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Polymodal Lattices and Polymodal Logic

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Abstract. A polymodal lattice is a distributive lattice carrying an n-place operator preserving top elements and certain finite meets. After exploring some of the basic properties of such structures, we investigate their freely generated instances and apply the results to the corresponding logical systems – polymodal logics – which constitute natural generalizations of the usual systems of modal logic familiar from the literature. We conclude by formulating an extension of Kripke semantics to classical polymodal logic and proving soundness and completeness theorems.

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0 Introduction

A *polymodal lattice* is a distributive lattice carrying, for some $n \geq 1$, an n-place operator (which we shall call a *polymodality*) preserving top elements and finite meets in a certain natural sense. The (dual of) the concept of polymodal lattice, which constitutes a natural generalization of the concept of *Boolean algebra with operators* first introduced by JÓNSSON and TARSKI in [4], has been studied (under the name *complex algebra*) by GOLDBLATT in [2], where a general representation theorem is proved. Our purpose in the present paper is to develop the theory of polymodal lattices in some rather different directions, namely, to investigate the relationship between polymodalities and a generalization ("filtroid") of the notion of filter to cartesian products of lattices, to analyze free polymodal lattices, and, finally, to apply the results obtained to the logical systems correlated with them, which we shall term *polymodal logics*, and which constitute natural generalizations of the usual systems of (propositional) modal logic familiar from the literature.

In Section 1 we introduce the concepts of filtroid and polymodal lattice and explore some of their basic properties and relationships. In Section 2 the connection between polymodalities and filtroids is exploited to establish properties of free polymodal lattices. In Section 3 the concept of (classical or intuitionistic propositional) polymodal logic is introduced and the results of Section 2 applied to show that these systems possess certain features (notably, versions of the socalled *disjunction property*) familiar from classical modal logic. To conclude, we formulate an extension of

Kripke semantics to classical polymodal logic and prove soundness and completeness theorems, the proof of the latter making essential use of the concept of filtroid introduced in Section 1 and, as a result, turning out to be more than just a routine variation on the usual proof of completeness for classical modal logic.

Notation. Given sets $X_1, \ldots, X_n$, we write $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_1, \ldots, y_n)$, etc. for members of the Cartesian product $X_1 \times \cdots \times X_n$. Given subsets $A_1, \ldots, A_n$ of $X_1, \ldots, X_n$, respectively, we write $A_1 + \cdots + A_n$ for $\{\boldsymbol{x} \in X_1 \times \cdots \times X_n : x_1 \in A_1 \text{ or } \ldots \text{ or } x_n \in A_n\}$. We write $\boldsymbol{x} \approx \boldsymbol{y}$ for $\exists j\, (\forall i \neq j).\, x_i = y_i$, and $\boldsymbol{x}|\boldsymbol{y}$ for $\exists j.\, x_j = y_j$. We write $P(X)$ for the power set of X.

1 Filtroids and polymodal lattices

By a *lattice* $L = (L, \wedge, \vee, \leq, 0, 1)$ we shall always mean a *distributive lattice* which is *bounded*, i.e. has both a bottom element 0 and a top element 1. We shall always assume that homomorphisms between lattices preserve 0 and 1. If L is a Heyting algebra (in particular, a Boolean algebra), we write $\rightarrow$, * for the pseudocomplementation operations in L: thus, for $x, y \in L$, $x \rightarrow y$ is the largest element $z \in L$ for which $x \wedge z \leq y$ and x^* is $x \rightarrow 0$. By a *filter* in a lattice L we mean a subset F of L such that

$$x, y \in F \Rightarrow x \wedge y \in F, \qquad x \in F, x \leq y \Rightarrow y \in F.$$

A filter F in L is *proper* if $F \neq L$, or equivalently if $0 \notin F$. Dually, an *ideal* in L is a subset I of L such that

$$x, y \in I \Rightarrow x \vee y \in I, \qquad x \in I, x \geq y \Rightarrow y \in I.$$

I is *proper* if $1 \notin I$. For each $X \subseteq L$ the set $\{y : (\exists x_1 \ldots x_n \in X).\, x_1 \wedge \cdots \wedge x_n \leq y\}$ (resp. $\{y : (\exists x_1 \ldots x_n \in X).\, y \leq x_1 \vee \cdots \vee x_n\}$) is the least filter (resp. ideal) containing X; it is called the filter (ideal) *generated* by X; it is proper iff for each finite subset $\{x_1, \ldots, x_n\}$ of X, $x_1 \wedge \cdots \wedge x_n \neq 0$ (resp. $x_1 \vee \cdots \vee x_n \neq 1$). A filter is *principal* if it is generated by a singleton $\{a\}$; in that case it is of the form $\{x : a \leq x\}$. A filter F is prime if it is proper and if $x \vee y \in F$ implies $x \in F$ or $y \in F$ for any x, y. If L is distributive and bounded, then every proper filter is the intersection of the family of prime filters that contain it. As a consequence, two elements are the same iff they are contained in the same prime filters. (For proofs of all these facts, see e.g. [1].)

We now extend some of these concepts to products of lattices. Given an n-tuple of lattices $\boldsymbol{\lambda} = (L_1, \ldots, L_n)$, the Cartesian product $\Pi\boldsymbol{\lambda} = L_1 \times \ldots \times L_n$ is a lattice in which $\wedge, \vee, \leq$ are defined "coordinatewise" in the obvious way. We write $\mathbf{1}$, $\mathbf{0}$ for the top and bottom elements of L^n, i.e., $\mathbf{1} = (1, \ldots, 1)$ and $\mathbf{0} = (0, \ldots, 0)$. By a *$\boldsymbol{\lambda}$-filtroid* we shall mean a subset F of $\Pi\boldsymbol{\lambda}$ satisfying, for all $\boldsymbol{x}$, $\boldsymbol{y} \in \Pi\boldsymbol{\lambda}$,

1) if $\boldsymbol{x} \in F$ and $\boldsymbol{x} \leq \boldsymbol{y}$, then $\boldsymbol{y} \in F$,
2) if $\boldsymbol{x}$, $\boldsymbol{y} \in F$ and $\boldsymbol{x} \approx \boldsymbol{y}$, then $\boldsymbol{x} \wedge \boldsymbol{y} \in F$,
3) $\{1\} + \cdots + \{1\} \subseteq F$.

A $\boldsymbol{\lambda}$-filtroid F is said to be *proper* if it does not contain $\mathbf{0}$ (i.e., does not coincide with $\Pi\boldsymbol{\lambda}$) and *prime* if it is proper and $\boldsymbol{x} \vee \boldsymbol{y} \in F$ implies $\boldsymbol{x} \in F$ or $\boldsymbol{y} \in F$ for any $\boldsymbol{x}$, $\boldsymbol{y}$. If $L_1 = \ldots = L_n = L$, a $\boldsymbol{\lambda}$-filtroid is called an n-filtroid over L. Clearly 1-filtroids coincide with filters. If $\boldsymbol{a} \in \Pi\boldsymbol{\lambda}$, the set $(\boldsymbol{a}) = \{\boldsymbol{x} \in \Pi\boldsymbol{\lambda} : \boldsymbol{a} \leq \boldsymbol{x} \text{ or } \boldsymbol{x}|\mathbf{1}\}$ is a filtroid called the *principal filtroid* generated by $\boldsymbol{a}$. Given filters $F_1, \ldots, F_n$ in $L_1, \ldots, L_n$, respectively, it is evident that $F_1 + \cdots + F_n$ is a $\boldsymbol{\lambda}$-filtroid; filtroids of this form will be called *basic*.

Proposition 1.1. *Let F be a $\boldsymbol{\lambda}$-filtroid. Then we have:*

(i) *F is basic iff, for any $\boldsymbol{x} \in F$, there is i such that $(0, \ldots, x_i, \ldots, 0) \in F$.*

(ii) *F is prime iff there exist prime filters $P_1, \ldots, P_n$ in $L_1, \ldots, L_n$, respectively, for which $F = P_1 + \cdots + P_n$.*

Proof. The proof of (i) is a simple exercise and is left to the reader. As for (ii), sufficiency is easy. For necessity, suppose that F is prime and define, for each i, $1 \leq i \leq n$, $P_i = \{x \in L_i : (0, \ldots, x, \ldots, 0) \in F\}$. It is readily checked that P_i is a prime filter in L_i. We claim that $P_1 + \cdots + P_n = F$. If $\boldsymbol{x} \in P_1 + \cdots + P_n$, then, for some i, $x_i \in P_i$, i.e., $(0, \ldots, x_i, \ldots, 0) \in F$. Since $(0, \ldots, x_i, \ldots, 0) \leq \boldsymbol{x}$ and F is a filtroid, it follows that $\boldsymbol{x} \in F$. Accordingly, $P_1 + \cdots + P_n \subseteq F$. For the reverse inclusion, observe that since, for any $\boldsymbol{x}$, we have $\boldsymbol{x} = (x_1, 0, \ldots, 0) \vee \cdots \vee (0, \ldots, 0, x_n)$, it follows from the primeness of F that if $\boldsymbol{x} \in F$, then $(0, \ldots, x_i, \ldots, 0) \in F$ for some i, i.e., $x_i \in P_i$. Thus $\boldsymbol{x} \in P_1 + \cdots + P_n$ and the reverse inclusion follows. □

Our next result generalizes a well-known property of filters.

