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# ON THE RELATIONSHIP BETWEEN WEAK COMPACTNESS IN $L_{\omega_1\omega}$, $L_{\omega_1\omega_1}$, AND RESTRICTED SECOND-ORDER LANGUAGES*

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*1. Introduction.* Let $\varkappa$ be a cardinal. A set of sentences $\Sigma$ of a language $L$ is said to be *$\varkappa$-consistent* if each subset of $\Sigma$ of power $< \varkappa$ has a model. Further, $L$ is said to be *$\varkappa$-compact* if each $\varkappa$-consistent set of sentences of $L$ of power $\varkappa$ has a model. In this paper, a sequel to [1], we investigate the property of $\varkappa$-compactness for the following languages:

(i) the infinitary languages $L_{\omega_1\omega}$ and $L_{\omega_1\omega_1}$;

(ii) the restricted second-order language $L(\aleph_1)$ (called $W_1$ in [1]) which has set variables ranging over countable sets of individuals; and

(iii) the language $L(WF)$ for well-founded relations.

Our main result is that, for inaccessible $\varkappa$, any one of these languages is $\varkappa$-compact if and only if the others are too. For $L_{\omega_1\omega}$ this result is perhaps somewhat surprising in view of its relatively limited expressive power and the observation that it behaves quite well when only countable sets of sentences are considered.

*2. Terminology and notation.* Throughout this paper the symbol "$L$" will be used to denote a first-order language with arbitrarily many nonlogical symbols, including in every case an identity symbol "$\approx$" and a membership symbol "$\in$". The infinitary extensions $L_{\omega_1\omega}$ and $L_{\omega_1\omega_1}$ of $L$ are obtained in the usual way [3]: in $L_{\omega_1\omega}$ one allows countable conjunctions and finite quantifications; $L_{\omega_1\omega_1}$ admits countable quantifications as well. In $L_{\omega_1\omega_1}$ a sentence $WF$ asserting the well-foundedness of $\varepsilon$ can be written down, namely

$$\neg \exists (x_n)_{n\in\omega} [\bigwedge_{n\in\omega} (x_{n+1} \in x_n)] .$$

The language $L(WF)$ for well-founded relations may now be defined as the smallest class $F$ of formulas of $L_{\omega_1\omega_1}$ containing $WF$ and every formula of $L$, and such that

$$\varphi, \psi \in F \Rightarrow \neg \varphi, \varphi \wedge \psi, \exists x \varphi \in F.$$

The restricted second-order language $L(\aleph_\alpha)$ is built up by adding to $L$ a set $\{V_n : n \in \omega\}$ of one-place predicate variables which are interpreted as ranging over sets of individuals of power $< \aleph_\alpha$. Notice that a sentence $WF'$ equivalent (under standard interpretations) to $WF$ can be written down in $L(\aleph_1)$, namely,

$$\forall V_0 [\exists x V_0(x) \to \exists x [V_0(x) \wedge \forall y [V_0(y) \to \neg (y \in x)]]].$$

Evidently $L(WF)$ is translatable into $L(\aleph_1)$ and $L(\aleph_1)$ into $L_{\omega_1\omega_1}$.

We use freely the following well-known theorem of Mostowski: if $A$ is a set, $R \subseteq A \times A$, and $\langle A, R\rangle$ is a model of $WF$ (or $WF'$) and the axiom of extensionality, then $\langle A, R\rangle$ is isomorphic to a transitive structure $\langle B, \in\rangle$ where $\in$ is the usual membership relation on $B$.
Finally, if $\varkappa$ is a cardinal, a family of sets $F$ is said to have the *$\varkappa$-intersection property* if $\cap A \neq \emptyset$ for all $A \subseteq F$ such that Card $(A) < \varkappa$. A field of subsets or a filter in such a field is said to be *$\varkappa$-complete* if it is closed under intersections of power $< \varkappa$.

