SETS AND CLASSES AS MANY

by

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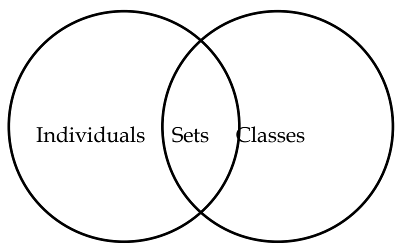
INTRODUCTION

Set theory is sometimes formulated by starting with two sorts of entities called *individuals* and *classes,* and then defining a *set* to be a *class as one*, that is, a *class which is at the same time an individual*, as indicated in the diagram:



If on the other hand we insist—as we shall here—that classes are to be taken in the sense of *multitudes, pluralities,* or *classes as many,* then no class can be an individual and so, in particular, the concept of set will need to be redefined. Here by "class as many" we have in mind what Erik Stenius refers to in [5] as *set of,* which he defines as follows:

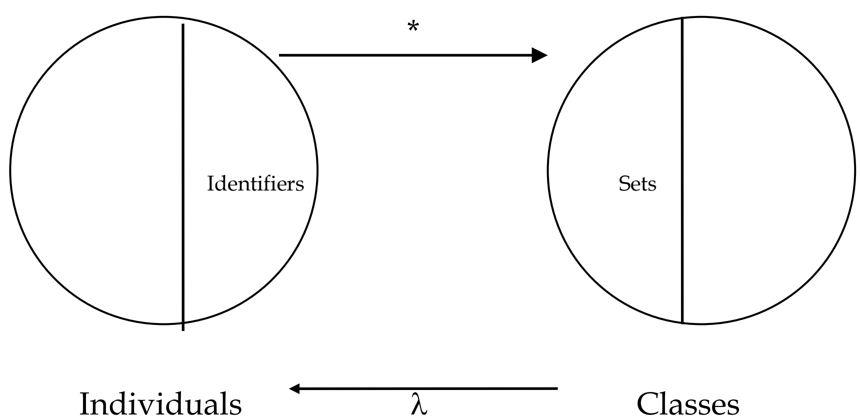
> *If we start from a Universe of Discourse given in advance, then we may define a set-of things as being many things in this UoD or just one thing - or even no things, if we want to introduce this way of speaking.*

Stenius draws a sharp distinction between this concept and that of *set as a thing*. As he says,

> *The distinction between a set-as-a-thing and a set of corresponds to the Russellian distinction between a "class as one" and a "class as many"* (Principles of Mathematics, p. 76). *Only I use the expressions 'set-as-a-thing" and "set-of" instead, in order to stress the (grammatical and)* ontological *character of the distinction; and also because of the difficulty that a set-of need not consist of many things; it can comprise just one thing or no things. But of course it is important that if a set-of comprises many things, it is a set as many and not as one, and that a set-as-a-thing is one and not many.*

Now while we shall require a set to be a class of *some* kind, construing the class concept as "class as many" entails that sets can no longer *literally* be taken as individuals. So instead we shall take sets to be classes that are represented, or *labelled,* by individuals in an appropriate way. For simplicity we shall suppose that labels are attached, not just to sets, but to all classes: thus each class X will be assigned an individual λX called its *label.* Now in view of Cantor's theorem that the number of classes of individuals exceeds the number of individuals, it is not possible for different classes always to be assigned distinct labels. This being the case, we single out a subdomain S of the domain of classes on which the labelling map λ is one-to-one. The classes falling under S will be identified as *sets*; and an individual which is the label of a set will be called an *identifier.*

For reasons of symmetry, it will be convenient (although not strictly necessary) to assume that, in addition to the operation of labelling each class by an individual, there is a reverse process—*colabelling*—which assigns a class to each individual. Thus we shall suppose that to each individual x there corresponds a unique class x^* called its *colabel*. Again, because of Cantor's theorem, not every class can be the colabel of an individual (although every individual can be the label of a class). However, it seems natural enough to stipulate that each *set* be the colabel of some individual, and indeed that this individual may be taken to be the label of the set in question. Thus we shall require that $X = \lambda(X)^*$ for every set X. In that event, for any identifier x in the above sense, we shall have $x = \lambda(x^*)$; that is, the colabel of an identifier is the set of which it is the label, or the set *labelled* by the identifier. Another way of putting this is to say that the restriction of the colabelling map to identifiers acts as an inverse to the restriction of the labelling map to sets. So the above diagram is to be replaced by the diagram:



