

# MATHEMATICAL LOGIC QUARTERLY

A Journal for Mathematical Logic,  
Foundations of Mathematics, and Logical Aspects  
of Theoretical Computer Science

M  
L  
Q

[www.mlq-journal.org](http://www.mlq-journal.org)

## The axiom of choice and the law of excluded middle in weak set theories

**John L. Bell**

Department of Philosophy, University of Western Ontario, London, Ontario, Canada N6A 3K7

Received 28 March 2007, revised 19 August 2007, accepted 22 August 2007

Published online 7 March 2008

**Key words** Axiom of choice, axiom of extensionality, weak set theory, law of excluded middle.

**MSC (2000)** 03E25, 03E55, 03F50

A weak form of intuitionistic set theory **WST** lacking the axiom of extensionality is introduced. While **WST** is too weak to support the derivation of the law of excluded middle from the axiom of choice, we show that beefing up **WST** with moderate extensionality principles or quotient sets enables the derivation to go through.

Math. Log. Quart. **54**, No. 2, 194–201 (2008) / DOI 10.1002/malq.200710030

## The axiom of choice and the law of excluded middle in weak set theories

John L. Bell\*

Department of Philosophy, University of Western Ontario, London, Ontario, Canada N6A 3K7

Received 28 March 2007, revised 19 August 2007, accepted 22 August 2007

Published online 7 March 2008

**Key words** Axiom of choice, axiom of extensionality, weak set theory, law of excluded middle.

**MSC (2000)** 03E25, 03E55, 03F50

A weak form of intuitionistic set theory **WST** lacking the axiom of extensionality is introduced. While **WST** is too weak to support the derivation of the law of excluded middle from the axiom of choice, we show that beefing up **WST** with moderate extensionality principles or quotient sets enables the derivation to go through.

© 2008 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim

In constructive mathematics the axiom of choice (**AC**) has a somewhat ambiguous status. On the one hand, in intuitionistic set theory, or the local set theory associated with a topos [2] it can be shown to entail the law of excluded middle (**LEM**) [3, 6]. On the other hand, under the “propositions-as types” interpretation which lies at the heart of constructive predicative type theories such as that of Martin-Löf [10], the axiom of choice is actually *derivable* (see, e. g. [12]), and so certainly cannot entail the law of excluded middle. This incongruity has been the subject of a number of recent investigations, for example [7, 8, 10, 13]. What has emerged is that for the derivation of **LEM** from **AC** to go through it is sufficient that sets (in particular, power sets), or functions, have a degree of extensionality which is, so to speak, built into the usual set theories but is incompatible with constructive type theories. Another condition, independent of extensionality, ensuring that the derivation goes through is that any equivalence relation determines a quotient set. **LEM** can also be shown to follow from a suitably extensionalized version of **AC**. The arguments establishing these intriguing results have mostly been formulated within type-theoretic frameworks. It is my purpose here to formulate and derive analogous results within a comparatively straightforward set-theoretic framework. The core principles of this framework form a theory – weak set theory **WST** – which lacks the axiom of extensionality<sup>1)</sup> and supports only minimal set-theoretic constructions. **WST** may be considered as a fragment both of (intuitionistic)  $\Delta_0$ -Zermelo set theory and Aczel’s constructive set theory [1]. In particular, **WST** is, like constructive type theories, too weak to allow the derivation of **LEM** from **AC**. But we shall see that, as with constructive type theories, beefing up **WST** with extensionality principles (even very moderate ones) or quotient sets enables the derivation to go through.

Let  $L$  be the first-order language of (intuitionistic) set theory which, in addition to the usual identity and membership symbols  $=$  and  $\in$  also contains a binary operation symbol  $\langle , \rangle$  permitting the formation of ordered pairs<sup>2)</sup>. At certain points various additional predicates and operation symbols will be introduced into  $L$ . The restricted quantifiers  $(\exists x \in a)$  and  $(\forall x \in a)$  are defined as usual, that is, as  $\exists x(x \in a \wedge \dots)$  and  $\forall x(x \in a \Rightarrow \dots)$ , respectively. A formula is *restricted* if it contains only restricted quantifiers.

\* e-mail: jbell@uwo.ca

1) Set theories (with classical logic) lacking the axiom of extensionality seem first to have been extensively studied in [4, 5, 11].

2) While the ordered pair  $\langle u, v \rangle$  could be defined in the customary way as  $\{\{u\}, \{u, v\}\}$ , here it is taken as a primitive operation – as it is in type theory – both for reasons of simplicity and to emphasize the fact that for our purposes it does not matter how (or indeed whether) it is defined set-theoretically.

