(ii) Note that $B \subseteq (A - B) \cup B$, so if not $B \subseteq A$, then $(A - B) \cup B \neq A$.

B2 Suppose $A \subseteq B$; then $(x, y) \in A \times C \Rightarrow x \in A \wedge y \in C \Rightarrow x \in B \wedge y \in \Rightarrow (x, y) \in B \times C$. Conversely suppose $C \neq \varnothing$ and fix an element $c \in C$. If $A \times C \subseteq B \times C$, then $x \in A \Rightarrow (x, c) \in A \times C \Rightarrow (x, c) \in B \times C \Rightarrow x \in C$.

B3 (i) $(x, y) \in A \times (B \cup C) \Leftrightarrow x \in A \wedge y \in B \cup C \Leftrightarrow x \in A \wedge (y \in B \vee y \in C \Leftrightarrow (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \Leftrightarrow (x, y) \in A \times B \vee (x, y) \in A \times C \Leftrightarrow (x, y) \in (A \times B) \cup (A \times C)$.

C3 $(x, y) \in (RS)^{-1} \Leftrightarrow (y, x) \in (RS) \Leftrightarrow$ *for some* $z(yRz \wedge zSx) \Leftrightarrow$ *for some* $z(z\, R^{-1}y \wedge x\, S^{-1}z) \Leftrightarrow (x, y) \in S^{-1} R^{-1}$.

D2. Let $U = a_R$, $V = b_R$ be two equivalence classes. If $U \cap V \neq \varnothing$, then there is c such that $c \in a_R \cap b_R$, i.e., $cRa \wedge cRb$, whence by symmetry $aRc \wedge cRb$, so that aRb by transitivity. So if $x \in U$, then xRa, and, since aRb, xRb follows by transitivity. Therefore $U \subseteq V$; similarly $V \subseteq U$, so that $U = V$.

E1. If $\leq$ is an ordering, then it is transitive, i.e., $x \leq y \wedge y \leq z \Rightarrow x \leq z$. This is equivalent to $y \leq^{-1} x \wedge z \leq^{-1} y \Rightarrow z \leq^{-1} x$, in other words to the transitivity of $\leq^{-1}$. Similarly for the remaining conditions.

E4. If R is a ranking, then S is easily verified to be reflexive and transitive. It is also obviously symmetric, and hence an equivalence relation.

F3. (ii) Suppose f and g are one-to-one. Then $(g \circ f)(x) = (g \circ f)(y) \Rightarrow g(f(x) = g(f(y)) \Rightarrow f(x) = f(y) \Rightarrow x = y$. So $g \circ f$ is one-to-one.

F4. (ii) Suppose $X \subseteq Y$. Then $y \in f[X] \Rightarrow \exists x \in X\ y = f(x) \Rightarrow \exists x \in Y\ y = f(x) \Rightarrow y \in f[Y]$. So $f[X] \subseteq f[Y]$. **(iii)** $z \in f[X \cup Y] \Leftrightarrow \exists x \in X \cup Y\ z = f(x) \Leftrightarrow \exists x \in X\ (z = f(x)) \vee \exists x \in Y\ z = f(x) \Leftrightarrow z \in f[X] \vee z \in f[Y] \Leftrightarrow z \in f[X] \cup f[Y]$. Hence $f[X \cup Y] = f[X] \cup f[Y]$. In general $f[X \cap Y] \subseteq f[X] \cap f[Y]$. But they are not always equal, for consider the function f: $\{0, 1\} \to \{0\}$ defined by $f(0) = f(1) = 0$, and let $X = \{0\}$, $Y = \{1\}$. Then $X \cap Y = \varnothing$, so $f[X \cap Y] = \varnothing$. But $f[X] = f[Y] = \{0\}$, so $f[X \cap Y] \neq f[X] \cap f[Y]$.

F5. (ii) $x \in g^{-1}[Y \cup Z] \Leftrightarrow g(x) \in Y \cup Z \Leftrightarrow g(x) \in Y \vee g(x) \in Z \Leftrightarrow x \in g^{-1}[Y] \vee x \in g^{-1}[Z] \Leftrightarrow x \in g^{-1}[Y \cup Z]$. **(iii)** $x \in X \Rightarrow g(x) \in g[X] \Rightarrow x \in g^{-1}[g[X]]$. Hence $X \subseteq g^{-1}[g[X]]$. Now suppose that g is one-to-one. We already know that $X \subseteq g^{-1}[g[X]]$. If $y \in g^{-1}[g[X]]$, then $g(y) \in g[X]$, so $g(y) = g(x)$ for some $x \in X$, whence $y = x \in X$ since g is one-to-one. Therefore $g^{-1}[g[X]] \subseteq X$, so $X = g^{-1}[g[X]]$.

