

Precovers, Modalities and Universal Closure Operators in a Topos

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Abstract. In this paper we develop the notion of formal precover in a topos by defining a relation between elements and sets in a local set theory. We show that such relations are equivalent to modalities and to universal closure operators. Finally we prove that these relations are well characterized by a convenient restriction to a particular set.

Mathematics Subject Classification: 03G30, 03B20, 03F55.

Keywords: Elementary Topos, Precover, Modalities.

1 Introduction

The concept of *precover* is an essential constituent of that of *pretopology*, which was formulated by GIOVANNI SAMBIN in [4] in order to provide a semantics for linear logic. The concept of pretopology may itself be regarded as a generalization of the notion of *formal topology*, which was introduced by SAMBIN as part of a program for developing an intuitionistic approach to *pointless topology*, that is, topology in which the basic concept is that of neighbourhood and points are defined as particular filters of neighbourhoods (see [3] and [4]).

In this paper we develop the notion of precover in an *elementary topos*. Our arguments will in fact be formulated within the so-called *local set theories* and the *linguistic toposes* associated with them (see [1]). After a brief review of the relevant definitions, we show how each modality within a local set theory induces a relation possessing the reflexivity and transitivity properties of a precover. This relation, defined by JOHN BELL in [2] and called *universal precover*, extends the notion of precover to local set theories, and hence to elementary toposes. We show that the concept of universal precover is in fact *equivalent* to that of modality, and hence to that of *universal closure operation* in a local set theory. We also observe that the notion of universal precover is a natural generalization of the *membership relation* within a local set theory.

¹⁾ We would like to express our gratitude to GIOVANNI SAMBIN for his assistance.

To define a universal precover in a local set theory we have to assign a relation to each “set” (or to each object of the associated topos). We show that, when the set is a monoid, the assigned relation actually becomes a precover in the original sense of SAMBIN. Finally, we show that a universal precover may be obtained by assigning such a relation just to the *truth-value set* of the local set theory.

2 Precovers and pretopologies

Definition. Let $(S, \cdot, 1)$ be a commutative monoid. A *precovering relation*, or *precover*, on S is a relation \triangleleft between elements and subsets of S , satisfying the following conditions:

- (i) If $a \in U$, then $a \triangleleft U$ (reflexivity);
- (ii) if $a \triangleleft U$ and $U \triangleleft V$, then $a \triangleleft V$, where $U \triangleleft V \equiv \forall b \in U(b \triangleleft V)$ (transitivity);
- (iii) if $a \triangleleft U$ and $b \triangleleft V$, then $a \cdot b \triangleleft U \cdot V$ (stability).

A *pretopology* is a quadruple $\mathcal{F} \equiv (S, \cdot, 1, \triangleleft_{\mathcal{F}})$, where $(S, \cdot, 1)$ is a commutative monoid and $\triangleleft_{\mathcal{F}}$ is a precovering relation on it.

A precover on S can equivalently be presented as an operator $\mathcal{F} : P(S) \rightarrow P(S)$ defined by

$$\mathcal{F}(U) = \{a \in S : a \triangleleft_{\mathcal{F}} U\}.$$

It follows from properties (i) and (ii) that \mathcal{F} is a closure operator on S (see [4]). $\mathcal{F}(U)$ is called the *\mathcal{F} -saturation* of U and U is called *\mathcal{F} -saturated* if $\mathcal{F}(U) = U$.

The collection $\text{Sat}(\mathcal{F})$ of \mathcal{F} -saturated subsets of S is then a complete lattice with respect to \subseteq in which arbitrary infima and suprema are defined by

$$\bigwedge_{i \in I} U_i = \bigcap_{i \in I} U_i \quad \text{and} \quad \bigvee_{i \in I} U_i = \mathcal{F}(\bigcup_{i \in I} U_i).$$

Furthermore, if we define the operation $\cdot_{\mathcal{F}}$ by

$$U \cdot_{\mathcal{F}} V = \mathcal{F}(U \cdot V), \quad \text{for any } U, V \in \text{Sat}(\mathcal{F}),$$

$(\text{Sat}(\mathcal{F}), \cdot_{\mathcal{F}})$ becomes a commutative monoid. The only link between the algebraic and the lattice structure is infinite distributivity, that is to say

$$V \cdot_{\mathcal{F}} \bigvee_{i \in I} U_i = \bigvee_{i \in I} (V \cdot_{\mathcal{F}} U_i).$$

If $\cdot_{\mathcal{F}}$ coincides with \wedge or when we require $(\text{Sat}(\mathcal{F}), \cdot_{\mathcal{F}})$ to be a semilattice with ordering \subseteq , $\text{Sat}(\mathcal{F})$ is a complete Heyting algebra [4].

3 Local set theories and linguistic toposes

A *linguistic topos* is a topos built up from a language in a way resembling the construction of Lindenbaum algebras. It may be regarded as a “model” for a *local set theory*, which is a type theoretic system built up from the primitive symbols \in , $=$, and $\{ : \}$ in a language whose basic axioms and rules are formulated so as to yield exactly the theorems of intuitionistic logic (see [1]).