*3. The main theorem.* We now prove the

*Theorem.* Let $\varkappa$ be a (strongly) inaccessible cardinal $> \aleph_0$. The following conditions are equivalent:

(i) $L_{\omega_1\omega_1}$ is $\varkappa$-compact;
(ii) $L(\aleph_1)$ is $\varkappa$-compact;
(iii) $L(WF)$ is $\varkappa$-compact;
(iv) $L_{\omega_1\omega}$ is $\varkappa$-compact;

(v) if $M$ is a transitive set of power $\varkappa$ and $X$ is a subset of $M$ with the $\varkappa$-intersection property, then there is an elementary embedding $j$ of $\langle M, \varepsilon\rangle$ into a transitive structure $\langle N, \varepsilon\rangle$ such that $\cap j``X \neq \emptyset$;
(vi) if $A$ is a set of power $\varkappa$, and $B$ is a $\varkappa$-complete field of subsets of $A$ of power $\varkappa$, then each $\varkappa$-complete filter in $B$ can be extended to an $\omega_1$-complete ultrafilter in $B$.

*Proof.*

(1) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious as is (i) $\Rightarrow$ (iv). We prove (iv) $\Rightarrow$ (vi), and (iii) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (i).
(iv) $\Rightarrow$ (vi). Assume (iv); let $A$ be a set of power $\varkappa$, let $B = \{B_\xi : \xi < \varkappa\}$ be a field of subsets of $A$ of power $\varkappa$, and suppose $F$ is a $\varkappa$-complete filter in $B$. Let $\Sigma$ be the union of the $L_{\omega_1\omega}$ theory of $\langle A, (B_\xi)_{\xi<\varkappa}, (a)_{a\in A}\rangle$ with the sentences $\{P_\xi(c) : B_\xi \in F\}$ where $P_\xi$ is a unary predicate corresponding to $B_\xi$ and $c$ is a new constant. Since $F$ has the $\varkappa$-intersection property, $\Sigma$ is $\varkappa$-consistent. Certainly Card $(\Sigma) = \varkappa$, so $\Sigma$ has a model $\mathfrak{C} = \langle C, (C_\xi)_{\xi<\varkappa}\rangle$. (We have suppressed reference to the constants of $L_{\omega_1\omega}$; similar suppressions are made afterwards.) Let $t$ be the interpretation of $c$ in $\mathfrak{C}$. Since $\mathfrak{A} <_{\omega_1\omega} \mathfrak{C}$ up to isomorphism, it is easy to see that the set $F' = \{B_\xi : t \in C_\xi\}$ is an $\omega_1$-complete ultrafilter in $B$ extending $F$.
(iii) $\Rightarrow$ (v). Assume (iii); let $A$ be a transitive set of power $\varkappa$, and $X$ a subset of $A$ with the $\varkappa$-intersection property. For each $a \in A$ let $c_a$ be a constant denoting $a$ in $A$, and let $c$ be a new constant. Let $\Sigma$ denote the union of the $L(WF)$ theory of $\langle A, \varepsilon, (a)_{a\in A}\rangle = \mathfrak{A}$ with the sentences $\{(c \in c_x) : x \in X\}$. Then $\Sigma$ is $\varkappa$-consistent and of power $\varkappa$, so has a model $\mathfrak{B}$. Since $\mathfrak{B}$ is a model of $WF$ and the axiom of extensionality, by Mostowski's theorem we may suppose $\mathfrak{B} = \langle B, \varepsilon\rangle$ where $B$ is transitive. The map $j : \mathfrak{A} \to \mathfrak{B}$ defined by $j(a) = c_a^{(\mathfrak{B})}$ is an elementary embedding, and $c^{(\mathfrak{B})} \in j(x)$ for each $x \in X$, i.e. $\cap j``X \neq \emptyset$ as required.

