

## Quantum Incompatibility and Noncommutativity

Here is a way of looking at quantum incompatibility of properties, suggested by Basil Hiley, which brings noncommutativity into full focus.

We are given a bunch of objects each of which is either a sphere or a cube, and each of which is either red or blue. Thus the objects are classified under two attributes SHAPE and COLOUR. SHAPE further breaks down into the two properties **spherical** and **cubical**, and COLOUR into the properties **red** and **blue**.

We suppose that each of the four properties **spherical**, **cubical**, **red**, **blue** comes with a test for determining whether an object has the property in question. Thus, for example, if an object passes the test for **red**, it is deemed to have the property **red**; if not, it is deemed to have the property **blue**; in other words, it would have passed the **blue** test. We assume that, if an object passes a test for a given property, it will also pass a test for that property applied immediately afterwards (i.e. , without any intervening application of a test for a different property).

Now suppose that we test the objects for the property **spherical**, and assemble the objects passing the test into an ensemble **S**. Next, we subject the objects in **S** to the test for **red**, and assemble the objects passing this subsequent test into an ensemble which we shall write as **S ★ R**.

Now perform this procedure in the opposite order, first testing for **red** and then for **spherical**. The resulting ensemble will then, naturally, be  $\mathbf{R} \star \mathbf{S}$ .

If the attributes SHAPE and COLOUR behave *classically*, we would expect the ensembles  $\mathbf{S} \star \mathbf{R}$  and  $\mathbf{R} \star \mathbf{S}$  to be identical, since then the operation  $\star$  would simply correspond to the logical conjunction ( $\wedge$ ) of properties (and of course  $\wedge$  is commutative). In other words, from a classical standpoint,  $\star$  would be a commutative operation.

But in the quantum world  $\star$  will not necessarily be a commutative operation, *and so will not necessarily correspond to the logical operation of conjunction*. For in the quantum world attributes such as SHAPE and COLOUR may be *incompatible*, that is, like position and momentum of electrons, not simultaneously ``testable``. In that case  $\mathbf{S} \star \mathbf{R}$  and  $\mathbf{R} \star \mathbf{S}$  may *differ*, so that the  $\star$  operation is *noncommutative*.

What this means is that, if we take an arbitrary object from  $\mathbf{S} \star \mathbf{R}$  and subject it to the **red** test, it will always pass the test. But if we subject an arbitrary object from  $\mathbf{R} \star \mathbf{S}$  to the same, **red**, test, sometimes it will not pass the test and *will then have to be deemed blue*, that is, it would have passed the **blue** test. In other words,  $\mathbf{S} \star \mathbf{R}$  consists entirely of red objects, but  $\mathbf{R} \star \mathbf{S}$  *will contain some blue objects*.

The startling nature of this conclusion can be made further evident by tracking the history of an object,  $a$ , in  $\mathbf{R} \star \mathbf{S}$  which fails to pass the test for **red** and so is deemed **blue**. Now  $a$  initially passed the **red** test and so was, correctly, included in  $\mathbf{R}$ . At this point  $a$  was definitely **red**. But then, for  $a$  to be included in  $\mathbf{R} \star \mathbf{S}$ , it must subsequently pass the **spherical** test. Once  $a$  passes this test, it is then tested for **blue** and passes that

test as well. In other words, applying the **spherical** test to the **red** object  $a$  has caused it to change its colour to **blue**!