To sketch the construction of a linguistic topos, we begin by describing a local language. A *local language* is a typed language with no logical operations among

its primitive symbols, in which are specified two particular type symbols: $\mathbf{1}$ and Ω , called the *unity type* and the *truth value type*, respectively. Its terms must include the following:

- 1) $*$ of type $\mathbf{1}$;
- 2) $\{x_A : \alpha\}$ of type $P(\mathbf{A})$, where x_A is a variable of any type \mathbf{A} and α a term of type Ω ;
- 3) $\sigma = \tau$ of type Ω , where σ and τ are terms of the same type;
- 4) $\sigma \in \tau$ of type Ω , where σ and τ are terms of type \mathbf{A} and $P(\mathbf{A})$, respectively;
- 5) $\langle \tau_1, \dots, \tau_n \rangle$ of type $\mathbf{A}_1 \times \dots \times \mathbf{A}_n$, where τ_i is of type \mathbf{A}_i ($i = 1, \dots, n$);
- 6) $(\tau)_i$ of type \mathbf{A}_i , where τ is of type $\mathbf{A}_1 \times \dots \times \mathbf{A}_n$.

An occurrence of a variable x in a term τ is *bound* if it appears within a context of the form $\{x : \alpha\}$, otherwise it is *free*. A term of type Ω is called a *formula*.

The logical operations among the terms in the language can be defined as follows, where α, β are terms of type Ω .

$$\begin{array}{ll}
 \alpha \leftrightarrow \beta & \text{for } \alpha = \beta, \\
 \text{true} & \text{for } * = *, \\
 \alpha \wedge \beta & \text{for } \langle \alpha, \beta \rangle = \langle \text{true}, \text{true} \rangle, \\
 \alpha \rightarrow \beta & \text{for } (\alpha \wedge \beta) = \alpha, \\
 \forall x \alpha & \text{for } \{x : \alpha\} = \{x : \text{true}\}, \\
 \text{false} & \text{for } \forall \omega. \omega, \quad \omega \text{ a variable of type } \Omega, \\
 \neg \alpha & \text{for } \alpha \rightarrow \text{false}, \\
 \alpha \vee \beta & \text{for } \forall \omega [((\alpha \rightarrow \omega) \wedge (\beta \rightarrow \omega)) \rightarrow \omega], \\
 & \quad \omega \text{ a variable of type } \Omega \text{ not occurring in } \alpha \text{ or } \beta, \\
 \exists x \alpha & \text{for } \forall \omega [\forall x (\alpha \rightarrow \omega) \rightarrow \omega], \\
 & \quad \omega \text{ a variable of type } \Omega \text{ not occurring in } \alpha.
 \end{array}$$

To describe the axioms and the inference rules of a local set theory we use the sequent notation $\Gamma : \alpha$ in which α is a formula and Γ is a (possibly empty) finite set of formulas.

The *basic axioms* of a local set theory are the following:

$$\begin{array}{ll}
 \text{tautology} & \emptyset : \alpha = \alpha, \\
 \text{unity} & \emptyset : x_{\mathbf{1}} = *, \\
 \text{equality} & x = y, \alpha(z/x) : \alpha(z/y), \text{ with } x \text{ and } y \text{ free for } z \text{ in } \alpha, \\
 \text{products} & \emptyset : \langle (x_1, \dots, x_n) \rangle_i = x_i \quad \text{and} \quad \emptyset : x = \langle (x)_1, \dots, (x)_n \rangle, \\
 \text{comprehension} & \emptyset : x \in \{x : \alpha\} \leftrightarrow \alpha.
 \end{array}$$

The *rules of inference* of a local set theory are the following: thinning, cut (with the limitation that any free variable of the cut-formula is free in the antecedents or

in the succedents of the sequents, different from α), and

$$\begin{array}{l} \text{substitution} \quad \frac{\Gamma : \alpha}{\Gamma(x/\tau) : \alpha(x/\tau)}, \tau \text{ free for } x \text{ in } \Gamma \text{ and } \alpha, \\ \text{extensionality} \quad \frac{\Gamma : x \in \sigma \leftrightarrow x \in \tau}{\Gamma : \sigma = \tau}, x \text{ not free in } \Gamma, \sigma, \tau, \\ \text{equivalence} \quad \frac{\alpha, \Gamma : \beta \quad \beta, \Gamma : \alpha}{\Gamma : \alpha \leftrightarrow \beta}. \end{array}$$

A *proof* from S is defined in the same way as in the sequent calculus and a sequent $\Gamma : \alpha$ is said to be *derivable* from S if there is a proof from S of which $\Gamma : \alpha$ is the conclusion. We will write $\Gamma \vdash_S \alpha$ to indicate that $\Gamma : \alpha$ is derivable from S , and $\vdash_S \alpha$ for $\emptyset \vdash_S \alpha$. A *local set theory* is a theory in a local language, that is to say a collection of sequents, S , which is closed under derivability.

Linguistic toposes are constructed as follows. In a local language the terms of power type (if \mathbf{A} is a type symbol $\mathbf{P}(\mathbf{A})$ is called the *power type* of \mathbf{A}) are called *set-like terms* and closed set-like terms are called \mathcal{L} -sets or simply *sets*. We consider the equivalence classes, called S -sets, of \mathcal{L} -sets under the relation \sim_S defined in the following way:

$$X \sim_S Y \quad \text{iff} \quad \vdash_S X = Y.$$

An S -map $f : X \rightarrow Y$ is a triple (f, X, Y) of S -sets such that $\vdash_S f \in Y^X$, where Y^X is defined as

$$\{u : u \subseteq X \times Y \wedge \forall x(x \in X \rightarrow \exists! y(y \in Y \wedge \langle x, y \rangle \in u))\}.$$

The collection of S -sets and maps then forms a topos which is called a *linguistic topos*. It can be shown (see [1]) that any elementary topos is equivalent to a linguistic topos.