Chapter VI

B1. (i) <u>∃x (Cx → Tx)</u>

∃x (Tx ∧ Cx) Not valid. Counterex: Domain = {1}, C= { }, T = { }

(iii) $\exists x\, Txr \rightarrow Tgr$

$\underline{Tgr \rightarrow \forall y\, Tyr}$

$\neg(\exists x\, Txr \wedge \exists x\, \neg Txr)$ Valid. No counterex.

B2. (a) False, because parallel lines don't intersect.

B3. For the knight case, you can take:

Domain={Arthur, Lancelot}, Knaves={Lancelot}, Knights={Arthur}

For the knave case, you can take:

Domain={Arthur, Lancelot}, Knaves={Lancelot, Arthur}, Knights={ }

B4. (i) false; (iii) true; (v) true

B5. (a) False, because the number 1 is odd but there are *no* positive even numbers less than it!

B6. (a) With Domain = {1} and Pxy taken to mean x=y, trivially satisfiable!

(c) <u>un</u>satisfiable

B7. (a) $\forall x(Fx \leftrightarrow \forall y Hyx)$ $\therefore$ $\forall x \forall y((Fx \wedge Fy) \rightarrow Hxy)$, valid.

(c) $\forall x \forall y((Lx \wedge By) \rightarrow Txy)$, $\forall x \forall y(x \neq y \rightarrow (Lx \leftrightarrow \neg Ly))$, $\therefore$ $\exists x \exists y(x \neq y \wedge Txy)$ not valid. Counterex: Domain = {a}, L={ }, B={ }, T={ }

(always shoot for a simple counterexample by trial and error first!!!)

B8. (a) $\exists x(Sx \wedge \forall y(Cy \rightarrow y=x))$, Cm, Tm $\therefore \forall x(Cx \rightarrow Tx)$, valid.

(c) $\forall x(\forall y(Osy \rightarrow Oxy) \rightarrow Osx)$ $\therefore$ $Oss \wedge \forall x(Osx \rightarrow x=s)$, not valid,

counterex: Domain ={s,a}, O ={(s,s), (a,s), (s,a)}

B9. (i) False, because shape a is not a triangle.

(iii) False, because g is neither a triangle nor square.

(v) False, because g has nothing left of it.

(vii) True ; (ix) True

B10. (1) $\forall x \exists y Ixy$, which is true because every statement implies itself!

(3) $\forall x \forall y(\forall z(Ixz \leftrightarrow Iyz) \rightarrow Exy)$, which is true because if x and y have the same implications, then since x implies itself, y must imply x too, and also (by the same argument) x must imply y, which means x and y must be equivalent.

(5) This says: 'Every statement fails to imply some statement', which is equivalent to saying: 'No statement implies all statements'. But contradictions do! So the stated claim is false.

(7) This says: 'Some statements are equivalent to anything they imply'— Yes, that's right: just consider any tautology!

B11. 1. False. To make it True, remove a's feather.

3. False. To make it True, increase a's and c's heights to the second line.

5. False. To make it True, give d's hat to b and remove d's feather.

B12. (a) false; (c) true; (e) true; (g) true; (i) true; (k) false

B13. (a) Case where true: Domain = {1}, R = { };

Case where false: Domain = {1}, R= {(1,1,1)}

*Note: These are obviously not the only possible answers!

(c) Case where true: Domain = {1,2}, P = {1,2}, Q = {1,2};

Case where false: Domain = {1}, P= { }, Q = { }

B14. (i) False; (iii) True; (v) True; (vii) True

B15. (i) false (iii) false (v) false (vii) false (ix) true

C1. (i) valid; (iii) valid; (v) invalid

C2. (a) $\forall x(Lx \to Nx), \neg\exists x(Vx \wedge Nx) \therefore \neg\exists x(Vx \wedge Lx)$, valid.

(c) $\exists x(Bxm \wedge \neg\exists yByx) \therefore \neg\exists Bmx$, invalid.

C3. valid

C4. (a) asserts that there is something which is *most likely* to have *P*: if *something* has *P,* it has *P*. (b) asserts that asserts that there is something which is *least likely* to have *P*: if *it* has *P, everything* has *P*. The validity of (b) leads to the so-called *drinker's paradox: there is someone in the pub such that, if he is drinking, then everyone in the pub is drinking*.