Weak set theory **WST** is the theory in  $L$  with the following basic axioms (in which the free variables are understood to be universally quantified, and similarly below):

**Unordered Pair:**  $\exists u \forall x [x \in u \Leftrightarrow x = a \vee x = b]$ ,

**Ordered Pair:**  $\langle a, b \rangle = \langle c, d \rangle \Leftrightarrow a = c \wedge b = d$ ,

**Binary Union:**  $\exists u \forall x [x \in u \Leftrightarrow x \in a \vee x \in b]$ ,

**Cartesian Product:**  $\exists u \forall x [x \in u \Leftrightarrow (\exists y \in a)(\exists z \in b)(x = \langle y, z \rangle)]$ ,

**Restricted Subsets:**  $\exists u \forall x [x \in u \Leftrightarrow x \in a \wedge \varphi]$ ,

where in this last axiom  $\varphi$  is any restricted formula in which the variable  $u$  is not free.

We introduce into  $L$  new predicates and operation symbols as indicated below and adjoin to **WST** the following “definitional” axioms:

$$\begin{aligned}
a \subseteq b &\Leftrightarrow \forall x [x \in a \Rightarrow x \in b], & a \approx b &\Leftrightarrow \forall x [x \in a \Leftrightarrow x \in b], \\
\text{Ext}(a) &\Leftrightarrow (\forall x \in a)(\forall y \in a)[x \approx y \Rightarrow x = y], \\
x \in a \cup b &\Leftrightarrow x \in a \vee x \in b, & x \in \{a, b\} &\Leftrightarrow x = a \vee x = b, & \{a\} &= \{a, a\}, & xry &\Leftrightarrow \langle x, y \rangle \in r, \\
y \in \{x \in a : \varphi(x)\} &\Leftrightarrow y \in a \wedge \varphi(y) & (\varphi \text{ restricted}), \\
\neg x \in 0, & 1 = \{0\}, & 2 = \{0, 1\}, \\
x \in a \times b &\Leftrightarrow (\exists u \in a)(\exists v \in b)[x = \langle u, v \rangle], \\
x \in a + b &\Leftrightarrow (\exists u \in a)(\exists v \in b)[x = \langle u, 0 \rangle \vee x = \langle v, 1 \rangle], \\
f : a \longrightarrow b &\Leftrightarrow f \subseteq a \times b \wedge (\forall x \in a)(\exists y \in b)(xfy) \wedge \forall x \forall y \forall z [(xfy \wedge xfz) \Rightarrow y = z], \\
\text{Fun}(f) &\Leftrightarrow \exists a \exists b (f : a \longrightarrow b), \\
f : a \longrightarrow b &\wedge x \in a \Rightarrow xff(x), \\
f : a \longrightarrow b \wedge g : b \longrightarrow c &\Rightarrow g \circ f : a \longrightarrow c \wedge (\forall x \in a)[(g \circ f)(x) = g(f(x))], \\
f : a \longrightarrow b &\Leftrightarrow f : a \longrightarrow b \wedge (\forall y \in b)(\exists x \in a)[y = f(x)], \\
\pi_1 : a + b \longrightarrow a \cup b &\wedge (\forall x \in a)[\pi_1(\langle x, 0 \rangle) = x] \wedge (\forall y \in b)[\pi_1(\langle y, 1 \rangle) = y], \\
\pi_2 : a + b \longrightarrow 2 &\wedge (\forall x \in a)[\pi_2(\langle x, 0 \rangle) = 0] \wedge (\forall y \in b)[\pi_2(\langle y, 1 \rangle) = 1], \\
\text{Eq}(s, a) &\Leftrightarrow s \subseteq a \times a \wedge (\forall x \in a)(xsx) \\
&\quad \wedge (\forall x \in a)(\forall y \in a)(xsy \Rightarrow ysx) \\
&\quad \wedge (\forall x \in a)(\forall y \in a)(\forall z \in a)[(xsy \wedge ysz) \Rightarrow xsz], \\
\text{Comp}(r, s) &\Leftrightarrow \forall x \forall x' \forall y [(xsx' \wedge x'ry) \Rightarrow xry], \\
\text{Comp}(r) &\Leftrightarrow \forall x \forall x' \forall y [(x \approx x' \wedge x'ry) \Rightarrow xry], \\
\text{Extn}(f, s) &\Leftrightarrow \text{Fun}(f) \wedge \forall x \forall x' [xsx' \wedge \exists y \exists y' (xfy \wedge x'fy') \Rightarrow f(x) = f(x')], \\
\text{Ex}(f) &\Leftrightarrow \text{Fun}(f) \wedge \forall x \forall x' [x \approx x' \wedge \exists y \exists y' (xfy \wedge x'fy') \Rightarrow f(x) = f(x')].
\end{aligned}$$