4 Modalities

In this paragraph we will show how, given a local set theory S and a modality on S (defined below), we can obtain a new relation between elements and S -sets which generalizes the membership relation. We will also see that saturating the S -sets with respect this new relation yields a universal closure operation.

Definition 4.1. Let S be a local set theory and Ω the S -set $\{\omega : \text{true}\}$. A *modality* is an S -map $\mu : \Omega \rightarrow \Omega$ such that

- (i) $\alpha \vdash_S \mu(\alpha)$;
- (ii) if $\alpha \vdash_S \beta$, then $\mu(\alpha) \vdash_S \mu(\beta)$;
- (iii) $\mu(\mu(\alpha)) \vdash_S \mu(\alpha)$.

We shall usually write α^* for $\mu(\alpha)$.

Note that this definition does not specify any relation between a modality and the logical operators, but it is consistent with them as we state in the next proposition, which is Corollary 5.3 of [1].

Proposition 4.2. *If μ is a modality in a local set theory S , we have:*

1. $\vdash_S (\alpha \wedge \beta)^* = \alpha^* \wedge \beta^*$;
2. $(\alpha \rightarrow \beta)^* \vdash_S \alpha^* \rightarrow \beta^*$;
3. $\alpha \rightarrow \beta \vdash_S \alpha^* \rightarrow \beta^*$;
4. $\exists x \alpha^* \vdash_S (\exists x \alpha)^*$, whence $(\exists x \alpha^*)^* \vdash_S (\exists x \alpha)^*$;
5. $(\forall x \alpha)^* \vdash_S \forall x \alpha^*$. □

Modalities enable us to generalize the membership relation. Thinking of $\mu(\alpha)$ as meaning “ α is possibly true”, $\mu(x \in U)$ means “it is possible that x is in U ”. Thus, given a modality μ on a local set theory S , for each S -set X we define a relation $\triangleleft_X \subseteq X \times P(X)$ by

$$x \triangleleft_X U \equiv \mu(x \in U),$$

for any S -set U such that $\vdash_S U \subseteq X$.

Theorem 4.3. *The relation \triangleleft_X has the following properties:*

1. If $\vdash_S U \subseteq X$, then $x \in U \vdash_S x \triangleleft_X U$.
2. If $\vdash_S U \subseteq X$ and $\vdash_S V \subseteq X$, then $y \triangleleft_X U, \forall x (x \in U \rightarrow x \triangleleft_X V) \vdash_S y \triangleleft_X V$.
3. If $f : X \rightarrow Y$ is an S -function and $\vdash_S U \subseteq Y$, then

$$x \in X \vdash_S f(x) \triangleleft_Y U \leftrightarrow x \triangleleft_X f^{-1}[U],$$

where $f^{-1}[U]$ is the term $\{x : x \in X \wedge f(x) \in U\}$.

Proof.

1. This follows immediately from the first property of modalities.
2. According to the definition of \triangleleft_X , we need to show:

$$(*) \quad \forall x [x \in U \rightarrow (x \in V)^*] \vdash_S (y \in U)^* \rightarrow (y \in V)^*.$$

Now $y \in U \rightarrow (y \in V)^* \vdash_S (y \in U)^* \rightarrow (y \in V)^*$ by 4.2.3, so, since $(y \in V)^{**} \vdash_S (y \in V)^*$, it follows that $y \in U \rightarrow (y \in V)^* \vdash_S (y \in U)^* \rightarrow (y \in V)^*$, from which (*) follows immediately.

3. As $x \in X \vdash_S (x \in f^{-1}[U])^* \leftrightarrow (f(x) \in U)^*$ follows from

$$x \in X, (x \in f^{-1}[U])^* : (f(x) \in U)^* \quad \text{and} \quad x \in X, (f(x) \in U)^* : (x \in f^{-1}[U])^*$$

by the equivalence rule, we need to infer the last two sequents. Regarding the first one, we have, by substituting into the equality axiom,

$$x \in f^{-1}[U] = (f(x) \in U \wedge x \in X), (x \in f^{-1}[U])^* \vdash_S (f(x) \in U \wedge x \in X)^*$$

so that, using 4.2.1,

$$x \in f^{-1}[U] = (f(x) \in U \wedge x \in X), (x \in f^{-1}[U])^* \vdash_S (f(x) \in U)^* \wedge (x \in X)^*.$$

Hence, by the comprehension axiom we get $(x \in f^{-1}[U])^* \vdash_S (f(x) \in U)^* \wedge (x \in X)^*$ so that $(x \in f^{-1}[U])^* \vdash_S (f(x) \in U)^*$. Hence, $x \in X, (x \in f^{-1}[U])^* \vdash_S (f(x) \in U)^*$.

Similarly, we show

From the first property of modalities we have $x \in X \vdash_S (x \in X)^*$ so concluding that $(x \in X), (f(x) \in U)^* \vdash_S (x \in f^{-1}[U])^*$. \square

The three properties in Theorem 4.3 embody the minimal conditions which a putative membership-relation has to satisfy. We also note that when the S -set is a monoid, the reflexive and transitive properties correspond to those of a precover.