Theorem 1.2. *Each proper $\boldsymbol{\lambda}$-filtroid is the intersection of the family of prime $\boldsymbol{\lambda}$-filtroids that contain it.*

Proof. We may assume $n > 1$. Let F be a proper $\boldsymbol{\lambda}$-filtroid and suppose $\boldsymbol{a} = (a_1, \ldots, a_n) \notin F$. We define sequences $F_1, \ldots, F_n$ of filters and $P_1, \ldots, P_n$ of prime filters in $L_1, \ldots, L_n$, respectively, by recursion as follows.

First, we put $F_1 = \{x \in L_1 : (x, a_2, \ldots, a_n) \in F\}$. Clearly F_1 is a filter in L_1 and $a_1 \notin F_1$. Let P_1 be a prime filter in L_1 containing F_1 but not a_1.

Now assume for $k \geq 1$ the inductive hypothesis that $F_1, \ldots, F_k$ and $P_1, \ldots, P_k$ have been defined, where $P_1, \ldots, P_k$ are prime, satisfy $F_i \subseteq P_i$, $a_i \notin P_i$ for $i = 1, \ldots, k$ and

$$(1) \quad F_k = \{x \in L_k : (\exists x_1 \notin P_1) \ldots (\exists x_{k-1} \notin P_{k-1}).\, (x_1, \ldots, x_{k-1}, x, a_{k+1}, \ldots, a_n) \in F\}.$$

Now define

$$F_{k+1} = \{x \in L_{k+1} : (\exists x_1 \notin P_1) \ldots (\exists x_k \notin P_k).\, (x_1, \ldots, x_k, x, a_{k+2}, \ldots, a_n) \in F\}.$$

We claim that $F_{k+1} = G$ is a filter in L_{k+1}. For, to begin with, G clearly contains 1 and is accordingly nonempty. Next, it is evident that if $x \in G$ and $x \leq y$, then $y \in G$. Finally, if $x, y \in G$, then for some x_i, y_i not in P_i $(i = 1, \ldots, k)$, the n-tuples $(x_1, \ldots, x_k, x, a_{k+2}, \ldots, a_n)$ and $(y_1, \ldots, y_k, y, a_{k+2}, \ldots, a_n)$ are both in F. Then, by the first filtroid condition, $(x_1 \vee y_1, \ldots, x_k \vee y_k, x, a_{k+2}, \ldots, a_n)$ and $(x_1 \vee y_1, \ldots, x_k \vee y_k, y, a_{k+2}, \ldots, a_n)$ are also both in F, so that, by the second filtroid condition, $(x_1 \vee y_1, \ldots, x_k \vee y_k, x \wedge y, a_{k+2}, \ldots, a_n) \in F$. Since $P_1, \ldots, P_k$ are prime, $x_i \vee y_i \notin P_i$ $(i = 1, \ldots, k)$. It follows that $x \wedge y \in G$ and so F_{k+1} is a filter as claimed.

Since $F_k \subseteq P_k$, we deduce from (1) that

$$\forall x[(\exists x_1 \notin P_1) \ldots (\exists x_{k-1} \notin P_{k-1}).\, (x_1, \ldots, x_{k-1}, x, a_{k+1}, \ldots, a_n) \in F \Rightarrow x \in P_k],$$

i.e.,

$$\neg[(\exists x_1 \notin P_1)\dots(\exists x_{k-1} \notin P_{k-1})(\exists x \notin P_k).\,(x_1,\dots x_{k-1},x,a_{k+1},\dots,a_n)\in F].$$

But this means that $a_{k+1} \notin F_{k+1}$. To complete the inductive step we choose a prime filter P_{k+1} in L_{k+1} containing F_{k+1} but not a_{k+1}.

We thus obtain filters $F_1,\dots,F_n$ and prime filters $P_1,\dots,P_n$ such that $a_i \notin P_i$ $(i=1,\dots,n)$ and

(2) $P_n \supseteq F_n = \{x \in L_n : (\exists x_1 \notin P_1)\dots(\exists x_{n-1} \notin P_{n-1}).\,(x_1,\dots,x_{n-1},x)\} \in F.$

Clearly $\boldsymbol{a} \notin P_1 + \dots + P_n$. Also, if $\boldsymbol{x} = (x_1,\dots,x_n) \in F$ and $x_i \notin P_i$ $(i=1,\dots,n-1)$, then $x_n \in F_n$ and so $x_n \in P_n$ by (2). This means that $\boldsymbol{x} \in P_1 + \dots + P_n$. We have therefore shown that $F \subseteq P_1 + \dots + P_n$. Thus we have produced, for each $\boldsymbol{a} \notin F$, a prime filtroid containing F but not $\boldsymbol{a}$, proving the theorem. □

We now introduce the central concept of the paper. Let L be a distributive lattice with 0 and 1. For $n \geq 1$, an *n-modality* on L is a map $\Box : L^n \longrightarrow L$ satisfying:

1) $\Box\boldsymbol{x} = 1$ for any $\boldsymbol{x} \in L^n$ such that $\boldsymbol{x}|1$,

2) $\Box(\boldsymbol{x} \wedge \boldsymbol{y}) = \Box\boldsymbol{x} \wedge \Box\boldsymbol{y}$ for any $\boldsymbol{x}, \boldsymbol{y} \in L^n$ such that $\boldsymbol{x} \approx \boldsymbol{y}$.

A 1-modality will be called simply a *modality*. Clearly any modality $\Box$ on L satisfies $\Box 1 = 1$ and $\Box(x \wedge y) = \Box x \wedge \Box y$ for arbitrary $x, y \in L$. A $\Box_n$*-lattice* (or $\Box_n$*-Boolean algebra* or $\Box_n$*-Heyting algebra*) is a pair consisting of a lattice (or Boolean algebra or Heyting algebra) and an n-modality on it. We shall call an n-modality for arbitrary $n \geq 1$ a *polymodality*, and a $\Box_n$-lattice for arbitrary $n \geq 1$ a *polymodal lattice*.

A $\Box$*-morphism* between $\Box_n$-lattices $(L,\Box)$ and $(M,\triangle)$ is a lattice homomorphism $h : L \longrightarrow M$ such that $h(\Box\boldsymbol{x}) = \triangle h(\boldsymbol{x})$, where $h(x_1,\dots,x_n) = (h(x_1),\dots,h(x_n))$. If L and M are Heyting algebras, h is called a *Heyting* $\Box$*-morphism* if in addition it preserves the relative pseudocomplement operation $\rightarrow$.

The following facts are easily checked: Polymodalities preserve order; for any n-modality $\Box$ on L and any filter F in L, the set

$$\Box^{-1}F := \{\boldsymbol{x} : \Box\boldsymbol{x} \in F\}$$

is an n-filtroid over L; moreover for each i the map $\Box_i : L \longrightarrow L$ given by

$$\Box^i x := \Box(0,\dots,x,\dots,0) \quad (x \text{ in the } i^{\text{th}} \text{ place})$$

is a modality on L: the $\Box^i$ are called the modalities on L *induced* by $\Box$.

Examples.

1. The maps $\boldsymbol{x} \mapsto x_1 \vee \dots \vee x_n$ and $\boldsymbol{x} \mapsto x_i$ are n-modalities for any n and $i \leq n$.

2. Let $\varphi_1(x),\dots,\varphi_n(x)$ be formulas, and T a theory in a first order language L such that $T \vdash \sigma^{(\varphi_i)}$ for any $i = 1,\dots,n$ and any $\sigma \in T$. For sentences $\sigma_1,\dots,\sigma_n$ of L define

$$\Box(\sigma_1,\dots,\sigma_n) := \sigma_1^{(\varphi_1)} \vee \dots \vee \sigma_n^{(\varphi_n)}.$$

Then $\Box$ induces in the obvious way an n-modality on the Lindenbaum-Tarski algebra of T.