C5. (a) $\exists xPx \to Pu, \forall x(Lx \to Px), Ld \therefore Pu$; valid.

(c) $\forall x(Dx \to Cx) \therefore \forall x(\exists y(Dy \wedge Lxy) \to \exists z(Cz \wedge Lxz))$; valid.

C6. (i)
$$\frac{\begin{array}{c}\forall xCx \\ Cw \to G\end{array}}{G} \quad \text{valid}$$

(iii)
$$\frac{\begin{array}{c}\forall w[\exists xLwx \to \forall zLzw] \\ Lrj\end{array}}{Liy} \quad \text{valid}$$

C7. (i) $\{\forall x[Rx \to (Lx \wedge Vx)], \neg Lj \wedge Vj\}$ satisfiable

(iii) $\{\forall w[\exists xLwx \to \forall zLzw], Lyy, \neg Lym\}$ unsatisfiable

C8. (a)
$$\frac{\exists x[Hx \wedge \forall y[By \to Pxy]]}{\forall y[By \to \exists x(Hx \wedge Pxy)]}$$
valid

(c)
$$\frac{\begin{array}{c}\forall x(Sx \to Ux) \\ \neg\exists x(Wx \wedge Ux) \\ \neg\exists x(\neg Sx \wedge Vx)\end{array}}{\neg\exists x(Wx \wedge Vx)}$$
valid

C9. (i)
$$\frac{\begin{array}{c}\neg\forall x\forall y((Cx \wedge Cy \wedge x \neq y) \to (\neg Txy \wedge \neg Tyx)) \\ \neg\exists x(Cx \wedge Txx)\end{array}}{\exists x\exists y(Cx \wedge Cy \wedge x \neq y)} \quad \text{Valid.}$$

C10. (i) equivalent

C11. $\exists!xPx$ means there's exactly one thing with property P. (i) valid; (iii) <u>in</u>valid

C15. (a) $\forall x(Cx \rightarrow Dtx)$

$\neg\exists x(Cx \wedge Dxt)$

$\underline{Cs}$

$\therefore \exists x(Dtx \wedge Dxt)$ NOT Valid.

(c) $\exists x(x < f \wedge Ex)$

$\underline{\forall x \forall y(x<y \rightarrow \exists z(x <z \wedge z <y))}$

$\therefore \exists x(x <f \wedge \exists y(y < x \wedge Ey))$

Valid.

C16. (a) $\forall x(Cx \rightarrow Lnx)$

$\neg\exists x(Cx \wedge Lxn)$

$\underline{Cw}$

$\therefore \exists x(Lnx \wedge \neg Lxn)$

Valid.

(c) $\forall x \forall y (\exists z Lyz \rightarrow Lxy)$

$\underline{Lrj}$

$\therefore Lfw$

Valid.

D1. (i) $m = gfn$; (iii) $fm=fn \wedge gm=gn \wedge m \neq n$; (v) $\exists x (m=fgx \vee m =ffx)$

D2. (a) m and n are siblings

(c) m is n's brother; (e) n is m's father

F1.

$\forall P(Pm \rightarrow Pn)$

$\neg\forall P(Pn \rightarrow Pm)$

|

$\exists P\neg(Pn \rightarrow Pm)$

|

Qn

$\neg Qm$

$\neg Qm \rightarrow \neg Qn$

/ \

Qm $\quad$ Qn

× $\quad$ ×

Chapter VII

A1.

$\neg(\Box (p \wedge q) \to \Box p \wedge \Box q)$

$\Box (p \wedge q)$

$\neg(\Box p \wedge \Box q)$

$\neg\Box p \qquad \neg\Box q$

$\Diamond\neg p \qquad \Diamond\neg q$

$\neg p \qquad \neg q$

$p \wedge q \qquad p \wedge q$

$p \qquad q$

$\times \qquad \times$

$\neg(\Box p \wedge \Box q \to \Box(p \wedge q))$

$\Box p \wedge \Box q$

$\neg\Box (p \wedge q)$

$\Box p$

$\Box q$

$\Diamond (\neg p \wedge q)$

$\neg(p \wedge q)$

$\neg p \qquad \neg q$

$p \qquad q$

$\times \qquad \times$

$\neg[(\Box p \to \Diamond q) \to \neg\Box \mathbf{f}]$

$\Box p \to \Diamond q$

$\nearrow \mathbf{f}$

$\neg\Box p$ $\quad\quad$ $\Diamond q$

$\Diamond\neg p$ $\quad\quad$ q

$\neg p$ $\quad\quad$ **f**

f $\quad\quad$ ×

×

B1 (iii)

$\neg(\Box A \rightarrow \Diamond A)$

$\Box A$

$\neg\Diamond A$

$\Box\neg A$

t or f

.