The relation \triangleleft_X enables us to generalize the subset-relation by writing $U \triangleleft_X V$ for $\forall x(x \in U \rightarrow x \triangleleft_X V)$. With this new definition we can rewrite Theorem 4.3.2 as follows:

2'. If $\vdash_S U \subseteq X$ and $\vdash_S V \subseteq X$, then $x \triangleleft_X U, U \triangleleft_X V \vdash_S x \triangleleft_X V$.

In particular, we can “saturate” an S -set $U, \vdash_S U \subseteq X$, with respect to the relation \triangleleft_X writing U^Δ for $\{x : x \triangleleft_X U\}$ and calling it the *saturation* of U . It follows easily from Theorem 4.3 that the saturation operation $^\Delta$ is a *universal closure operation* in the sense of [1].

5 Universal precovers

We are now going to define a relation between elements and S -sets in a local set theory which will be characterized by the properties established in Theorem 4.3.

Definition 5.1. A *universal precover* \triangleleft on a local set theory S is an assignment of an S -set \triangleleft_X to each S -set X such that $\vdash_S \triangleleft_X \subseteq X \times P(X)$ and

- (i) if $\vdash_S U \subseteq X$, then $x \in U \vdash_S x \triangleleft_X U$,
- (ii) if $\vdash_S U \subseteq X$ and $\vdash_S V \subseteq X$, then $x \triangleleft_X U, U \triangleleft_X V \vdash_S x \triangleleft_X V$,
- (iii) if $f : X \rightarrow Y$ is an S -function and $\vdash_S U \subseteq Y$, then

$$x \in X \vdash_S f(x) \triangleleft_Y U \leftrightarrow x \triangleleft_X f^{-1}[U].$$

We will show that, given a type \mathbf{A} , a universal precover is “uniform” on every S -set of type $\mathbf{P}(\mathbf{A})$, and so also on every subset of a given S -set X . For this reason we can write $\triangleleft_{\mathbf{A}}$ in place of \triangleleft_X if \mathbf{A} is the type of X , and we can omit the subscript whenever it is clear from the context.

Proposition 5.2. If X is an S -set of type $\mathbf{P}(\mathbf{A})$ and $\vdash_S U \subseteq X$, then

$$x \in X \vdash_S x \triangleleft_X U \leftrightarrow x \triangleleft_{\mathbf{A}} U,$$

where we write \mathbf{A} for the S -set $\{x_{\mathbf{A}} : \text{true}\}$.

Proof. Consider the insertion $j : X \rightarrow \mathbf{A}$ defined as $j(x) \equiv x$. Since $\vdash_S U \subseteq \mathbf{A}$ it follows from the third property of universal precovers that

$$x \in X \vdash_S x \triangleleft_X j^{-1}[U] \leftrightarrow j(x) \triangleleft_{\mathbf{A}} U.$$

It remains to prove $\vdash_S j^{-1}[U] = U$, yielding $x \in X \vdash_S x \triangleleft_X U \leftrightarrow x \triangleleft_{\mathbf{A}} U$. Noting that $j^{-1}[U] = \{x : x \in X \wedge x \in U\}$ we infer $x \in X, x \in U \vdash_S x \in j^{-1}[U]$, whence $x \in U \vdash_S x \in j^{-1}[U]$, since $\vdash_S U \subseteq X$. From $x \in j^{-1}[U] \vdash_S x \in X \wedge x \in U$ we obtain $x \in j^{-1}[U] \vdash_S x \in U$, and $\vdash_S j^{-1}[U] = U$ follows. \square

From this we easily obtain

Corollary 5.3. If X is an S -set of type $\mathbf{P}(\mathbf{A})$ and U is an S -set such that $\vdash_S U \subseteq X$, then $x \in U \vdash_S x \triangleleft_U U \leftrightarrow x \triangleleft_X U$. \square

We want next to show that universal precovers respect the operations of function image and cartesian product. To do this we require the following

Lemma 5.4.

- (a) $x \triangleleft X, y \in Y \vdash_S \langle x, y \rangle \triangleleft X \times Y, \quad x \in X, y \triangleleft Y \vdash_S \langle x, y \rangle \triangleleft X \times Y.$
- (b) $x \triangleleft X, y \triangleleft Y \vdash_S \langle x, y \rangle \triangleleft X^\Delta \times Y, \quad x \triangleleft X, y \triangleleft Y \vdash_S \langle x, y \rangle \triangleleft X \times Y^\Delta.$
- (c) $\vdash_S X^\Delta \times Y \triangleleft X \times Y, \quad \vdash_S X \times Y^\Delta \triangleleft X \times Y.$

Proof.

(a) Let X be of type \mathbf{A} and consider $\sigma : A \times Y \rightarrow A$ defined by $\sigma(\langle x, y \rangle) \equiv x$. The third property of universal precovers yields

$$X \subseteq A, \langle x, y \rangle \in A \times Y, x \triangleleft X \vdash_S \langle x, y \rangle \triangleleft X \times Y$$

and so, since the cartesian product is defined by $A \times Y \equiv \{\langle x, y \rangle : x \in A \wedge y \in Y\}$, we obtain $X \subseteq A, x \in A, y \in Y, x \triangleleft X \vdash_S \langle x, y \rangle \triangleleft X \times Y$. From the hypothesis $\vdash_S X \subseteq A$, we obtain $x \in A, y \in Y, x \triangleleft X \vdash_S \langle x, y \rangle \triangleleft X \times Y$, i.e. $y \in Y, x \triangleleft X \vdash_S \langle x, y \rangle \triangleleft X \times Y$, recalling that $A = \{x_A : true\}$ so that $\vdash_S x \in A$. The second sequent in (a) is an easy consequence.