Most of these definitions are standard. The functions  $\pi_1$  and  $\pi_2$  are projections of ordered pairs onto their 1<sup>st</sup> and 2<sup>nd</sup> coordinates, respectively: clearly, for  $u, v \in a + b$  we have

$$(\text{proj}) \quad u = v \Leftrightarrow [\pi_1(u) = \pi_1(v) \wedge \pi_2(u) = \pi_2(v)].$$

The relation  $\approx$  is that of *extensional equality*.  $\text{Ext}(a)$  expresses the extensionality of the members of the set  $a$ .  $\text{Eq}(s, a)$  asserts that  $s$  is an equivalence relation on  $a$ . If  $r$  is a relation between  $a$  and  $b$ , and  $s$  an relation on  $a$ ,  $\text{Comp}(r, s)$  expresses the compatibility of  $r$  with  $s$ , and  $\text{Comp}(r)$  the compatibility of  $r$  with extensional equality. If  $f : a \longrightarrow b$ , and  $s$  is an equivalence relation on  $a$ ,  $\text{Extn}(f, s)$  expresses the idea that  $f$  treats the relation  $s$  as if it were the identity relation: we shall then say that  $f$  is *s-extensional*.  $\text{Ex}(f)$  asserts that  $f$  is extensional in the sense of treating extensional equality as if it were the identity.

In addition to the axioms of **WST**, we formulate the following axioms additional to those of **WST** (recalling that in stating axioms all free variables are universally quantified):

**Extensionality:**  $\text{Ext}(\{a, b\})$ ,

**Extsub(1):**  $a \subseteq 1 \wedge b \subseteq 1 \Rightarrow \text{Ext}(\{a, b\})$ ,

**Extsub(2):**  $a \subseteq 2 \wedge b \subseteq 2 \Rightarrow \text{Ext}(\{a, b\})$ .

While **Extensionality** asserts that all extensionally equal sets are identical, **Extsub(1)** and **Extsub(2)** are weak versions confining the assertion just to subsets of 1 or 2. Explicitly, **Extsub(1)** and **Extsub(2)** assert that each doubleton composed of subsets of 1, or of 2, is extensional. Notice that since  $1 \subseteq 2$ , **Extsub(1)** is a consequence of **Extsub(2)**.

Following [1], we define a set  $a$  to be a *base* if every relation with domain  $a$  includes a function with domain  $a$ , i. e.

$$\text{Base}(a) \Leftrightarrow \forall b \forall r [r \subseteq a \times b \wedge (\forall x \in a)(\exists y \in b)(xry) \Rightarrow (\exists f : a \longrightarrow b)(\forall x \in a)(xrf(x))].$$

We call  $a$  an *extensional base* if every relation with domain  $a$  compatible with extensional equality includes an *extensional* function with domain  $a$ , i. e.

$$\begin{aligned} \text{Extbase}(a) &\Leftrightarrow \forall b \forall r [r \subseteq a \times b \wedge \text{Comp}(r) \wedge (\forall x \in a)(\exists y \in b)(xry) \\ &\Rightarrow (\exists f : a \longrightarrow b)[\text{Ex}(f) \wedge (\forall x \in a)(xrf(x))]. \end{aligned}$$

We use these notions to state a number of versions of the axiom of choice (again recalling that in stating axioms all free variables are universally quantified):

**(Intensional) Axiom of choice AC:**  $\text{Base}(a)$ ,

**Weak axiom of choice 1 WAC(1):**  $a \subseteq 1 \wedge a' \subseteq 1 \wedge b \subseteq 1 \wedge b' \subseteq 1 \Rightarrow \text{Base}(\{\langle a, a' \rangle, \langle b, b' \rangle\})$ ,

**Weak axiom of choice 2 WAC(2):**  $a \subseteq 2 \wedge a' \subseteq 2 \Rightarrow \text{Base}(\{a, a'\})$ ,

**Universal extensional axiom of choice UEAC:**

$$\begin{aligned} \text{Eq}(s, a) \wedge r \subseteq a \times b \wedge \text{Comp}(r, s) \wedge (\forall x \in a)(\exists y \in b)(xry) \\ \Rightarrow (\exists f : a \longrightarrow b)[\text{Extn}(f, s) \wedge (\forall x \in a)(xrf(x))], \end{aligned}$$

**Extensional axiom of choice EAC:**  $\text{Extbase}(a)$ .

In asserting that every set is a base, **AC** means, as usual, that a choice function always exists under the appropriate conditions on an arbitrarily given relation. **WAC(1)** and **WAC(2)** restrict the existence of such choice functions to relations whose domains are doubletons of a certain form.<sup>3)</sup> **UEAC** asserts that, in the presence of an equivalence relation  $s$  with which a given relation  $r$  is compatible, the choice function can be taken to be  $s$ -extensional. **EAC** is the special case of **UEAC** in which the equivalence relation is that of extensional equality. In view of the fact that **AC** can be seen to be the special case of **UEAC** in which the equivalence relation is the identity relation, **AC** is sometimes known as *the intensional axiom of choice*.