1

(v)

$\neg(\Box(\Box A \rightarrow A) \rightarrow \Box A)$

$\Box(\Box A \rightarrow A)$

$\neg\Box A$

$\Diamond\neg A$

$\neg A$

$\Box A \rightarrow A$

$\neg\Box A$ $\quad\quad$ A

×

$\Diamond\neg A$

$\neg A$

f

f

. $\rightarrow$.

$\rightarrow$.

1 2

3

C1.

$\neg(\Diamond\Box A \to A)$

$\Diamond\Box A$

$\neg A$

$\Box A$

A

f t

. → .

1 2

C5. (ii)

$\neg(\Box\Diamond\Box\Diamond p \to \Box\Diamond p)$

$\Box\Diamond\Box\Diamond p$

$\neg\Box\Diamond p$

$\Diamond\neg\Diamond p$

$\neg\Diamond p$

$\Diamond\Box\Diamond p$

$\Box\neg p$

$\Box\Diamond p$

$\Box\neg p$

$\Diamond p$

p

$\neg p$

×

$\neg(\Box\Diamond p \to \Box\Diamond\Box\Diamond p)$

$\Box\Diamond p$

$\neg\Box\Diamond\Box\Diamond p$

$\Diamond\neg\Diamond\Box\Diamond p$

$\neg\Diamond\Box\Diamond p$

$\Diamond p$

$\Box\neg\Box\Diamond p$

p

$\neg\Box\Diamond p$

$\Box\Diamond p$

×

(iii)

$\neg\Box\Diamond(\Diamond\Box p \to \Box p)$

$\Diamond\neg\Diamond(\Diamond\Box p \to \Box p)$

$\neg\Diamond(\Diamond\Box p \to \Box p)$

$\Box\neg(\Diamond\Box p \to \Box p)$

$\neg(\Diamond\Box p \to \Box p)$

$\Diamond\Box p$

$\neg\Box p$

$\Box p$

$\Box\neg(\Diamond\Box p \to \Box p)$

$\neg(\Diamond\Box p \to \Box p)$

$\Diamond\Box p$

$\neg\Box p$

C6.

$\neg(\Box\Diamond A \to A)$

$\Box\Diamond A$

$\neg A$

$\Diamond A$

A

f t

. $\to$.

1 2

C8.

$\neg(\Box\Diamond\Box\Diamond p \to \Box\Diamond p)$

$\Box\Diamond\Box\Diamond p$

$\neg\Box\Diamond p$

$\Diamond\neg\Diamond p$

$\neg\Diamond p$

$\Diamond\Box\Diamond p$

$\Box\neg p$

$\Box\Diamond p$

$\times$

$\neg(\Box\Diamond p \to \Box\Diamond\Box\Diamond p)$

$\Box\Diamond p$

$\neg\ \Box\Diamond\Box\Diamond p$

$\Diamond\neg\ \Diamond\Box\Diamond p$

$\neg\Diamond\Box\Diamond p$

$\Box\neg\Box\Diamond p$

$\times$

Chapter VIII

A1 (x)

?¬¬¬p → ¬p

¬¬¬p

?¬p

p

¬¬¬p

?¬¬p

¬p

p

×

?¬p → ¬¬¬p

¬p

?¬¬¬p

¬¬p

?¬p

p

¬p

×

(xv)

?(¬p ∨ ¬q) → ¬(p ∧ q)

$\neg p \vee \neg q)$

$?\neg(p \wedge q)$

$p \wedge q$

$\neg p \vee \neg q$

$\neg p$ $\quad$ $\neg q$

p $\quad$ q

$\times$ $\quad$ $\times$

B2. (i)

$?\neg\neg A \rightarrow A$

$\neg\neg A$

$?A$

$?\neg A$

A

f f t

$. \rightarrow . \rightarrow .$

1 2 3

(ii)

$?A \vee \neg A$

$?A$

$?\neg A$

A

f t

$. \rightarrow .$

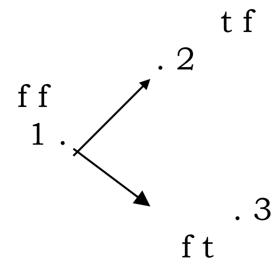
1 2

(iii)

?(A → B) ∨ (B → A)

?A → B
?B → A

A	B
?B	?A



Appendix A

A1. Justification for the sequence: in initial set, Axiom 1, Axiom 1, Axiom 2, Modus Ponens, Modus Ponens, Axiom 9, Axiom 10, Modus Ponens, Axiom 1, Modus Ponens, in initial set, Modus Ponens.