3. Let $T_1, \ldots, T_n, T$ be theories in a first-order language L capable of encoding its own syntax such that (i) $T \subseteq T_1 \cap \cdots \cap T_n$ and (ii) for each i there is in L a T_i-provability predicate Pr_i for which $T \vdash \mathrm{Pr}_i(\langle\sigma\rangle)$ whenever $T_i \vdash \sigma$, where $\langle\sigma\rangle$ is the code for σ. Define, for sentences $\sigma_1, \ldots, \sigma_n$ of L,

$$\Box(\sigma_1, \ldots, \sigma_n) := \mathrm{Pr}_1(\langle\sigma_1\rangle) \vee \cdots \vee \mathrm{Pr}_n(\langle\sigma_n\rangle).$$

Then $\Box$ induces in the obvious way an n-modality on the Lindenbaum-Tarski algebra of T.

4. Let each of $C_1, \ldots, C_n$ be a ZF-definable class of models of Zermelo-Fraenkel set theory ZF. Then the operator $\Box$ defined on sentences $\sigma_1, \ldots, \sigma_n$ of the language of ZF by

$$\Box(\sigma_1, \ldots, \sigma_n) := (\forall M \in C_1).\, M \models \sigma_1 \vee \cdots \vee (\forall M \in C_n).\, M \models \sigma_n$$

induces an n-modality on the Lindenbaum-Tarski algebra of sentences of ZF in the evident way.

5. If $\triangle_1, \ldots, \triangle_n$ are (1-)modalities on a fixed lattice L, then $\Box : L^n \longrightarrow L$ defined by

$$\Box \boldsymbol{x} := \triangle_1 x_1 \vee \cdots \vee \triangle_n x_n$$

is an n-modality on L. A polymodality which can be represented in this way will be called *simple*; if the modalities $\triangle_1, \ldots, \triangle_n$ in the representation are all identical, the polymodality will be called *rudimentary*.

Rudimentary and simple polymodalities may be characterized as follows.

Proposition 1.3. *Let $\Box$ be an n-modality on a lattice L. Then the following conditions are equivalent:*

(i) *$\Box$ is simple;*

(ii) *for all $\boldsymbol{x}$, $\Box\boldsymbol{x} = \Box(x_1, 0, \ldots, 0) \vee \cdots \vee \Box(0, \ldots, 0, x_n)$;*

(iii) *$\Box^{-1}P$ is basic for any prime filter P in L.*

Moreover, $\Box$ is rudimentary iff it satisfies any of (i) – (iii) *and is, in addition, symmetric, i. e. the value of $\Box\boldsymbol{x}$ is invariant under permutation of the x_i.*

Proof. The equivalence of (i) and (ii), and the final assertion, are left as simple exercises to the reader. If (ii) holds, and $\Box\boldsymbol{x}$ is a member of a prime filter P, then, for some i, $\Box(0, \ldots, x_i, \ldots, 0) \in P$, so that $(0, \ldots, x_i, \ldots, 0) \in \Box^{-1}P$. It now follows from 1.1(i) that $\Box^{-1}P$ is basic. Conversely, assume (iii). Writing $\nabla\boldsymbol{x}$ for the expression on the right-hand-side of the equation in (ii), clearly we have $\nabla\boldsymbol{x} \leq \Box\boldsymbol{x}$, and to prove the reverse inequality we need only show that any prime filter containing $\Box\boldsymbol{x}$ also contains $\nabla\boldsymbol{x}$. If $\Box\boldsymbol{x}$ is in a prime filter P, then $\boldsymbol{x} \in \Box^{-1}P$, so that, by (iii) and 1.1(i), there is i for which $(0, \ldots, x_i, \ldots, 0) \in \Box^{-1}P$. It follows that $\nabla\boldsymbol{x} \geq \Box(0, \ldots, x_i, \ldots, 0) \in P$. Therefore $\nabla\boldsymbol{x} \in P$; hence (ii). $\Box$

We shall call an n-filtroid F over L *$\Box$-prime* if there is a prime filter P in L for which $F = \Box^{-1}P$. Write $\Box\hat{\ } F$ for the filter in L generated by the set $\{\Box\boldsymbol{x} : \boldsymbol{x} \in F\}$. We say that F is *$\Box$-disjunctive* if, for any $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in L^n$,

$$\Box\boldsymbol{x}_1 \vee \cdots \vee \Box\boldsymbol{x}_k \in \Box\hat{\ } F \Rightarrow \boldsymbol{x}_i \in F \text{ for some } i.$$

If $\{1\} + \cdots + \{1\}$ is $\Box$-disjunctive, we shall say that $(L, \Box)$ is *disjunctive*. We note the following

Proposition 1.4. *For any proper n-filtroid F over L, the following are equivalent:*

(i) *F is $\Box$-prime;*

(ii) *F is $\Box$-disjunctive.*

Proof.

(i) $\Rightarrow$ (ii). Suppose that $F = \Box^{-1}P$ for some prime filter P and assume that $a = \Box x_1 \vee \cdots \vee \Box x_k \in \Box\hat{\ }F$. Now $\Box\hat{\ }F = \Box\hat{\ }(\Box^{-1}P) \subseteq P$, so $a \in P$, whence $\Box x_i \in P$ for some i, so that $x_i \in F$.

(ii) $\Rightarrow$ (i). Suppose that F is $\Box$-disjunctive. Then the set $\{\Box x : x \notin F\}$ generates a proper ideal I disjoint from $\Box\hat{\ }F$. By a well-known result (see, e. g., [1, 9.13]), there is a prime filter P in L containing $\Box\hat{\ }F$ and disjoint from I. Then $\Box^{-1}P = F$, since if $x \in \Box^{-1}P$, then $\Box x \in P$, so that $\Box x \notin I$, whence $x \in F$, while if $x \in F$, then $\Box x \in \Box\hat{\ }F \subseteq P$, whence $x \in \Box^{-1}P$. □

In this connection we also have

Proposition 1.5. *For any polymodal lattice $(L, \Box)$ the following are equivalent:*

(i) *Every $\Box$-prime filtroid over L is prime;*

(ii) *$\Box(x \vee y) = \Box x \vee \Box y$ for all x, y in L.*

Proof.

(i) $\Rightarrow$ (ii). Assume (i) and let P be a prime filter in L. Then $\Box^{-1}P$ is prime and we have

$$\begin{aligned} \Box(x \vee y) \in P \quad & \text{iff} \quad x \vee y \in \Box^{-1}P \\ & \text{iff} \quad x \in \Box^{-1}P \text{ or } y \in \Box^{-1}P \\ & \text{iff} \quad \Box x \in P \text{ or } \Box y \in P \\ & \text{iff} \quad \Box x \vee \Box y \in P. \end{aligned}$$

Therefore $\Box(x \vee y)$ and $\Box x \vee \Box y$ are contained in the same prime filters; they are, accordingly, equal.

(ii) $\Rightarrow$ (i). Assuming (ii), if P is prime and $x \vee y \in \Box^{-1}P$, then $\Box x \vee \Box y = \Box(x \vee y) \in P$ so that $\Box x \in P$ or $\Box y \in P$, i. e. $x \in \Box^{-1}P$ or $y \in \Box^{-1}P$. Accordingly $\Box^{-1}P$ is prime. □

A $\Box_n$-lattice $(L, \Box)$ will be said to be *well-primed* (resp. *properly primed, principally primed, weakly primed*) if every n-filtroid over L (resp. every proper filtroid, every proper principal filtroid, $\{1\} + \cdots + \{1\}$) is $\Box$-prime.

Proposition 1.6. *Let $(L, \Box)$ be a $\Box_n$-lattice.*

(i) *$(L, \Box)$ is weakly primed iff it is disjunctive.*

(ii) *$(L, \Box)$ is principally primed iff, for any $a, x_1, \ldots, x_k \in L^n$,*

$$\Box a \leq \Box x_1 \vee \cdots \vee \Box x_k \Rightarrow a \leq x_i \text{ or } x_i | 1 \text{ for some } i.$$

(If $n = 1$, the phrase "or $x_i|1$" is redundant.)

(iii) *If $n = 1$, $(L, \Box)$ is principally primed iff it is properly primed.*

(iv) *If $(L, \Box)$ is principally primed, then each induced modality $\Box^i$ on L is injective.*

(v) *If* $(L,\Box)$ *is well-primed, then* $\Box\boldsymbol{x}$ *is never* 0 *and the map* $n \mapsto (\Box^1)^n 0$ *(where* $(\Box^1)^n$ *is the* n^{th} *iterate of* $\Box^1$ *) is an injection of the set of natural numbers into* L, *so the latter is infinite.*

Proof.