Singletons and the *empty class*—"multitudes" with just one, or no members respectively—are here regarded, like the "numbers" 1 and 0, as "ideal" entities introduced to enable the theory to be developed smoothly. (It may help to think of singletons as "clubs with one member" and the empty set as a club all of whose members are deceased.) Thus, as is customary in standard set theory, we distinguish between an individual a and the singleton-class $\{a\}$ whose only member is a. But, of course, there is a difference, for while in standard set theory $\{a\}$ is, like a, merely another individual which could, in principle at least, coincide with a (and in nonwellfounded set theories sometimes does), in the theory to be developed here $\{a\}$ and a, being of different sorts, can *never* coincide.

The *membership relation* $\in$ between individuals and classes is a primitive of our system. It will be taken as an *objective* relation in the sense suggested, for example, by the assertion that Lazare Carnot was a member of the Committee of Public Safety, or Polaris is a member of the constellation Ursa Minor. The fact that $\in$ is not iterable—there are no "$\in$-chains"—means that it can have very few intrinsic properties. This is to be contrasted with the relation ε of "membership" between *individuals*, defined by $x \mathrel{\varepsilon} y \leftrightarrow x \in y^*$: x is a member of the class *labelled by* y. This relation links entities of the same sort and is, accordingly, iterable. It should be noted, however, that the presence of the colabelling map * in the definition of ε gives the latter a purely formal, arbitrary character.

The formal character of ε renders the scheme to be put forward here particularly apposite for presenting *nonwellfounded* set theories. In the usual set theories it is difficult to grasp the nature of a set which is, for example, identical with its own singleton since such a set cannot be "formed" by assembling individuals. In the present scheme, on the other hand, the assertion $a = \{a\}$—which is, as remarked above, not well-formed—is replaced by the assertion

$$\forall x(x \,\varepsilon\, a \leftrightarrow x = a),$$

that is,

$$a^* = \{a\},$$

which asserts that $\{a\}$ is identical, not with a itself, but rather with its colabel. Similarly, the self-membership assertion $a \in a$ is transformed into the statement

$$a \,\varepsilon\, a,$$

that is,

$$a \in a^*,$$

which asserts that a belongs, not to itself, but merely to its colabel. And an assertion of cyclic membership $a \in b \in a$ is transformed into the assertion

$$a \,\varepsilon\, b \,\varepsilon\, a,$$

or

$$a \in b^* \;\&\; b \in a^*,$$

that is, "a (resp. b) is a member of the colabel of b (resp. a)." These rephrasings appear much more natural in that they only impute the possession of curious properties to the colabelling map, rather than to the objective membership relation $\in$ itself.

We shall also see that, in addition to nonwellfounded set theories, a number of other theories familiar from the literature can be provided with natural formulations within the system to be presented here. These include second-order arithmetic, the set theories of Zermelo-Fraenkel, Morse-Kelley, and Ackermann, as well as a system in which Frege's construction of the natural numbers can be carried out. Each of these theories can therefore be seen as the result of imposing a particular condition on a common apparatus of labelling classes by individuals.

1. THE THEORY M

The theory **M** of *multitudes* or *classes as many* will be presented informally, employing the usual logical notation. It will be evident, however, that the theory can be readily formalized in a two-sorted first-order language L.

We assume that we are given two distinct *sorts* of entity, *individuals,* and *classes* (or *multitudes*). We use lower-case letters $a, b, c, x, y, z, \ldots$ to denote individuals, and upper-case letters $A, B, C, X, Y, Z, \ldots$ to denote classes. Letters from the end of the alphabet will normally be used as variables ranging over the domain of entities of the appropriate sort.