(b) This follows from (a) by substituting X^Δ for X and observing that

$$\vdash_S x \in X^\Delta \leftrightarrow x \triangleleft X$$

follows from the comprehension axiom, so that $x \triangleleft X \vdash_S x \in X^\Delta$. In the same way we deduce the second sequent in (b).

(c) The comprehension axiom implies $\langle x, y \rangle \in X^\Delta \times Y \vdash_S x \in X^\Delta \wedge y \in Y$, and by (a), $x \triangleleft X, y \in Y \vdash_S \langle x, y \rangle \triangleleft X \times Y$, whence $\langle x, y \rangle \in X^\Delta \times Y \vdash_S \langle x, y \rangle \triangleleft X \times Y$. Hence $\vdash_S \forall x, y (\langle x, y \rangle \in X^\Delta \times Y \rightarrow \langle x, y \rangle \triangleleft X \times Y)$ which may be written in the form $\vdash_S X^\Delta \times Y \triangleleft X \times Y$. In the same way we deduce $\vdash_S X \times Y^\Delta \triangleleft X \times Y$. \square

Proposition 5.5. *If \triangleleft is a universal precover on a local set theory S , then*

- (i) *if $f : X \rightarrow Y$ is an S -function and $\vdash_S U \subseteq X$, then*
 - $x \in X, x \triangleleft_X U \vdash_S f(x) \triangleleft_Y f[U];$
- (ii) $\vdash_S \langle x, y \rangle \triangleleft X \times Y \leftrightarrow (x \triangleleft X \wedge y \triangleleft Y).$

Proof.

(i) Let $f : X \rightarrow Y$ be an S -function. Since $U \subseteq X \vdash_S U \subseteq f^{-1}[f[U]]$ it follows that $U \subseteq X \vdash_S U \triangleleft f^{-1}[f[U]]$. By the second property of universal precovers, we have $x \triangleleft_X U, U \triangleleft f^{-1}[f[U]] \vdash_S x \triangleleft_X f^{-1}[f[U]]$ which yields

$$(1) \quad U \subseteq X, x \triangleleft_X U \vdash_S x \triangleleft_X f^{-1}[f[U]].$$

The third property of universal precovers gives

$$x \in X, f[U] \subseteq Y, x \triangleleft_X f^{-1}[f[U]] \vdash_S f(x) \triangleleft_Y f[U]$$

so that, since $U \subseteq X \vdash_S f[U] \subseteq Y$, we infer

$$(2) \quad U \subseteq X, x \in X, x \triangleleft_X f^{-1}[f[U]] \vdash_S f(x) \triangleleft_Y f[U].$$

From (1) and (2) follows

$$U \subseteq X, x \in X, x \triangleleft_X U \vdash_S f(x) \triangleleft_Y f[U],$$

whence $x \in X, x \triangleleft_X U \vdash_S f(x) \triangleleft_Y f[U]$.

(ii) Let X, Y be of types $P(A), P(B)$, respectively, write A, B for $\{x_A : true\}, \{x_B : true\}$, respectively, and $\pi_1 : A \times B \rightarrow A, \pi_2 : A \times B \rightarrow B$ for the S -functions $\langle x, y \rangle \mapsto x, \langle x, y \rangle \mapsto y$, respectively. We first show $\langle x, y \rangle \triangleleft X \times Y \vdash_S (x \triangleleft X \wedge y \triangleleft Y)$. As we saw in (i) we can get

$$X \times Y \subseteq A \times B, \langle x, y \rangle \in A \times B, \langle x, y \rangle \triangleleft X \times Y \vdash_S \pi_1(\langle x, y \rangle) \triangleleft \pi_1(X \times Y)$$

so that

$$(3) \quad X \times Y \subseteq A \times B, \langle x, y \rangle \triangleleft X \times Y \vdash_S \pi_1(\langle x, y \rangle) \triangleleft \pi_1(X \times Y),$$

recalling the definitions of A and B and observing that $\vdash_S \langle x, y \rangle \in A \times B$. From the two hypotheses $\vdash_S X \subseteq A$ and $\vdash_S Y \subseteq B$ we infer $\vdash_S X \times Y \subseteq A \times B$, and $\langle x, y \rangle \triangleleft X \times Y \vdash_S x \triangleleft X$ follows from (3). In the same way, using π_2 in place of π_1 , we have $\langle x, y \rangle \triangleleft X \times Y \vdash_S y \triangleleft Y$, whence $\langle x, y \rangle \triangleleft X \times Y \vdash_S x \triangleleft X \wedge y \triangleleft Y$. Lastly, we have $(x \triangleleft X \wedge y \triangleleft Y) \vdash_S \langle x, y \rangle \triangleleft X \times Y$. For from Lemma 5.4.(c), we obtain $\vdash_S X^\Delta \times Y^\Delta \triangleleft X \times Y$ from which follows

$$\langle x, y \rangle \in X^\Delta \times Y^\Delta \vdash_S \langle x, y \rangle \triangleleft X \times Y$$

so that $x \triangleleft X, y \triangleleft Y \vdash_S \langle x, y \rangle \triangleleft X \times Y$. □

We finally show that a universal precover is uniquely determined by the relation between formulas and the S -set $Tr \equiv \{\omega : \omega = true\}$.