Our next axiom is

**Quotients:**  $\text{Eq}(s, a) \Rightarrow \exists u \exists f [f : a \longrightarrow u \wedge (\forall x \in a)(\forall y \in a)[f(x) = f(y) \Leftrightarrow xsy]]$ .

This axiom asserts that each equivalence relation determines a quotient set. In **WST + Quotients**, we introduce operation symbols  $\cdot / \cdot$ ,  $[\cdot]$ , and adjoin the “definitional” axiom

$$\begin{aligned} \text{(Q)} \quad \text{Eq}(s, a) \Rightarrow [(\forall x \in a)([x]_s \in a/r) \wedge (\forall u \in a/s)(\exists x \in a)(u = [x]_s) \\ \wedge (\forall x \in a)(\forall y \in a)[[x]_s = [y]_s \Leftrightarrow xsy]]. \end{aligned}$$

Here,  $a/s$  is the *quotient* of  $a$  by  $s$  and, for  $x \in a$ ,  $[x]_s$  is *the image of  $x$  in  $a/s$* .

Reminding the reader that our background logic is intuitionistic, we finally introduce the following logical scheme:

**Restricted Excluded Middle REM:**  $\varphi \vee \neg \varphi$  for any restricted formula  $\varphi$ .

We are going to prove the following results.

**Theorem 1** REM is derivable in:

- (a) **WST + Extsub(1) + WAC(1)**;
- (b) **WST + Extsub(2) + WAC(2)**;
- (c) **WST + EAC**.

<sup>3)</sup> Using the fact that subsets of 2 are in natural bijective correspondence with pairs of subsets of 1, it can be shown that **WAC(1)** and **WAC(2)** are in fact equivalent. Nevertheless for our purposes it will be useful to have both principles.

**Theorem 2** REM is derivable in WST + AC + Quotients.

**Theorem 3** AC  $\Leftrightarrow$  UEAC is derivable in WST + Quotients.

Thus, while in the absence of extensionality for doubletons of subsets of 1 or 2, the intensional axiom of choice does not entail the law of excluded middle, with that degree of extensionality the law of excluded middle becomes a consequence of very weak versions of the intensional axiom of choice. Moreover, the extensional axiom of choice entails the law of excluded middle without additional extensionality assumptions. And finally, when quotients are present, the intensional axiom of choice is no weaker than its universal extensional version and entails the law of excluded middle without additional extensionality assumptions.

**Proof of Theorem 1.**

(a) We argue in WST + Extsub(1) + WAC(1). Given an arbitrary restricted formula  $\varphi$ , we define

$$s = \{x \in \{0\} : \varphi\} \quad \text{and} \quad a = \{\langle s, \{0\} \rangle, \langle \{0\}, s \rangle\}.$$

It is then easily shown that

$$\langle u, v \rangle \in a \Rightarrow (\exists x \in 2)[(x = 0 \Rightarrow 0 \in u) \wedge (x = 1 \Rightarrow 0 \in v)].$$

So WAC(1) gives  $f : a \rightarrow 2$  such that, for  $\langle u, v \rangle \in a$ ,

$$(1) \quad f(\langle u, v \rangle) = 0 \Rightarrow 0 \in u, \quad f(\langle u, v \rangle) = 1 \Rightarrow 0 \in v.$$

Since  $f$  maps to 2, we have

$$[f(\langle s, \{0\} \rangle) = 0 \vee f(\langle s, \{0\} \rangle) = 1] \wedge [f(\langle \{0\}, s \rangle) = 0 \vee f(\langle \{0\}, s \rangle) = 1].$$

From this, together with (1), it follows that

$$[0 \in s \vee f(\langle s, \{0\} \rangle) = 1] \wedge [f(\langle \{0\}, s \rangle) = 0 \vee 0 \in s],$$

whence, using the distributive law,

$$0 \in s \vee [f(\langle s, \{0\} \rangle) = 0 \wedge f(\langle \{0\}, s \rangle) = 1].$$