(i) and (ii) are straightforward consequences of 1.4.

(iii) Suppose $n = 1$; it suffices to show that principally primed implies properly primed. So suppose $(L,\Box)$ principally primed, and let F be a proper filter in L for which $x = \Box x_1 \vee \cdots \vee \Box x_k \in \Box^{\hat{}} F$. Then there are $a_1, \ldots, a_m \in F$ such that $\Box(a_1 \wedge \cdots \wedge a_m) = \Box a_1 \wedge \cdots \wedge \Box a_m \leq x$. Then $0 \neq a = a_1 \wedge \cdots \wedge a_m \in F$, and since $(L,\Box)$ has been assumed principally primed, there is i such that $a \leq x_i$. But then $x_i \in F$, and we conclude that $(L,\Box)$ is properly primed.

(iv) If $(L,\Box)$ is principally primed, it follows from (ii) that $\Box^1 x \leq \Box^1 y$ implies $x \leq y$. Injectivity is an immediate consequence.

(v) If $(L,\Box)$ is well-primed, there is a prime filter P such that $L^n = \Box^{-1}P$. In particular $\mathbf{0} \in \Box^{-1}P$ so that $\Box\mathbf{0} \in P$, whence $\Box\mathbf{0} \neq 0$. The first claim in (v) is an immediate consequence. By (iv), $\Box^1$ is injective (and order preserving), so $0 < \Box^1 0 < (\Box^1)^2 0 < \cdots$. □

We now temporarily confine attention to Boolean algebras. Let $B_1, \ldots, B_n, B$ be Boolean algebras. A map $h : B_1 \times \cdots \times B_n \longrightarrow B$ is called a *hemimorphism* if $h(\boldsymbol{x}) = 1$ for any $\boldsymbol{x}|1$ and $h(\boldsymbol{x} \wedge \boldsymbol{y}) = h(\boldsymbol{x}) \wedge h(\boldsymbol{y})$ for any $\boldsymbol{x} \approx \boldsymbol{y}$. It is easily shown that the inverse image of a filter under a hemimorphism is a filtroid.

We are going to construct, for any Boolean algebras $B_1, \ldots, B_n$, a Boolean algebra $B_1\Box\cdots\Box B_n$ such that hemimorphisms with domain $B_1 \times \cdots \times B_n$ correspond bijectively to (Boolean) homomorphisms with domain $B_1\Box\cdots\Box B_n$. In particular, for any Boolean algebra B, polymodalities on B correspond to homomorphisms $B\Box\cdots\Box B$, so that the study of polymodalities on Boolean algebras reduces in principle to the study of special kinds of homomorphisms.

Let $\text{Filt}(B_1, \ldots, B_n) = \text{Filt}$ be the set of all $(B_1, \ldots B_n)$-filtroids; for each $\boldsymbol{x}$ in $B_1 \times \cdots \times B_n$ define $\mathcal{F}_{\boldsymbol{x}} = \{F \in \text{Filt} : \boldsymbol{x} \in F\}$. Let $B_1\Box\cdots\Box B_n$ be the Boolean subalgebra of $P(\text{Filt})$ generated by the $\mathcal{F}_{\boldsymbol{x}}$. Let $i : B_1 \times \cdots \times B_n \longrightarrow B_1\Box\cdots\Box B_n$ be the map $\boldsymbol{x} \mapsto \mathcal{F}_{\boldsymbol{x}}$. Clearly i is a hemimorphism.

Theorem 1.7. *For any Boolean algebra* $B_1, \ldots, B_n, B$ *and each hemimorphism* $f : B_1 \times \cdots \times B_n \longrightarrow B$ *there is a unique homomorphism* $g : B_1\Box\cdots\Box B_n \longrightarrow B$ *such that* $f = g \circ i$.

Proof. Since the image K of $B_1 \times \cdots \times B_n$ under i generates $B_1\Box\cdots\Box B_n$, if the g corresponding to a given f exists, it must be unique. To establish its existence we need to show that the map $k : K \longrightarrow B$ defined by $k(i(x)) = f(x)$ can be extended to a homomorphism $B_1\Box\cdots\Box B_n \to B$. By [8, 12.2], this will be the case provided that, for any $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_m \in B_1 \times \cdots \times B_n$,

$$(3) \qquad \begin{aligned} & i(\boldsymbol{x}_1) \cap \cdots \cap i(\boldsymbol{x}_k) \cap i(\boldsymbol{y}_1)^* \cap \cdots \cap i(\boldsymbol{y}_m)^* = \emptyset \\ & \Rightarrow f(\boldsymbol{x}_1) \wedge \cdots \wedge f(\boldsymbol{x}_k) \wedge f(\boldsymbol{y}_1)^* \wedge \cdots \wedge f(\boldsymbol{y}_m)^* = 0. \end{aligned}$$

Now (3) is equivalent to

$$(4) \quad f(\boldsymbol{x}_1)\wedge\cdots\wedge f(\boldsymbol{x}_k)\not\leq f(\boldsymbol{y}_1)\vee\cdots\vee f(\boldsymbol{y}_m) \\ \Rightarrow (\exists F\in \mathrm{Filt}).\,\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k\}\subseteq F\,\&\,\{\boldsymbol{y}_1,\ldots,\boldsymbol{y}_m\}\cap F=\emptyset.$$

Let G be the filter in B generated by the set $\{f(\boldsymbol{x}_1)\wedge\cdots\wedge f(\boldsymbol{x}_k)\}$. Then $F = f^{-1}[G]\in$ Filt. Clearly $\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k\}\subseteq F$. If $f(\boldsymbol{x}_1)\wedge\cdots\wedge f(\boldsymbol{x}_k)\not\leq f(\boldsymbol{y}_1)\vee\cdots\vee f(\boldsymbol{y}_m)$, then $f(\boldsymbol{x}_1)\wedge\cdots\wedge f(\boldsymbol{x}_k)\not\leq f(\boldsymbol{y}_i)$ so that $\boldsymbol{y}_i\notin F$ for all $i=1,\ldots,m$. This proves (4) and the theorem. □

We conclude this section with a discussion of some of the properties of the set $\mathrm{Mod}_n(L)$ of all n-modalities on a bounded distributive lattice L. We define a partial ordering on $\mathrm{Mod}_n(L)$ by $\Box\leq\triangle$ iff $\Box\boldsymbol{x}\leq\triangle\boldsymbol{x}$ for all $\boldsymbol{x}\in L^n$ This turns $\mathrm{Mod}_n(L)$ into a bounded lower semilattice with top (bottom) element the constant map on L^n with value 1 (0), and in which the meet $\Box\wedge\triangle$ is given by $(\Box\wedge\triangle)\boldsymbol{x}=\Box\boldsymbol{x}\wedge\triangle\boldsymbol{x}$. When L is a complete Heyting algebra we can assert much more.

Proposition 1.8. *If L is a complete Heyting algebra, so is* $\mathrm{Mod}_n(L)$.

Proof. Clearly, if L is complete, $\mathrm{Mod}_n(L)$ is a complete lattice in which the meet operation $\bigwedge$ is given by

$$\textstyle\bigwedge_{i\in I}(\Box_i)\boldsymbol{x}=\bigwedge_{i\in I}\Box_i\boldsymbol{x}.$$

We need to show that, when L is in addition a Heyting algebra, then the relative pseudocomplement $\Box\to\triangle$ exists in $\mathrm{Mod}_n(L)$ for any $\Box,\triangle\in\mathrm{Mod}_n(L)$. To this end, define $\Gamma: L^n\longrightarrow L$ by

$$\Gamma\boldsymbol{x} := \bigwedge\{\Box\boldsymbol{z}\to\triangle(\boldsymbol{x}\vee\boldsymbol{z}):\boldsymbol{z}\in L^n\}\quad\text{for } \boldsymbol{x}\in L^n.$$