Proposition 5.6. *If U is an S -set of type $P(A)$ and $\vdash_S U \subseteq X$, then*

$$\vdash_S x \triangleleft_A U \leftrightarrow (x \in U) \triangleleft_\Omega Tr.$$

Proof. Let $\mathcal{X} : A \rightarrow \Omega$ be the S -function defined by $\mathcal{X}(x) = x \in U$. Then $\vdash_S \mathcal{X}^{-1}[Tr] = U$ so that $\vdash_S x \triangleleft_A U \leftrightarrow x \triangleleft_A \mathcal{X}^{-1}[Tr]$ and by property (iii) of universal precovers, $\vdash_S x \triangleleft_A \mathcal{X}^{-1}[Tr] \leftrightarrow \mathcal{X}(x) \triangleleft_\Omega Tr$. Putting these equivalences together gives the required result. □

6 Equivalence between modalities and universal precovers

Our next task is to show that a modality is equivalent to a universal precover. In fact we will show that a universal precover induces a modality which, in its turn, gives rise to a universal precover which turns out to be the same as the original one. We will prove the same thing starting with a modality.

First, we show how to obtain a modality from a universal precover.

Proposition 6.1. *Let \triangleleft be a universal precover on a local set theory S and $\mu : \Omega \rightarrow \Omega$ the S -function defined by $\mu(\omega) \equiv \omega \triangleleft_\Omega Tr$. Then μ is a modality.*

Proof. We have to prove that μ satisfies the three properties of a modality.

(i) We need to show that $\alpha \vdash_S \alpha \triangleleft_\Omega Tr$. Since $\vdash_S \omega = (\omega = true)$, it follows that $\vdash_S \omega \leftrightarrow \omega \in Tr$ so that $\omega \vdash_S \omega \in Tr$. From the first property of a universal precover we have $\omega \in Tr \vdash_S \omega \triangleleft_\Omega Tr$, whence $\omega \vdash_S \omega \triangleleft_\Omega Tr$, as required.

(ii) We have to show that from $\alpha \vdash_S \beta$ we can infer $\alpha \triangleleft_\Omega Tr \vdash_S \beta \triangleleft_\Omega Tr$. From the hypothesis $\alpha \vdash_S \beta$ we infer $\vdash_S \alpha \rightarrow \beta$ and hence, recalling the definition of $\alpha \rightarrow \beta$, the equality axiom yields $\alpha \triangleleft_\Omega Tr \vdash_S \alpha \wedge \beta \triangleleft_\Omega Tr$. So our task reduces to showing that $\alpha \wedge \beta \triangleleft_\Omega Tr \vdash_S \beta \triangleleft_\Omega Tr$, and, to this end, we consider the S -function

$\& : \Omega \times \Omega \longrightarrow \Omega$ defined by $\&(\langle \omega, \omega' \rangle) \equiv \omega \wedge \omega'$. By the third property of universal precovers, we have $\vdash_S (\omega \wedge \omega') \triangleleft_{\Omega} Tr \leftrightarrow \langle \omega, \omega' \rangle \triangleleft \&^{-1}[Tr]$. Now, if we can show that

$$(4) \quad \vdash_S \&^{-1}[Tr] = Tr \times Tr,$$

we may infer $\vdash_S (\omega \wedge \omega') \triangleleft_{\Omega} Tr \leftrightarrow \langle \omega, \omega' \rangle \triangleleft Tr \times Tr$, so that

$$(\omega \wedge \omega') \triangleleft_{\Omega} Tr \vdash_S \langle \omega, \omega' \rangle \triangleleft Tr \times Tr.$$

Proposition 5.5.(ii) then implies $(\omega \wedge \omega') \triangleleft_{\Omega} Tr \vdash_S \omega \triangleleft_{\Omega} Tr \wedge \omega' \triangleleft_{\Omega} Tr$ from which follows $(\omega \wedge \omega') \triangleleft_{\Omega} Tr \vdash_S \omega' \triangleleft_{\Omega} Tr$, so that, substituting α for ω and β for ω' , we get what we want.

So we turn our attention to proving (4). To do this, we observe that, by the comprehension axiom, $\vdash_S \langle \omega, \omega' \rangle \in \&^{-1}[Tr] \leftrightarrow \langle \omega, \omega' \rangle \in \Omega \times \Omega \wedge (\omega \wedge \omega') \in Tr$, so that $\vdash_S \langle \omega, \omega' \rangle \in \&^{-1}[Tr] \leftrightarrow \langle \omega, \omega' \rangle \in \Omega \times \Omega \wedge (\omega \wedge \omega') = true$. Therefore, we have $\vdash_S \langle \omega, \omega' \rangle \in \&^{-1}[Tr] \leftrightarrow \langle \omega, \omega' \rangle \in \Omega \times \Omega \wedge (\omega \wedge \omega')$ from $\vdash_S \omega \wedge \omega' = ((\omega \wedge \omega') = true)$. Now, $\omega \wedge \omega'$ is $\langle \omega, \omega' \rangle = \langle true, true \rangle$, whence $\vdash_S \omega \wedge \omega' \leftrightarrow \omega \in Tr \wedge \omega' \in Tr$. So we have $\vdash_S \langle \omega, \omega' \rangle \in \&^{-1}[Tr] \leftrightarrow \langle \omega, \omega' \rangle \in \Omega \times \Omega \wedge (\omega \in Tr \wedge \omega' \in Tr)$, whence

$$\vdash_S \langle \omega, \omega' \rangle \in \&^{-1}[Tr] \leftrightarrow \langle \omega, \omega' \rangle \in Tr \times Tr,$$

from which we infer $\vdash_S \&^{-1}[Tr] = Tr \times Tr$ by extensionality.