Writing  $\psi(s)$  for the second disjunct in this last formula, it becomes

$$(2) \quad 0 \in s \vee \psi(s).$$

Now from  $s \subseteq 1$  we deduce  $0 \in s \Rightarrow s \approx \{0\}$ , whence using Extsub(1),  $0 \in s \Rightarrow s = \{0\}$ . Hence

$$[0 \in s \wedge \psi(s)] \Rightarrow [s = \{0\} \wedge \psi(s)] \Rightarrow \psi(\{0\}) \Rightarrow 0 = 1.$$

Since clearly  $0 \neq 1$ , we conclude that  $\psi(s) \Rightarrow \neg 0 \in s$  and (2) then yields

$$(3) \quad 0 \in s \vee \neg 0 \in s.$$

But obviously  $0 \in s \Leftrightarrow \varphi$ , so (3) gives  $\varphi \vee \neg \varphi$ , as required.

(b)<sup>4)</sup> We argue in WST + Extsub(2) + WAC(2). Given a formula  $\varphi$ , define

$$a = \{x \in 2 : x = 0 \vee \varphi\}, \quad b = \{x \in 2 : x = 1 \vee \varphi\}.$$

Since  $0 \in a$  and  $1 \in b$ , we have

$$(\forall x \in \{a, b\})(\exists y \in 2)(y \in x).$$

<sup>4)</sup> Using the observation in the previous footnote that WAC(1) and WAC(2) are equivalent, it can be seen that (b) actually follows from (a). However, the direct proof given here, based on that given in [6], is, in the author's view, considerably more illuminating.

Hence **WAC(2)** applied to the relation  $r = \{\langle x, y \rangle \in \{a, b\} \times 2 : y \in x\}$  yields a function  $f : \{a, b\} \rightarrow 2$  for which  $(\forall x \in \{a, b\})(f(x) \in x)$ . It follows that  $f(a) \in a \wedge f(b) \in b$ , whence

$$[f(a) = 0 \vee \varphi] \wedge [f(b) = 1 \vee \varphi].$$

Applying the distributive law, we then get

$$\varphi \vee [f(a) = 0 \wedge f(b) = 1],$$

whence

$$(4) \quad \varphi \vee f(a) \neq f(b).$$

Now clearly  $\varphi \Rightarrow a \approx b$ , and from this and **Extsub(2)** we deduce  $\varphi \Rightarrow a = b$ , whence

$$(5) \quad \varphi \Rightarrow f(a) = f(b).$$

It follows that  $f(a) \neq f(b) \Rightarrow \neg\varphi$ , and we conclude from (4) that  $\varphi \vee \neg\varphi$ , as required.

(c) Here the argument in **WST + EAC** is the same as that given in (b) except that in deriving (5) above we invoke **EAC** in place of **Extsub(2)**. To justify this step it suffices to show that  $\text{Comp}(r)$ , where  $r$  is the relation defined in the proof of (b). This, however, is clear.  $\square$

**Proof of Theorem 2.** For  $b \subseteq a$ , we say that  $b$  is *detachable (in a)* if

$$(\forall x \in a)[x \in b \vee x \notin b].$$

An *indicator for b (in a)* is a function  $g : a \times 2 \rightarrow 2$  satisfying

$$(\forall x \in a)[x \in b \Leftrightarrow g(\langle x, 0 \rangle) = g(\langle x, 1 \rangle)].$$

It is easy to show that a subset is detachable if and only if it has an indicator. For if  $b \subseteq a$  is detachable, then the function  $g : a \times 2 \rightarrow 2$  defined by

$$\begin{aligned} g(\langle x, 0 \rangle) = g(\langle x, 1 \rangle) &= 0 && \text{if } x \in b, \\ g(\langle x, 0 \rangle) = 0 \wedge g(\langle x, 1 \rangle) &= 1 && \text{if } x \notin b \end{aligned}$$

is an indicator for  $b$ . Conversely, for any function  $g : a \times 2 \rightarrow 2$ , we have

$$g(\langle x, 0 \rangle) = g(\langle x, 1 \rangle) \vee g(\langle x, 0 \rangle) \neq g(\langle x, 1 \rangle),$$

so if  $g$  is an indicator for  $b$ , we infer  $(\forall x \in a)[x \in b \vee x \notin b]$ , and  $b$  is detachable.