We claim that Γ is $\Box\to\triangle$. First, Γ is in $\mathrm{Mod}_n(L)$ since clearly $\Gamma\boldsymbol{x}=1$ whenever $\boldsymbol{x}|1$ and if $\boldsymbol{x}\approx\boldsymbol{y}$, then, noting that $\boldsymbol{x}\vee\boldsymbol{z}\approx\boldsymbol{y}\vee\boldsymbol{z}$ for any $\boldsymbol{z}\in L^n$, we have

$$\begin{aligned}\Gamma(\boldsymbol{x}\wedge\boldsymbol{y}) &= \bigwedge\{\Box\boldsymbol{z}\to\triangle((\boldsymbol{x}\wedge\boldsymbol{y})\vee\boldsymbol{z}):\boldsymbol{z}\in L^n\}\\ &= \bigwedge\{\Box\boldsymbol{z}\to\triangle((\boldsymbol{x}\vee\boldsymbol{z})\wedge(\boldsymbol{y}\vee\boldsymbol{z})):\boldsymbol{z}\in L^n\}\\ &= \bigwedge\{\Box\boldsymbol{z}\to(\triangle((\boldsymbol{x}\vee\boldsymbol{z})\wedge\triangle(\boldsymbol{y}\vee\boldsymbol{z})):\boldsymbol{z}\in L^n\}\\ &= \bigwedge\{(\Box\boldsymbol{z}\to\triangle(\boldsymbol{x}\vee\boldsymbol{z}))\wedge(\Box\boldsymbol{z}\to\triangle(\boldsymbol{y}\vee\boldsymbol{z})):\boldsymbol{z}\in L^n\}\\ &= \Gamma\boldsymbol{x}\wedge\Gamma\boldsymbol{y}.\end{aligned}$$

Finally, suppose $\Sigma\in\mathrm{Mod}_n(L)$ satisfies $\Sigma\leq\Gamma$. Then for any $\boldsymbol{x}\in L^n$,

$$\Sigma\boldsymbol{x}\wedge\Box\boldsymbol{x}\leq\Gamma\boldsymbol{x}\wedge\Box\boldsymbol{x}\leq(\Box\boldsymbol{x}\to\triangle\boldsymbol{x})\wedge\Box\boldsymbol{x}\leq\triangle\boldsymbol{x},$$

whence $\Sigma\wedge\Box\leq\triangle$. Conversely, if $\Sigma\wedge\Box\leq\triangle$, then for any $\boldsymbol{x},\boldsymbol{z}\in L^n$, we have

$$\Sigma\boldsymbol{x}\wedge\Box\boldsymbol{z}\leq\Sigma(\boldsymbol{x}\vee\boldsymbol{z})\wedge\Box(\boldsymbol{x}\vee\boldsymbol{z})\leq\triangle(\boldsymbol{x}\vee\boldsymbol{z}).$$

Therefore $\Sigma\boldsymbol{x}\leq\Box\boldsymbol{z}\to\triangle(\boldsymbol{x}\vee\boldsymbol{z})$, whence

$$\Sigma\boldsymbol{x}\leq\bigwedge\{\Box\boldsymbol{z}\to\triangle(\boldsymbol{x}\vee\boldsymbol{z}):\boldsymbol{z}\in L^n\},$$

so that $\Sigma\leq\Gamma$. □

2 Free polymodal lattices and algebras

A $\Box_n$-lattice ($\Box_n$-Boolean algebra) $(L, \Box)$ is said to be *freely generated* by a set I (or *free on* I) if there is an (injective) map $p : I \longrightarrow L$ (called the *canonical injection*) such that, for any $\Box_n$-lattice (resp. $\Box_n$-Boolean algebra) $(M, \triangle)$ and any map $f : I \longrightarrow M$ there is a unique $\Box$-morphism $h : (L, \Box) \longrightarrow (M, \triangle)$ such that $f = h \circ p$. If in this definition we replace the terms "$\Box_n$-lattice" and "$\Box$-morphism" by the terms "$\Box_n$-Heyting algebra" and "Heyting $\Box$-morphism", respectively, we obtain the notion of $\Box_n$-Heyting algebra freely generated by I. Standard arguments of universal algebra (see, e. g. [3, ch. 4]) show that, for any I, there exists a (unique up to isomorphism) $\Box_n$-lattice, $\Box_n$-Boolean algebra and $\Box_n$-Heyting algebra freely generated by I.

Let C be a collection of properties of $\Box_n$-lattices or algebras expressible in terms of universally quantified equational conditions (e. g. $\forall \boldsymbol{x}. \Box \boldsymbol{x} \leq x_1 \vee \cdots \vee x_n$, equivalent to $\forall \boldsymbol{x}. (x_1 \vee \cdots \vee x_n) \wedge \Box \boldsymbol{x} = \Box \boldsymbol{x}$). A $\Box_n$-lattice or algebra possessing all the properties in C will be called a C-$\Box_n$-lattice or algebra. Standard arguments again establish the existence, for any set I, of the C-$\Box_n$-lattice, the C-$\Box_n$-Heyting algebra and the C-$\Box_n$-Boolean algebra free on I.

A list of properties selected from the conditions

$$\Box 0 = 0, \quad \Box \boldsymbol{x} \leq x_1 \vee \cdots \vee x_n, \quad \Box \boldsymbol{x} \leq \Box(\Box \boldsymbol{x}, \ldots, \Box \boldsymbol{x})$$

will be called a *standard list*. These are the natural extensions to polymodalities of the familiar conditions

$$\Box 0 = 0, \qquad \Box x \leq x, \qquad \Box x \leq \Box\Box x$$

sometimes imposed on modalities.

We now make the following definitions. Write 2 for the lattice (resp. Boolean algebra) $\{0, 1\}$. For $\boldsymbol{x} = (x_1, \ldots, x_n) \in L^n$, $\boldsymbol{i} = (i_1, \ldots, i_n) \in 2^n$, write $\boldsymbol{x} \otimes \boldsymbol{i}$ for $((x_1, i_1), \ldots, (x_n, i_n))$. Given a $\Box_n$-lattice $(L, \Box)$ and an n-filtroid F over L, define the maps $\Box_F$, $\Box_* : (L \times 2)^n \longrightarrow L \times 2$ by

$$\Box_F(\boldsymbol{x} \otimes \boldsymbol{i}) = \begin{cases} (\Box \boldsymbol{x}, 1) & \text{if } \boldsymbol{x} \in F, \\ (\Box \boldsymbol{x}, 0) & \text{otherwise;} \end{cases}$$

$$\Box_*(\boldsymbol{x} \otimes \boldsymbol{i}) = \begin{cases} (1, 1) & \text{if } (x_j, i_j) = (1, 1) \text{ for some } j, \\ (\Box \boldsymbol{x}, 0) & \text{otherwise.} \end{cases}$$

It is easily verified that $\Box_F$ and $\Box_*$ are $\Box_n$-modalities over the lattice $L \times 2$ (which is a Heyting algebra or Boolean algebra if L is).

The next result will be instrumental in demonstrating that free polymodal lattices possess some of the properties formulated in the previous section.

Theorem 2.1. *Let $(L, \Box)$ be a $\Box_n$-lattice.*

(i) *Let F be an n-filtroid over L. Then the following conditions are equivalent:*

(a) *F is $\Box$-prime.*

(b) *There is a $\Box$-morphism $h : (L, \Box) \longrightarrow (L \times 2, \Box_F)$ such that $\pi_1 \circ h$ is the identity on L. (Here π_1 is the projection of $L \times 2$ to L.)*

(ii) *The following are equivalent:*

(a) $(L, \Box)$ *is disjunctive.*

(b) *There is a* $\Box$*-morphism* $h : (L, \Box) \longrightarrow (L \times 2, \Box_*)$ *such that* $\pi_1 \circ h$ *is the identity on* L.

These equivalences also hold, mutatis mutandis, for $\Box_n$*-Boolean algebras.*

Proof.

(i)(a) $\Rightarrow$ (i)(b). Suppose $F = \Box^{-1} P$ for some prime filter P. Define $h : L \longrightarrow L \times 2$ by

$$h(x) = \begin{cases} (x, 1) & \text{if } x \in P, \\ (x, 0) & \text{if } x \notin P. \end{cases}$$

It is easily checked that h is a homomorphism, and clearly $\pi_1 \circ h$ is the identity on L. Moreover, h is a $\Box$-morphism since if $x \in F$, then $\Box x \in P$, so, for some $i \in 2^n$,

$$\Box_F h(x) = \Box_F(x \otimes i) = (\Box x, 1) = h(\Box x),$$

while if $x \notin F$, then $\Box x \notin P$, so, for some $i \in 2^n$,

$$\Box_F h(x) = \Box_F(x \otimes i) = (\Box x, 0) = h(\Box x).$$

(i)(b) $\Rightarrow$ (i)(a). Suppose h meets the conditions laid down in (b). Let P be the prime filter $\{x : \pi_2 h(x) = 1\}$ (where π_2 is the projection of $L \times 2$ to 2) in L. Then we have, for $x \in L^n$

$$\begin{aligned} \Box x \in P \quad & \text{iff} \quad \pi_2 h(\Box x) = 1 \\ & \text{iff} \quad \pi_2 \Box_F h(x) = 1 \\ & \text{iff} \quad \pi_2 \Box_F (x \otimes i) = 1 \text{ with } h(x) = x \otimes i \\ & \text{iff} \quad x \in F. \end{aligned}$$

So $F = \Box^{-1} P$ as required.