(iii) Finally, we must prove $\mu(\mu(\alpha)) \vdash_S \mu(\alpha)$, i. e. $(\alpha \triangleleft_{\Omega} Tr) \triangleleft_{\Omega} Tr \vdash_S \alpha \triangleleft_{\Omega} Tr$. Since $\vdash_S \alpha \triangleleft_{\Omega} Tr \leftrightarrow \alpha \in Tr^{\Delta}$ it follows that

$$(\alpha \triangleleft_{\Omega} Tr) \triangleleft_{\Omega} Tr \vdash_S (\alpha \in Tr^{\Delta}) \triangleleft_{\Omega} Tr.$$

By proposition 5.6, we have $\vdash_S (\alpha \in Tr^{\Delta}) \triangleleft_{\Omega} Tr \leftrightarrow \alpha \triangleleft_{\Omega} Tr^{\Delta}$. Hence

$$(5) \quad \vdash_S (\alpha \triangleleft_{\Omega} Tr) \triangleleft_{\Omega} Tr \leftrightarrow \alpha \triangleleft_{\Omega} Tr^{\Delta}.$$

From the second property of universal precovers, we have

$$\alpha \triangleleft_{\Omega} Tr^{\Delta}, Tr^{\Delta} \triangleleft Tr \vdash_S \alpha \triangleleft_{\Omega} Tr.$$

Since clearly $\vdash_S Tr^{\Delta} \triangleleft Tr$, we obtain $\alpha \triangleleft_{\Omega} Tr^{\Delta} \vdash_S \alpha \triangleleft_{\Omega} Tr$, which, together with (5) gives the required conclusion. \square

Now we establish the equivalence between modalities and universal precovers.

Theorem 6.2. *Every modality in a local set theory S is equivalent to a universal precover in S , and further, every universal precover is equivalent to a modality.*

Proof. Let μ be a modality in S . As we saw in Section 4, we can define a universal precover, $x \triangleleft U \equiv \mu(x \in U)$, which, in its turn, induces a modality μ_a as follows:

$$\mu_a(\alpha) \equiv \alpha \triangleleft Tr \equiv \mu(\alpha \in Tr).$$

We show that $\vdash_S \mu(\alpha) \leftrightarrow \mu_a(\alpha)$. Since $\vdash_S \alpha \leftrightarrow \alpha \in Tr$, we can apply the second property of modalities and obtain

$$\vdash_S \mu(\alpha) \leftrightarrow \mu(\alpha \in Tr),$$

as claimed.

Conversely, given a universal precover \triangleleft , we can define a modality μ as in the previous proposition and then a universal precover \triangleleft_μ by

$$x \triangleleft_\mu U \equiv \mu(x \in U) \equiv (x \in U) \triangleleft_\Omega Tr.$$

By Proposition 5.6 we have $\vdash_S x \triangleleft_X U \leftrightarrow (x \in U) \triangleleft_\Omega Tr$ which we can rewrite as $\vdash_S x \triangleleft_X U \leftrightarrow x \triangleleft_\mu U$ so that \triangleleft_μ is the same as \triangleleft . \square

7 Stability

We have defined a universal precover on a local set theory by assigning to each S -set X a relation \triangleleft_X satisfying certain properties. When the S -set X is a monoid, we have noted that the first two properties correspond to the reflexive and transitive properties of a precover, respectively. Here we will show that, when X is a monoid, \triangleleft_X also satisfies stability so that \triangleleft_X is a *precover* on X .

Let \mathcal{L} be a local language in which we can write the monoid axioms and let \mathbf{M} be a type of a monoid. We write M for the S -set $\{x_M : true\}$ of type $\mathbf{P}(\mathbf{M})$, and if U and V are S -sets of type $\mathbf{P}(\mathbf{M})$, we write $U \cdot V$ for the term $\{x \cdot y : x \in U \wedge y \in V\}$.

We will say that a universal precover satisfies *stability* if

$$a \triangleleft_M U, b \triangleleft_M V \vdash_S a \cdot b \triangleleft_M U \cdot V.$$

Proposition 7.1. *If \triangleleft is a universal precover on a local set theory S , then \triangleleft_M satisfies stability.*

Proof. Let $f : M \times M \rightarrow M$ be the S -function defined by $f(\langle x, y \rangle) \equiv x \cdot y$. From 5.5.(i) we infer $\langle x, y \rangle \in M \times M, \langle x, y \rangle \triangleleft U \times V \vdash_S f(\langle x, y \rangle) \triangleleft_M f(U \times V)$ and so, since $\vdash_S \langle x, y \rangle \triangleleft U \times V \leftrightarrow x \triangleleft_M U \wedge y \triangleleft_M V$ by Proposition 5.5.(ii), we obtain

$$\langle x, y \rangle \in M \times M, x \triangleleft_M U \wedge y \triangleleft_M V \vdash_S x \cdot y \triangleleft_M U \cdot V.$$

Since clearly $\vdash_S \langle x, y \rangle \in M \times M$, the conclusion follows. \square

8 Precovers on Ω

We have seen in Theorem 6.2 that specifying a modality on a local set theory S is equivalent to specifying a universal precover, but to specify a universal precover we have to define a relation for *every* S -set X whereas, to specify a modality we have to define an S -function *only* on the S -set Ω . This fact leads one to suspect that a universal precover may be obtained just by defining a relation on the S -set Ω , which is what we finally show.