Now we show in **WST + AC + Quotients** that every subset of a set has an indicator, and is hence detachable. Given  $b \subseteq a$ , let  $s$  be the binary relation on  $a + a$  given by:

$$\begin{aligned} s = \{ \langle \langle x, 0 \rangle, \langle x, 0 \rangle \rangle : x \in a \} &\cup \{ \langle \langle x, 1 \rangle, \langle x, 1 \rangle \rangle : x \in a \} \\ &\cup \{ \langle \langle x, 0 \rangle, \langle x, 1 \rangle \rangle : x \in b \} \\ &\cup \{ \langle \langle x, 1 \rangle, \langle x, 0 \rangle \rangle : x \in b \}. \end{aligned}$$

It is easily checked that  $\text{Eq}(s, a + a)$ . Also, it is clear that, for  $z, z' \in a + a$ ,

$$(6) \quad zsz' \Rightarrow \pi_1(z) = \pi_1(z')$$

and, for  $x \in a$ ,

$$(7) \quad x \in b \Leftrightarrow \langle x, 0 \rangle s \langle x, 1 \rangle.$$

Invoking axiom **(Q)** above, we introduce the quotient  $(a + a)/s$  of  $a + a$  by  $s$  and the image  $[u]_s$  of an element  $u$  of  $a + a$  in  $(a + a)/s$  for which we then have

$$(8) \quad (\forall z \in (a + a)/s)(\exists u \in a + a)(z = [u]_s)$$

and

$$(9) \quad (\forall u \in a + a)(\forall v \in a + a)[[u]_s = [v]_s \Leftrightarrow usv].$$

Applying **AC** to (7) yields a function  $f : (a + a)/s \longrightarrow a + a$  for which

$$(10) \quad (\forall z \in (a + a)/s)(z = [f(z)]_s).$$

Clearly  $f$  is one-one, that is, we have

$$(11) \quad f(z) = f(z') \Leftrightarrow z = z'.$$

Next, observe that, for  $i = 0, 1$ , and  $x \in a$ ,

$$(12) \quad \pi_1([f(\langle x, i \rangle)]_s) = x.$$

For from (10) we have  $[f(\langle x, i \rangle)]_s = [f([\langle x, i \rangle]_s)]_s$ , whence by (9),  $\langle x, i \rangle sf([\langle x, i \rangle]_s)$ . Hence by (6),

$$\pi_1(\langle x, i \rangle) = \pi_1(f([\langle x, i \rangle]_s)).$$

(12) now follows from this and the fact that  $\pi_1(\langle x, i \rangle) = x$ .

We have also

$$(13) \quad x \in b \Leftrightarrow f([\langle x, 0 \rangle]_s) = f([\langle x, 1 \rangle]_s).$$

For we have

$$\begin{aligned} x \in b &\Leftrightarrow \langle x, 0 \rangle s \langle x, 1 \rangle && \text{using (7)} \\ &\Leftrightarrow [\langle x, 0 \rangle]_s = [\langle x, 1 \rangle]_s && \text{using (9)} \\ &\Leftrightarrow f([\langle x, 0 \rangle]_s) = f([\langle x, 1 \rangle]_s) && \text{using (11)}. \end{aligned}$$

Now define  $g : a \times 2 \longrightarrow 2$  by

$$g(\langle x, i \rangle) = \pi_2(f([\langle x, i \rangle]_s)).$$

We claim that  $g$  is an indicator for  $b$ . This can be seen from the following equivalences:

$$\begin{aligned} x \in b &\Leftrightarrow f([\langle x, 0 \rangle]_s) = f([\langle x, 1 \rangle]_s) && \text{(by (13))} \\ &\Leftrightarrow \pi_1(f([\langle x, 0 \rangle]_s)) = \pi_1(f([\langle x, 1 \rangle]_s)) \wedge \pi_2(f([\langle x, 0 \rangle]_s)) = \pi_2(f([\langle x, 1 \rangle]_s)) && \text{(by (proj))} \\ &\Leftrightarrow \pi_2(f([\langle x, 0 \rangle]_s)) = \pi_2(f([\langle x, 1 \rangle]_s)) && \text{(using (12))} \\ &\Leftrightarrow g(\langle x, 0 \rangle) = g(\langle x, 1 \rangle). \end{aligned}$$

So we have shown that in **WST + AC + Quotients** every subset of a set has an indicator, and is accordingly detachable. This latter fact easily yields **REM**. For, given a restricted  $\varphi$ , for any  $a$ , the set  $b = \{x \in a : \varphi\}$  is then a detachable subset of  $a$ , from which  $(\forall x \in a)(\varphi \vee \neg\varphi)$  immediately follows. By taking  $a = \{x\}$  we obtain that  $\varphi \vee \neg\varphi$ .  $\square$

**Proof of Theorem 3.** It suffices to derive **UEAC** from **AC** in **WST + Quotients**. With the assumption of **Eq**( $s, a$ ), we use **AC** as in the proof of Theorem 2 to obtain a function  $p : a/s \longrightarrow a$  such that  $u = [p(u)]_s$  for all  $u \in a/s$ . From this we deduce  $[x]_s = [p([x]_s)]_s$ , whence

$$(14) \quad xsp([x]_s)$$

for all  $x \in a$ .