(ii) We know from 1.6 that (a) is equivalent to the assertion that $(L, \Box)$ is weakly primed. An argument similar to that in (i) (taking $F = \{1\} + \cdots + \{1\}$) demonstrates the equivalence of this latter condition with (b): we leave this to the reader. $\Box$

This result has the following consequences.

Corollary 2.2.

(i) *Any free* $\Box_n$*-lattice or* $\Box_n$*-Heyting algebra or* $\Box_n$*-Boolean algebra is well primed.*

(ii) *Write* P_0 *for the property of polymodalities* $\Box 0 = 0$. *Then any free* $\{P_0\}$-$\Box_n$*-lattice or* $\{P_0\}$-$\Box_n$*-Heyting algebra or* $\{P_0\}$-$\Box_n$*-Boolean algebra is properly primed.*

(iii) *If* C *is a standard list then any free* C-$\Box_n$*-lattice or* C-$\Box_n$*-Heyting algebra or* C-$\Box_n$*-Boolean algebra is weakly primed (and hence disjunctive).*

Proof. We prove just the lattice versions; the proofs for Heyting algebras and Boolean algebras are similar.

(i) Write $(L, \Box)$ for the free $\Box_n$-lattice on I and let $p : I \longrightarrow L$ be the canonical injection. Define $k : I \longrightarrow L \times 2$ by $k(i) := (p(i), 1)$ for $i \in I$. Then for any n-filtroid

F over L there is, by the freedom of $(L,\Box)$, a $\Box$-morphism $h:(L,\Box)\longrightarrow(L\times 2,\Box_F)$ such that $k=h\circ p$. Then $\pi_1\circ h\circ p=\pi_1\circ k=p$ so that $\pi_1\circ h$ is the identity on L by the uniqueness of factorizations through p. So condition (i)(b) of 2.1 is satisfied by any n-filtroid F and it follows that $(L,\Box)$ is well-primed.

(ii) It is easily verified that, if $(L,\Box)$ is any $\Box_n$-lattice satisfying P_0, then for any *proper* n-filtroid F over L, $(L\times 2,\Box_F)$ also satisfies P_0. Writing $(M,\Box)$ for the free $\{P_0\}$-$\Box_n$-lattice on I, the same argument as in (i) shows that, for any proper n-filtroid over M, there is a $\Box$-morphism $h:(M,\Box)\longrightarrow(M\times 2,\Box_F)$ such that $\pi_1\circ h$ is the identity on M. It now follows from 2.1(i) that $(M,\Box)$ is properly primed.

(iii) Let C be a standard list. If $(L,\Box)$ is any $\Box_n$-lattice having the properties in C, it is readily verified that $(L\times 2,\Box_*)$ also has the properties in C. Writing $(M,\Box)$ for the free C-$\Box_n$-lattice on I, the same argument as in (i) shows that there is a $\Box$-morphism $h:(M,\Box)\longrightarrow(M\times 2,\Box_*)$ such that $\pi_1\circ h$ is the identity on M. It follows now from 2.1(ii) that $(M,\Box)$ is weakly primed. $\Box$

Corollary 2.3. *Any free $\Box_n$-lattice (or $\Box_n$-Heyting algebra or $\Box_n$-Boolean algebra), and any $\{P_0\}$-$\Box_n$-lattice (or $\{P_0\}$-$\Box_n$-Heyting algebra or $\{P_0\}$-$\Box_n$-Boolean algebra) free on a nonempty set, is infinite.*

Proof. The first assertion follows from 2.2(i) and 1.6(v). To prove the second, we note that if $(L,\Box)$ is a $\Box_n$-lattice free on a nonempty set I and $\Box^1$ is injective, then L is infinite. This may be shown as follows. Writing $p:I\longrightarrow L$ for the canonical injection, we have $p(i)\notin\mathrm{range}(\Box^1)$ for any $i\in I$. To see this, consider the $\Box_n$-lattice $(L,\#)$ with $\#x=1$ for all $x\in L^n$. There must be a $\Box$-morphism $h:(L,\Box)\longrightarrow(L,\#)$ such that $h\circ p=p$. Thus if $p(i)=\Box^1 a$ for some $a\in L$, it would follow that

$$p(i)=h(p(i))=h(\Box^1 a)=\#h((a,0,\ldots,0))=1,$$

which is clearly impossible since $p(i)$ is a free generator. So if $\Box^1$ is injective, then since it carries L into the proper subset $L-p[I]$, it follows that L is infinite. This applies in particular when $(L,\Box)$ is any $\{P_0\}$-$\Box_n$-lattice free on a nonempty set, since by 2.2(ii) and 1.6(iv), $\Box^1$ is injective in such lattices. For $\{P_0\}$-$\Box_n$-Boolean algebras or $\{P_0\}$-$\Box_n$-Heyting algebras the argument is similar. $\Box$

We see from this result that the free $\Box^1$- and $\{P_0\}$-$\Box^1$-lattices and Boolean algebras on a single generator are all infinite. But if we write E for the list of conditions $\forall x.\Box x\le x$, $\forall x.\Box x\le\Box\Box x$, it is easy to see that the free E-$\Box^1$-lattice on 1 generator has 4 elements, while according to [6, Theorem 5.2] the corresponding free Boolean algebra is infinite. Whether all finitely generated free E-$\Box^1$-lattices are finite seems to be an open question.

3 Applications to logic

We now apply some of the results of previous sections to the logical systems – *polymodal (propositional) logics* – which are the natural generalizations of the usual modal logics to incorporate polymodal operators.

Let $\mathcal{L}$ be the language of propositional logic (which we will suppose contains a symbol $\bot$ for "the false") and, for $n\ge 1$ let $\mathcal{L}(n,\Box)$ be obtained from $\mathcal{L}$ by adding the new n-ary logical operator $\Box$ and extending the class of formulas by the clause

if $\varphi_1, \ldots, \varphi_n$ are formulas, then so is $\Box(\varphi_1, \ldots, \varphi_n)$.

We employ boldface letters $\boldsymbol{\varphi}, \boldsymbol{\psi}, \ldots$ for n-tuples of formulas of $\mathcal{L}(n, \Box)$.

If $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n)$, $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_n)$, we write

$$\boldsymbol{\varphi} \to \boldsymbol{\psi} \quad \text{for} \quad (\varphi_1 \to \psi_1, \ldots, \varphi_n \to \psi_n),$$
$$\boldsymbol{\varphi} \wedge \boldsymbol{\psi} \quad \text{for} \quad (\varphi_1 \wedge \psi_1, \ldots, \varphi_n \wedge \psi_n),$$

and $\bigvee \boldsymbol{\varphi}$ for $\varphi_1 \vee \cdots \vee \varphi_n$. For any formula φ we write (φ) for the n-tuple $(\varphi, \ldots, \varphi)$. Recall that $\boldsymbol{\varphi} \approx \boldsymbol{\psi}$ stands for $\exists i\, (\forall j \neq i).\, \varphi_j = \psi_j$.

Now let K_n, $\mathsf{K}_n^{\text{int}}$ be the logical systems in $\mathcal{L}(n, \Box)$ obtained by adding to the usual classical and intuitionistic systems, respectively, the *axiom schemes*

$$(\varphi_1 \leftrightarrow \psi_1) \wedge \cdots \wedge (\varphi_n \leftrightarrow \psi_n) \to (\Box(\varphi_1, \ldots, \varphi_n) \leftrightarrow \Box(\psi_1, \ldots, \psi_n))$$

for any $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n$;

$$\Box(\boldsymbol{\varphi} \wedge \boldsymbol{\psi}) \to (\Box\boldsymbol{\varphi} \wedge \Box\boldsymbol{\psi})$$

for any $\boldsymbol{\varphi}, \boldsymbol{\psi}$;

$$(\Box\boldsymbol{\varphi} \wedge \Box\boldsymbol{\psi}) \to \Box(\boldsymbol{\varphi} \wedge \boldsymbol{\psi})$$

for any $\boldsymbol{\varphi}, \boldsymbol{\psi}$ such that $\boldsymbol{\varphi} \approx \boldsymbol{\psi}$; and the *rule of inference*

$$\varphi \,/\, \boldsymbol{\alpha},$$

where φ is any formula and $\boldsymbol{\alpha}$ is any n-tuple of formulas in which φ appears. Since K_1 and $\mathsf{K}_1^{\text{int}}$ are just (equivalent to) the usual minimal systems of classical and intuitionistic modal logic, respectively, it seems appropriate to call K_n and $\mathsf{K}_n^{\text{int}}$ systems of *polymodal logic*. It also seems natural to think of the operator $\Box$ in such systems as "generalized disjunction": we shall see below that the semantics for K_n provides support for this idea.