Definition 8.1. A *precover-on- Ω* in a local set theory S is an S -set \triangleleft_Ω such that $\triangleleft_\Omega \subseteq \Omega \times P(\Omega)$, and

- (i) if $\vdash_S U \subseteq \Omega$, then $x \in U \vdash_S x \triangleleft_\Omega U$;
- (ii) if $\vdash_S U \subseteq \Omega$ and $\vdash_S V \subseteq \Omega$, then $x \triangleleft_\Omega U, U \triangleleft_\Omega V \vdash_S x \triangleleft_\Omega V$;
- (iii) $\vdash_S (\omega \wedge \omega') \triangleleft_\Omega Tr \rightarrow \omega \triangleleft_\Omega Tr \wedge \omega' \triangleleft_\Omega Tr$;
- (iv) if $f : \Omega \rightarrow \Omega$ is an S -function and $\vdash_S U \subseteq \Omega$, then $\vdash_S \omega \triangleleft_\Omega f^{-1}[U] \leftrightarrow f(\omega) \triangleleft_\Omega U$.

We now show that this relation determines a universal precover.

Theorem 8.2. *Let S be a local set theory and \triangleleft_{Ω} a precover-on- Ω in S . If for each S -set X we define the relation $\triangleleft_X, \triangleleft_X \subseteq X \times P(X)$, by*

$$x \triangleleft_X U \equiv (x \in U) \triangleleft_{\Omega} Tr,$$

then we obtain a universal precover on S .

Proof.

(i) Since $\omega \vdash_S \omega \in Tr$, we obtain $x \in U \vdash_S (x \in U) \in Tr$ from which follows $x \in U \vdash_S (x \in U) \triangleleft_{\Omega} Tr$ by the first property of a precover-on- Ω , and if we rewrite the conclusion using the definition of \triangleleft_X , we get $x \in U \vdash_S x \triangleleft_X U$.

(ii) We must show

$$(y \in U) \triangleleft_{\Omega} Tr, \forall x ((x \in U) \rightarrow (x \in V) \triangleleft_{\Omega} Tr) \vdash_S (y \in V) \triangleleft_{\Omega} Tr.$$

Now, from the definition of $\alpha \rightarrow \beta$ we infer

$$(x \in U) \rightarrow (x \in V) \triangleleft_{\Omega} Tr, (x \in U) \triangleleft_{\Omega} Tr \vdash_S ((x \in U) \wedge (x \in V) \triangleleft_{\Omega} Tr) \triangleleft_{\Omega} Tr$$

and the third property of \triangleleft_{Ω} implies

$$(x \in U) \rightarrow (x \in V) \triangleleft_{\Omega} Tr, (x \in U) \triangleleft_{\Omega} Tr \vdash_S (x \in U) \triangleleft_{\Omega} Tr \wedge ((x \in V) \triangleleft_{\Omega} Tr) \triangleleft_{\Omega} Tr.$$

Hence $(x \in U) \rightarrow (x \in V) \triangleleft_{\Omega} Tr, (x \in U) \triangleleft_{\Omega} Tr \vdash_S ((x \in V) \triangleleft_{\Omega} Tr) \triangleleft_{\Omega} Tr$ so that

$$\forall x ((x \in U) \rightarrow (x \in V) \triangleleft_{\Omega} Tr), (y \in U) \triangleleft_{\Omega} Tr \vdash_S ((y \in V) \triangleleft_{\Omega} Tr) \triangleleft_{\Omega} Tr.$$

Now, from Proposition 5.6 we infer $((y \in V) \triangleleft_{\Omega} Tr) \triangleleft_{\Omega} Tr \vdash_S (y \in V) \triangleleft_{\Omega} Tr^{\Delta}$ and so, since $\vdash_S Tr^{\Delta} \triangleleft Tr$, we deduce

$$\forall x ((x \in U) \rightarrow (x \in V) \triangleleft_{\Omega} Tr), (y \in U) \triangleleft_{\Omega} Tr \vdash_S (y \in V) \triangleleft_{\Omega} Tr,$$

as required.

(iii) Let $f : X \rightarrow Y$ be an S -function and $\vdash_S U \subseteq Y$. The equality axiom gives us $(f(x) \in U) = (x \in f^{-1}[U])$, $(f(x) \in U) \triangleleft_{\Omega} Tr \vdash_S (x \in f^{-1}[U]) \triangleleft_{\Omega} Tr$, and then, observing that $x \in X \vdash_S f(x) \in U \leftrightarrow x \in f^{-1}[U]$ follows from the definition of $f^{-1}[U]$, we obtain $x \in X, (f(x) \in U) \triangleleft_{\Omega} Tr \vdash_S (x \in f^{-1}[U]) \triangleleft_{\Omega} Tr$. Similarly we infer $x \in X, (x \in f^{-1}[U]) \triangleleft_{\Omega} Tr \vdash_S (f(x) \in U) \triangleleft_{\Omega} Tr$ from which follows

$$x \in X \vdash_S (x \in f^{-1}[U]) \triangleleft_{\Omega} Tr \leftrightarrow (f(x) \in U) \triangleleft_{\Omega} Tr.$$

If we rewrite the conclusion using \triangleleft_X , we get $x \in X \vdash_S x \triangleleft_X f^{-1}[U] \leftrightarrow f(x) \triangleleft_Y U$. \square

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(Received: March 14, 1995)