So, yes, it is a legitimate, if lengthy deduction of $\neg\neg p \vdash p$. Simple deduction is: $\underline{\neg\neg p}$, $\underline{\neg\neg p \rightarrow} p$, $\underline{p}$: in initial set Ax.10 MP

A2. 1. axiom 2; 2. axiom 1; 3. in initial set; 4. modus ponens; 5. axiom 1; 6. axiom 5; 7. modus ponens; 8. in initial set; 9. axiom 9; 10. modus ponens; 11. modus ponens; 12. axiom 10; 13 modus ponens; 14. You ain't foolin' me!

A3. For first sequence, justification is: in S, in S, Axiom 5, MP, MP; so the sequence is indeed a legitimate deduction from S.

A4. tautology goes with theorem

unsatisfiable goes with formally inconsistent

valid argument goes with deduction

valid argument, no premises goes with proof

A5. (i) $S \vdash p \Rightarrow S \models p$. Every deduction in the propositional calculus generates a valid argument!

(iii) To show that logic (at least propositional logic without quantifiers) can be completely captured through formal rules of symbol manipulation without any reference to the external (and potentially problematic) notion of 'truth' or 'meaning'.

A6. (a) false; (c) true; (e) false; (g) true; (i) false

A7. Because if $p \vdash r$, then clearly $p,q \vdash r$ for any q, from which it follows by the deduction theorem that $p \vdash (q \rightarrow r)$.

A8. (a) The Deduction theorem says: $S, p \vdash q$ if and only if $S \vdash p \rightarrow q$.

Assuming completeness and soundness, all we have to show is:

$$S, p \models q \text{ iff } S \models p \rightarrow q \quad (*)$$

because we could then argue as follows:

$$S, p \vdash q \Leftrightarrow S, p \models q \Leftrightarrow S \models p \rightarrow q \Leftrightarrow S \vdash p \rightarrow q.$$

Argument for (*). Assuming $S, p \models q$, it follows that $\{S, p, \neg q\}$ is unsatisfiable. So if all statements in S are true, $p \rightarrow q$ <u>can't</u> be false; because if it were, both p and $\neg q$ would have to be true as well, contradicting the unsatisfiability of $\{S, p, \neg q\}$. So if all statements in S are true, so is $p \rightarrow q$, which is just to say that $S \models p \rightarrow q$. Now we need to argue the other way around. If $S \models p \rightarrow q$, then there's never a case where all the statements in S are true and $p \rightarrow q$ is false, i.e. p true, $\neg q$ true. So $\{S, p, \neg q\}$ is unsatisfiable, which implies $S, p \models q$.

A9. (i) If $S \vdash p \rightarrow q$ then there is a legitimate deduction sequence of form S, …, p. Similarly, if $p \vdash q$ then a legitimate sequence p, …, q exists. So now just concatenate these 2 sequences to yield the following legitimate sequence: S, …, p, p, …, q which justifies $S \vdash q$!

(iii) First note that $p,q \vdash p \wedge q$ which is justified by the sequence: $p, q, p \rightarrow (q \rightarrow (p \wedge q))$, $q \rightarrow (p \wedge q)$, $p \wedge q$. Now suppose $S \vdash p$, i.e. S, …, p is a legitimate deduction sequence, and $S \vdash q$, i.e., S, …,q is legitimate. Since p,q, …, $p \wedge q$ is legitimate (as we've just shown), concatenating these three sequences gives S, …, p, S, …, q, p, q,…, $p \wedge q$, which *also* is a legitimate deduction sequence, thus establishing $S \vdash (p \wedge q)$.

A10. (a) By completeness and (strengthened) soundness, all you have to argue is that: $\exists q$: $S \models q$ and $S \models \neg q$ if and only if $\forall p$: $S \models p$

But that's clear: if the left-hand side is true, then S can't be satisfiable, which means that the right-hand side is true. (The argument from right to left is trivial.)