(v) $\Rightarrow$ (vi). Suppose (v) holds; let $B$ be a $\varkappa$-complete field of subsets of power $\varkappa$ of a set $A$, also of power $\varkappa$, and let $F$ be a $\varkappa$-complete filter in $B$. Without loss of generality we may assume $A = \varkappa$. Let $M$ be the transitive closure of $B \cup {}^{\omega}B$; then Card $(M) = \varkappa$ and $\varkappa + 1 \subseteq M$. Let $j$ be an elementary embedding of $\langle M, \varepsilon\rangle$ into a transitive $\langle N, \varepsilon\rangle$ with some $a \in \cap j$" $F$. Defining $F' = \{x \in B : a \in j(x)\}$ yields an $\omega_1$-complete ultrafilter in $B$ containing $F$. For it is a simple matter to verify that $F'$ is an ultrafilter in $B$ containing $F$. To see that $F'$ is $\omega_1$-complete, suppose $\{x_n : n \in \omega\} \subseteq F'$. To show that $\bigcap_{n\in\omega} x_n \in F'$ it is clearly sufficient to show that $j(\bigcap_{n\in\omega} x_n) = \bigcap_{n\in\omega} j(x_n)$, i.e. $j(\bigcap_{n\in\omega} g(n)) = \bigcap_{n\in\omega} j(g(n))$ where $g \in {}^{\omega}B$ is defined by $g(n) = x_n$ for each $n \in \omega$. Since ${}^{\omega}B$ is a subset of $M$, $g \in M$. Let $b = \bigcap_{n\in\omega} x_n$; then, since $M$ is transitive, we have

$$\langle M, \varepsilon\rangle \models \forall x\, [x \in b \leftrightarrow \forall n \in \omega\, [x \in g(n)]]$$

so that, since $j$ is elementary,

$$\langle N, \varepsilon\rangle \models \forall x\, [x \in j(b) \leftrightarrow \forall\, x \in j(\omega)\, [x \in j(g)(n)]].$$

But since $N$ is transitive we must have $j(n) = n$ for each $n \in \omega$ and $j(\omega) = \omega$. (The natural numbers and $\omega$ are absolute.) Furthermore, by elementarity of $j$, we have

$$j(g)(n) = j(g)(j(n)) = j(g(n)),$$

for each $n \in \omega$. Hence

$$\langle N, \varepsilon\rangle \models \forall x\, [x \in j(b) \leftrightarrow \forall n \in \omega\, [x \in j(g(n))]].$$

By transitivity of $N$, the statement on the right holds absolutely i.e.

$j(b) = \bigcap_{n\in\omega} j(g(n))$.

$\omega_1$-completeness of $F'$ follows.

(vi) $\Rightarrow$ (i). Assume (vi), and let $\Sigma = \{\sigma_\xi : \xi < \varkappa\}$ be a $\varkappa$-consistent set of sentences of $L_{\omega_1\omega_1}$ of power $\varkappa$. By introducing (infinitary) Skolem functions we may suppose without loss of generality that $\Sigma$ is a set of universal sentences. For each $\Delta \in S_\varkappa(\varkappa)$ (here $S_\varkappa(\varkappa)$ is the family of subsets of $\varkappa$ of power $< \varkappa$) let $\mathfrak{A}_\Delta$ be a model of $\{\sigma_\xi : \xi \in \Delta\}$ in which each of the new function symbols denotes a function. Let $D$ be a subset of the domain of $\Pi\{\mathfrak{A}_\Delta : \Delta \in S_\varkappa(\varkappa)\}$ of power $\varkappa$ which is closed under the denotations of the Skolem functions (defined pointwise) in this product. For each open formula $\varphi(v_\xi)_{\xi<\alpha}$ with $\alpha < \omega_1$ of $L\omega_1\omega_1$ and each sequence $f = \langle f_\xi : \xi < \alpha\rangle$ from $D$ let

$$J_{\varphi,f} = \{\Delta \in S_\varkappa(\varkappa) : \mathfrak{A}_\Delta \models \varphi\, [\langle f_\xi(\Delta) : \xi < \alpha\rangle]\}.$$

And for each $\Delta \in S_\varkappa(\varkappa)$ let $\hat{\Delta} = \{\Delta' \in S_\varkappa(\varkappa) : \Delta \subseteq \Delta'\}$. Let $B$ the be $\varkappa$-complete field of subsets of $S_\varkappa(\varkappa)$ generated by all $\hat{\Delta}$ and all $J_{\varphi,f}$. The Card $(B)$ = Card $(S_\varkappa(\varkappa)) = \varkappa$. Now $\{\hat{\Delta} : \Delta \in S_\varkappa(\varkappa)\}$ has the $\varkappa$-intersection property and by (vi) the ultrafilter $D$ on a measurable cardinal. Also, it has been pointed out by P. Aczel that, by contrast, $ZF$ + "there exists a strongly compact cardinal" + "there exist arbitrarily large inaccessibles" $\vdash \neg P$.

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