Assuming the antecedent of **UEAC**, viz.

$$\text{Eq}(s, a) \wedge r \subseteq a \times b \wedge \text{Comp}(r, s) \wedge (\forall x \in a)(\exists y \in b)(xry),$$

define the relation  $r' \subseteq a/s \times b$  by

$$ur'y \Leftrightarrow p(u)ry.$$

Now use **AC** to obtain a function  $g : a/s \rightarrow b$  for which  $(\forall u \in a/s)(ur'g(u))$ , i. e.

$$(15) \quad (\forall u \in a/s)(p(u)rg(u)).$$

Define  $f : a \rightarrow b$  by  $f(x) = g([x]_s)$ . Then by (15),

$$(\forall x \in a)(p([x]_s)rg([x]_s)).$$

From this, (14), and  $\text{Comp}(r, s)$  it follows that  $(\forall x \in a)(xrg([x]_s))$ , i. e.

$$(16) \quad (\forall x \in a)(xrf(x)).$$

Moreover, for all  $x, x' \in a$ , we have

$$xx' \Rightarrow [x]_s = [x']_s \Rightarrow f(x) = g([x]_s) = g([x']_s) = f(x'),$$

whence  $\text{Ext}_n(f, s)$ . This, together with (16), establishes the consequent of **UEAC**.  $\square$

### Concluding remarks

1. The proof of Theorem 1(a) is an adaptation of the proof of the analogous result given in [2, pp. 144 – 146], and that of (b) is based on that given in [6].

2. The proof of Theorem 2 is an adaptation to a set-theoretical context of the argument in [3] that, in a topos which satisfies the axiom of choice, all subobjects are complemented. By weakening **Quotients** to the assertion **Quotients**(1 + 1) that quotient sets are determined just by equivalence relations on the set 1 + 1, the proof of Theorem 2 shows that **REM** is derivable in the theory **WST** + **AC** + **Quotients**(1 + 1).

3. **Quotients** can be derived within **WST** augmented by *the full extensional power set axiom*

$$\mathbf{Extpow}: \exists u[\text{Ext}(u) \wedge \forall x[x \in u \Leftrightarrow x \subseteq a]].$$

So (cf. [8]) adding extensional power sets to **WST** + **AC** yields **REM**.

4. Another version of the axiom of choice, easily derivable in **WST** from **AC**, is:

$$\mathbf{AC}^*: (\forall x \in a)[x \subseteq b \wedge \exists y(y \in x)] \Rightarrow (\exists f : a \rightarrow b)(\forall x \in a)(f(x) \in x).$$

If one adds to **WST** *the nonextensional power set axiom*, viz.

$$\mathbf{Pow}: \exists u \forall x[x \in u \Leftrightarrow x \subseteq a],$$

then **AC** becomes derivable from **AC**<sup>\*</sup>. Note that while **Extpow** entails **REM**, **Pow** is logically “harmless”, that is, it has no nonconstructive logical consequences such as **LEM**.

The extensional version of **AC**<sup>\*</sup>, viz.

$$\mathbf{EAC}^*: (\forall x \in a)[x \subseteq b \wedge \exists y(y \in x)] \Rightarrow (\exists f : a \rightarrow b)[\text{Ext}(f) \wedge (\forall x \in a)(f(x) \in x)]$$

is derivable in **WST** from **EAC**. In **WST** + **Pow**, **EAC** and **EAC**<sup>\*</sup> are equivalent.

5. A version of the axiom of choice considered in [9] is what we shall call

**Rep**(representatives):

$$\text{Eq}(s, a) \Rightarrow \exists f[f : a \rightarrow a \wedge (\forall x \in a)(xsf(x)) \wedge (\forall x \in a)(\forall y \in a)[xsy \Leftrightarrow f(x) = f(y)]].$$

**Rep** asserts that unique representatives can be chosen from the equivalence classes of any equivalence relation. Obviously, in **WST**, **Rep** implies **Quotients**. Moreover, the proof of Theorem 2 is easily adapted to show that, in **WST**, **Rep** yields **REM**. In **WST**, **AC** + **Quotients** entails **Rep**, and, in **WST** + **Pow**, conversely.