Consider the formula or formula schemes

$$\text{(a) } \neg\Box(\bot), \quad \text{(b) } \Box\boldsymbol{\varphi} \to \bigvee\boldsymbol{\varphi}, \quad \text{(c) } \Box\boldsymbol{\varphi} \to (\Box\boldsymbol{\varphi}, \ldots, \Box\boldsymbol{\varphi}).$$

We shall introduce a slight variation on the notation of LEMMON [5] and write $\mathsf{K}_n\mathsf{D}$, $\mathsf{K}_n\mathsf{T}$, K_n4, $\mathsf{K}_n\mathsf{T}4$ for the polymodal systems obtained by adding the axioms (a), (b), (c), (b) & (c), respectively, to K_n and analogously for $\mathsf{K}_n^{\text{int}}$. If Σ is any one of these systems, then we can form the Lindenbaum-Tarski algebra $\mathrm{LT}(\Sigma)$ of Σ in the usual way: this is the lattice of equivalence classes of formulas of $\mathcal{L}(n, \Box)$ under the equivalence relation of provable equivalence from Σ. Write I for the set of proposition letters of $\mathcal{L}$. Then $\mathrm{LT}(\mathsf{K}_n)$ (resp. $\mathrm{LT}(\mathsf{K}_n^{\text{int}})$) is a $\Box_n$-Boolean (resp. $\Box_n$-Heyting) algebra isomorphic to the free $\Box_n$-Boolean (resp. $\Box_n$-Heyting) algebra on I. If Σ is any of the systems $\mathsf{K}_n\mathsf{D}$, $\mathsf{K}_n\mathsf{T}$, K_n4, $\mathsf{K}_n\mathsf{T}4$ (resp. $\mathsf{K}_n^{\text{int}}\mathsf{D}$, $\mathsf{K}_n^{\text{int}}\mathsf{T}$, $\mathsf{K}_n^{\text{int}}4$, $\mathsf{K}_n^{\text{int}}\mathsf{T}4$), then $\mathrm{LT}(\Sigma)$ is isomorphic to the free C-$\Box_n$-Boolean (resp. C-$\Box_n$-Heyting) algebra on I, where C is the standard list of conditions (see Section 2) corresponding to the axioms of the form (a), (b), (c) appearing in Σ. (For a proof of the corresponding claim for the modal system $\mathsf{S4}$ ($= \mathsf{K}_1\mathsf{T}4$ in our notation), see [7, XI, 9.6]: the proofs of the claims made here are essentially the same.) It follows that the results concerning free $\Box_n$-algebras obtained in the previous section may be interpreted in terms of polymodal logical systems.

To do this we need to make some further definitions. If $\varphi = (\varphi_1, \ldots, \varphi_n)$, we write $\vdash_\Sigma \varphi$ for $\vdash_\Sigma \varphi_1 \wedge \cdots \wedge \varphi_n$, $\vdash^*_\Sigma \varphi$ for $\exists i.\ \vdash_\Sigma \varphi_i$ and, if Φ is a set of n-tuples of formulas of $\mathcal{L}(n, \Box)$, $\Phi \vdash_\Sigma \psi$ for $\exists \varphi \in \Phi.\ \vdash_\Sigma \varphi \to \psi$. We shall call Φ an *(n-)scheme over* Σ if it satisfies

1.) if $\Phi \vdash_\Sigma \varphi$, then $\varphi \in \Phi$;
2.) if $\vdash_\Sigma \varphi$, then $\alpha \in \Phi$ for any n-tuple α of formulas in which φ appears;
3.) if $\Phi \vdash_\Sigma \varphi$, $\vdash_\Sigma \psi$, and $\varphi \approx \psi$, then $\Phi \vdash_\Sigma \varphi \wedge \psi$.

An n-scheme Φ is *consistent* if $(\bot) \notin \Phi$ and *prime* if it is consistent and, for any n-tuples of formulas φ, ψ, if $\varphi \vee \psi \in \Phi$, then $\varphi \in \Phi$ or $\psi \in \Phi$. Clearly 1-schemes over Σ are just theories in Σ, and n-schemes are the logical counterparts of n-filtroids in the corresponding free algebras. Consistent schemes correspond to proper filtroids, and prime schemes to prime filtroids. Noting these definitions and correspondences, we derive immediately from 1.6 and 2.2 the following theorem and corollary.

Theorem 3.1.

(i) *For any n-scheme Φ over K_n ($\mathsf{K}_n^{\mathrm{int}}$) there is a prime theory Π in K_n (resp. $\mathsf{K}_n^{\mathrm{int}}$) such that $\Phi = \{\varphi : \Box\varphi \in \Pi\}$.*

(ii) *For any consistent n-scheme Φ over $\mathsf{K}_n\mathsf{D}$ ($\mathsf{K}_n^{\mathrm{int}}\mathsf{D}$) there is a prime theory Π in $\mathsf{K}_n\mathsf{D}$ (resp. $\mathsf{K}_n^{\mathrm{int}}\mathsf{D}$) such that $\Phi = \{\varphi : \Box\varphi \in \Pi\}$.*

(iii) *If Σ is any of the systems $\mathsf{K}_n\mathsf{T}$, K_n4, $\mathsf{K}_n\mathsf{T}4$, $\mathsf{K}_n^{\mathrm{int}}\mathsf{T}$, $\mathsf{K}_n^{\mathrm{int}}4$, $\mathsf{K}_n^{\mathrm{int}}\mathsf{T}4$, then there is a prime theory Π such that, for all $\varphi = (\varphi_1, \ldots, \varphi_n)$, $\Box\varphi \in \Pi$ if and only if $\vdash_\Sigma \varphi_i$ for some i.* □

Corollary 3.2.

(i) *If Σ is any of the systems K_n, $\mathsf{K}_n\mathsf{D}$, $\mathsf{K}_n^{\mathrm{int}}$, $\mathsf{K}_n^{\mathrm{int}}\mathsf{D}$, then for any $\varphi, \varphi_1, \ldots, \varphi_k$,*

$$\vdash_\Sigma \Box\varphi \to \varphi_1 \vee \cdots \vee \varphi_k \text{ iff for some } i,\ \vdash^*_\Sigma \varphi_i \text{ or } \vdash_\Sigma \varphi \to \varphi_i.$$

(ii) *If Σ is any of the systems $\mathsf{K}_n\mathsf{T}$, K_n4, $\mathsf{K}_n\mathsf{T}4$, $\mathsf{K}_n^{\mathrm{int}}\mathsf{T}$, $\mathsf{K}_n^{\mathrm{int}}4$, $\mathsf{K}_n^{\mathrm{int}}\mathsf{T}4$, then Σ has the disjunction property, viz., for any $\varphi_1, \ldots, \varphi_k$,*

$$\vdash_\Sigma \varphi_1 \vee \cdots \vee \varphi_k \text{ iff } \vdash^*_\Sigma \varphi_i \text{ for some } i. \qquad \Box$$

When $n = 1$, the results stated in this corollary for classical modal systems are well known (see [5]) but for intuitionistic modal systems they appear to be new.

In conclusion, we extend the usual Kripke semantics for (propositional) modal logic to classical polymodal systems, and use some of our previous results to establish the adequacy of this extended semantics.

Let At be the set of propositional atoms of $\mathcal{L}$: thus At consists of an infinite set I of proposition letters together with the symbol $\bot$. For $n \geq 1$ we define an *n-frame* to be a triple $\mathcal{U} = (U, R, \Pi)$ with U a nonempty set, $R \subseteq U^{n+1}$ and $\Pi : U \longrightarrow P(\mathrm{At} - \{\bot\})$. We define, for formulas φ of $\mathcal{L}(n, \Box)$ and $u \in U$, the assertion $\mathcal{U} \vDash^u \varphi$, read "$\varphi$ is true at u in $\mathcal{U}$", recursively as follows.