6. Finally, consider the following versions of **AC** which are closely related to that introduced by Zermelo [14] (see also [10]), namely

$$\mathbf{ACZ}: [(\forall x \in a)\exists y(y \in x) \wedge (\forall x \in a)(\forall y \in a)[\exists z(z \in x \wedge z \in y) \Rightarrow x = y]] \\ \Rightarrow \exists u(\forall x \in a)\exists!y(y \in x \wedge y \in u),$$

$$\mathbf{EACZ}: [(\forall x \in a)\exists y(y \in x) \wedge (\forall x \in a)(\forall y \in a)[\exists z(z \in x \wedge z \in y) \Rightarrow x \approx y]] \\ \Rightarrow \exists u(\forall x \in a)\exists!y(y \in x \wedge y \in u).$$

These are construals of the assertion that, given any collection of mutually disjoint nonempty sets, there is a set intersecting each member of the collection in exactly one element. Clearly **EACZ** implies **ACZ**; the former is readily derivable from **EAC** and the latter from **AC**. Since **REM** is not a consequence of **AC**, it cannot, a fortiori, be a consequence of **ACZ**. But, like **EAC**, **EACZ** can be shown to yield **REM**. We sketch the argument, which is similar to the proof of Theorem 2(b).

Given a restricted formula  $\varphi$ , we define  $b = \{x \in 2 : x = 0 \vee \varphi\}$ ,  $c = \{x \in 2 : x = 1 \vee \varphi\}$ , and  $a = \{b, c\}$ . A straightforward argument shows that  $a$  satisfies the antecedent of **EACZ**. So, if this last is assumed, its consequent yields a  $u$  with exactly one element in common with  $b$  and with  $c$ . Writing  $d$  and  $e$  for these elements, one easily shows that

$$(*) \quad \varphi \vee d \neq e.$$

Now since it is also easily shown that  $\varphi \Rightarrow d = e$ , it follows that  $d \neq e \Rightarrow \neg\varphi$ , and this, together with  $(*)$ , yields that  $\varphi \vee \neg\varphi$ .

**Acknowledgements** I am grateful to the referee for making a number of helpful suggestions for sharpening the paper.

## References

- [1] P. Aczel and M. Rathjen, Notes on constructive set theory. Institut Mittag-Leffler Report 40, 2000/2001. Available on [http://www.ml.kva.se/preprints/meta/AczelMon\\_Sep\\_24\\_09\\_16\\_56.rdf.html](http://www.ml.kva.se/preprints/meta/AczelMon_Sep_24_09_16_56.rdf.html).
- [2] J. L. Bell, Toposes and Local Set Theories: An Introduction (Clarendon Press, 1988). To be republished by Dover in 2008.
- [3] R. Diaconescu, Axiom of choice and complementation. Proc. Amer. Math. Soc. **51**, 176 – 178 (1975).
- [4] R. O. Gandy, On the axiom of extensionality, part I. J. Symbolic Logic **21**, 36 – 48 (1956).
- [5] R. O. Gandy, On the axiom of extensionality, part II. J. Symbolic Logic **24**, 287 – 300 (1959).
- [6] N. Goodman and J. Myhill, Choice implies excluded middle. Z. Math. Logik Grundlagen Math. **24**, 461 (1978).
- [7] M. E. Maietti, About effective quotients in constructive type theory. In: Types for Proofs and Programs, International Workshop “Types 98” (T. Altenkirch et al., eds.). Lecture Notes in Computer Science 1657, pp. 164 – 178 (Springer-Verlag, 1999).
- [8] M. E. Maietti and S. Valentini, Can you add power-set to Martin-Löf intuitionistic type theory? Math. Log. Quart. **45**, 521 – 532 (1999).
- [9] P. Martin-Löf, Intuitionistic Type Theory (Bibliopolis, 1984).
- [10] P. Martin-Löf, 100 years of Zermelo’s axiom of choice: what was the problem with it? Computer J. **49**, 345 – 350 (2006).
- [11] D. S. Scott, More on the axiom of extensionality. In: Essays on the Foundations of Mathematics, pp. 115 – 131 (Magnes Press, 1966).
- [12] W. W. Tait, The law of excluded middle and the axiom of choice. In: Mathematics and Mind (A. George, ed.), pp. 45 – 70 (Oxford University Press, 1994).
- [13] S. Valentini, Extensionality versus constructivity. Math. Log. Quart. **48**, 179 – 187 (2002).
- [14] E. Zermelo, Neuer Beweis für die Möglichkeit einer Wohlordnung. Mathematische Annalen **65**, 107 – 128 (1908). Translated in: J. van Heijenoort, From Frege to Gödel: A Source Book in Mathematical Logic 1879 – 1931, pp. 183 – 198 (Harvard University Press, 1967).