(a) $\mathcal{U} \vDash^u \varphi$ iff $\varphi \in \Pi(u)$ for atomic φ;

(b) $\mathcal{U} \models^u \neg\varphi$ iff not $\mathcal{U} \models^u \varphi$;

(c) $\mathcal{U} \models^u \varphi \wedge \psi$ iff $\mathcal{U} \models^u \varphi$ and $\mathcal{U} \models^u \psi$;

(d) $\mathcal{U} \models^u \Box(\varphi_1, \ldots, \varphi_n)$ iff $(\forall v_1 \ldots v_n)[Ruv_1 \ldots v_n \Rightarrow \exists i. \mathcal{U} \models^{v_i} \varphi_i]$.

Note how in clause (d) $\Box$ is interpreted "disjunctively".

We write $\mathcal{U} \models \varphi$ for $(\forall u \in U)(\mathcal{U} \models^u \varphi)$, $\mathcal{U} \models^u \Gamma$ for $(\forall \varphi \in \Gamma)(\mathcal{U} \models^u \varphi)$, $\mathcal{U} \models \Gamma$ for $(\forall u \in U)(\mathcal{U} \models^u \Gamma)$ and $\Gamma \models_n \varphi$ for $\forall \mathcal{U}\, (\forall u \in U)(U \models^u \Gamma \Rightarrow \mathcal{U} \models^u \Gamma)$. Here and in the sequel Γ denotes an arbitrary set of formulas of $\mathcal{L}(n, \Box)$.

We now prove the

Theorem 3.3 (Soundness Theorem). *If $\Gamma \vdash_{\mathsf{K}_n} \varphi$, then $\Gamma \models_n \varphi$.*

Proof. By induction on the proof of φ from Γ in K_n. We need only check the modal axioms and rule of inference. The soundness of the axiom $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$ is immediate from (c) in the definition of $\models^u$. Suppose now that $\varphi \approx \psi$ and $\mathcal{U} \models^u \Box\varphi \wedge \Box\psi$. Then $\mathcal{U} \models^u \Box\varphi$ and $\mathcal{U} \models^u \psi$ so that, for any $v_1, \ldots, v_n$ such that $Ruv_1 \ldots v_n$ there exist i, j such that $1 \leq i, j \leq n$ and $\mathcal{U} \models^{v_i} \varphi_i$, $\mathcal{U} \models^{v_j} \psi_j$. If $i = j$, then $\mathcal{U} \models^{v_i} \varphi_i \wedge \psi_i$. If $i \neq j$, then, since $\varphi \approx \psi$, either $\varphi_i = \psi_i$ or $\varphi_j = \psi_j$, whence $\mathcal{U} \models^{v_i} \varphi_i \wedge \psi_i$ or $\mathcal{U} \models^{v_j} \varphi_j \wedge \psi_j$. It follows that $\mathcal{U} \models^u \Box(\varphi \wedge \psi)$, thus verifying the soundness of the axiom $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$ when $\varphi \approx \psi$. Finally, to verify the soundness of the modal rule of inference, we need to show that if $\Gamma \models_n \varphi$, then $\Gamma \models_n \Box\alpha$ for any α in which φ appears. Supposing φ is α_i, if $\Gamma \models_n \varphi$, then for any $\mathcal{U} \models \Gamma$ we have $(\forall u \in U). \mathcal{U} \models^u \alpha_i$, so a *fortiori* $(\forall u \in U)(\forall v_1 \ldots v_n)\,[Ruv_1 \ldots v_n \Rightarrow \mathcal{U} \models^{v_i} \alpha_i]$, whence $\mathcal{U} \models_n \Box\alpha$. $\Box$

In order to prove the corresponding completeness theorem, i.e. the converse of 3.3, we first note that by applying the correspondence between prime filtroids over $\mathrm{LT}(\mathsf{K}_n)$ and prime schemes over K_n to 1.1(ii) we obtain

(A) The prime n-schemes over K_n are precisely those schemes of the form $\Phi_1 + \cdots + \Phi_n$, where $\Phi_1, \ldots, \Phi_n$ are prime theories in K_n.

By applying this same correspondence to 1.2, we obtain

(B) Each consistent n-scheme over K_n is the intersection of the family of prime n-schemes that contain it.

For any theory Φ in K_n, it is easily verified that $\Phi^\Box = \{\varphi : \Box\varphi \in \Phi\}$ is an n-scheme over K_n. We use this in defining, for any consistent set of formulas Γ in K_n, the *canonical n-frame* $\mathcal{U}_\Gamma = (U_\Gamma, R_\Gamma, \Pi_\Gamma)$ as follows. First, U_Γ is the set of prime theories in K_n containing Γ. (U_Γ is of course nonempty by standard arguments: it follows, for example, from the fact that in $\mathrm{LT}(\mathsf{K}_n)$ every proper filter is contained in a prime filter.) We define $R_\Gamma \subseteq U_\Gamma^{n+1}$ by

$$(\Phi, \Psi_1, \ldots, \Psi_n) \in R_\Gamma \quad \text{iff} \quad \Phi^\Box \subseteq \Psi_1 + \cdots + \Psi_n.$$

Finally we define Π_Γ by $\Pi_\Gamma(\Phi) = \Phi \cap \mathrm{At}$.

We can now prove the

Lemma 3.4.

(i) *$\mathcal{U}_\Gamma \models^\Phi \varphi$ iff $\varphi \in \Phi$, for $\Phi \in U_\Gamma$.*

(ii) *$\mathcal{U}_\Gamma \models \varphi$ iff $\Gamma \vdash_{\mathsf{K}_n} \varphi$; in particular $\mathcal{U}_\Gamma \models \Gamma$.*

Proof.

(i) By induction on φ. For atomic φ (i) holds by definition. The induction steps for $\wedge$ and $\neg$ are routine, the latter using the primeness of each member of U_Γ. The induction step for $\Box$ proceeds as follows: Suppose (i) holds for $\varphi_1, \dots, \varphi_n$. Then, writing $\boldsymbol{\varphi}$ for $(\varphi_1, \dots, \varphi_n)$, and suppressing the subscript Γ, we have

$$\begin{aligned}
\mathcal{U} \models^{\Phi} \Box\boldsymbol{\varphi} &\text{ iff } (\forall \Psi_1 \dots \Psi_n \in U)[R\Phi\Psi_1 \dots \Psi_n \to U \models^{\Psi_1} \varphi_1 \text{ or } \cdots \text{ or } U \models^{\Psi_n} \varphi_n] \\
&\text{ iff } (\forall \Psi_1 \dots \Psi_n \in U)[\Phi^{\Box} \subseteq \Psi_1 + \cdots + \Psi_n \Rightarrow \varphi_1 \in \Psi_1 \text{ or } \cdots \text{ or } \varphi_n \in \Psi_n] \\
&\text{ iff } (\forall \Psi_1 \dots \Psi_n \in U)[\Phi^{\Box} \subseteq \Psi_1 + \cdots + \Psi_n \Rightarrow \boldsymbol{\varphi} \in \Psi_1 + \cdots + \Psi_n] \\
&\text{ iff } \boldsymbol{\varphi} \text{ is in every prime } n\text{-scheme containing } \Phi^{\Box} \quad \text{(by (A))} \\
&\text{ iff } \boldsymbol{\varphi} \in \Phi^{\Box} \quad \text{(by (B))} \\
&\text{ iff } \Box\boldsymbol{\varphi} \in \Phi.
\end{aligned}$$

(ii) Clearly, $\Gamma \vdash_{\mathsf{K}_n} \varphi$ implies $\varphi \in \Phi$ for all $\Phi \in U_\Gamma$ such that $\mathcal{U}_\Gamma \models \varphi$ by (i). Conversely, if not $\Gamma \vdash_{\mathsf{K}_n} \varphi$, then there is, by standard arguments (or by (B) with $n = 1$), $\Gamma \in U_\Gamma$ such that $\varphi \notin \Phi$. Then not $\mathcal{U}_\Gamma \models^{\Phi} \varphi$ by (i) whence not $\mathcal{U}_\Gamma \models \varphi$. $\Box$

As an immediate consequence of this we have

Theorem 3.5 (Completeness Theorem). *If* $\Gamma \models_n \varphi$, *then* $\Gamma \vdash_{\mathsf{K}_n} \varphi$.

Finally, we note that these results can be extended to some of the other polymodal systems we have introduced. For example, call an n-frame *reflexive* if its relation contains every n-tuple of the form $(u, \dots, u)$. It is then not hard to show that deducibility in $\mathsf{K}_n\mathsf{T}$ is equivalent to validity in all reflexive n-frames. I have not, however, been able to formulate a simple condition S on n-frames (when $n > 1$) so that deducibility in K_n4 is equivalent to validity in all frames satisfying S. There seems to be a disanalogy here with the case $n = 1$, since it is well known that deducibility in $\mathsf{K}4$ is equivalent to validity in all transitive (1-)frames.

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