We assume given a relation $\in$ between individuals and classes called *membership*: thus, for an individual a and a class A, we write

$$a \in A,$$

which is to be read, as usual, *a belongs to A,* or *a* is an *element*, or a *member,* of *A*. Note the fact that an assertion of the form $\alpha \in \beta$ is meaningful only when α denotes an individual an β a class.

We assume that the equality relation = is defined on each sort of entity, so that an assertion of the form $\alpha = \beta$ is meaningful only when α and β denote entities of the same sort.

Equality of *classes* is governed by the

Axiom of Extensionality

$$\forall X \forall Y[X = Y \leftrightarrow \forall x(x \in X \leftrightarrow x \in Y)].$$

This expresses the idea that classes are uniquely determined by their elements.

Assigned to each class X is an individual $\lambda(X)$ or λX called its *label* and to each individual x a class x^* called its *colabel.* We also assume the presence of a pair of predicates I, S defined on individuals and classes respectively: $I(x)$ is to be read "x is an *identifier*" and $S(X)$ "X is a *set.*" These notions are subject to the

Labelling Axioms.

$$S(X) \to I(\lambda X),\ \ I(x) \to S(x^*),\ S(X) \to (\lambda X)^* = X,\ \ I(x) \to \lambda(x^*) = x.$$

The following assertions are then immediate consequences:

$$S(X) \wedge S(Y) \wedge \lambda X = \lambda Y \to X = Y, \quad I(x) \wedge I(y) \wedge x^* = y^* \to x = y.$$

By a *property* of individuals x, we mean any formula $\alpha(x)$ in which the variable x is free in the two-sorted first-order language whose nonlogical symbols are $\in$, I, S, λ, $*$. The formula α may contain other free individual or class variables: these are called *parameters.* Associated with each property $\alpha(x)$ is a class

$$\{x: \alpha(x)\}$$

called the class *defined* by α or the class of individuals *satisfying* α. The notion of property is enlarged in the obvious way to include expressions of the form $\{x: \alpha(x)\}$, and with any property $\varphi(x)$ in this enlarged sense we also associate a class $\{x: \varphi(x)\}$. Providing this concept of property and associated class with a precise recursive definition is a straightforward matter.

Classes defined by properties are governed by the

Axiom of Comprehension.

$$y \in \{x: \varphi(x)\} \leftrightarrow \varphi(y).$$

From this and the Axiom of Extensionality one deduces immediately:

$$\{x: \varphi(x)\} = \{y: \psi(y)\} \leftrightarrow \forall x[\varphi(x) \leftrightarrow \psi(x)], \qquad \forall X\ X = \{x: x \in X\}.$$

We now define **M** to be the theory in L based on the axioms of extensionality, labelling and comprehension.

In order to be able to handle *relations* in our framework it will later prove convenient to introduce *ordered pairs* into L. Thus we define $L^{(,)}$ to be the language obtained by adjoining to L a binary term (,) taking pairs of individuals to individuals, and $\mathbf{M}^{(,)}$ to be the theory obtained from **M** by adding the axiom

Ordered Pairs

$$\forall x \forall y \forall u \forall v[(x, y) = (u, v) \leftrightarrow x = u \wedge y = v].$$

Taking an ordered pair of individuals to be an individual might seem to be in conflict with the maintenance of a strict distinction between Aindividuals and classes. However, the concept of "ordered pair" introduced here is intended not to connote the set-theoretic notion but rather that of coordinate geometry, in which an "ordered pair" is a *single* point in the Cartesian plane representing a pair of points selected from the coordinate axes in a prescribed order.

The usual classes and relations and operations thereon can now be defined in L:

$V = \{x \colon x = x\}$

$\varnothing = \{x \colon x \neq x\}$

$\{a\} = \{x \colon x = a\}$

$\{a_1, \ldots, a_n\} = \{x \colon x = a_1 \vee \ldots \vee x = a_n\}$

$\{x \in X \colon \varphi(x)\} = \{x \colon x \in X \wedge \varphi(x)\}$

$X \cup Y = \{x \colon x \in X \vee x \in Y\}$

$X \cap Y = \{x \colon x \in X \wedge x \in Y\}$

$X - Y = \{x \in X \colon x \notin Y\}$

$-X = V - X$

$X \subseteq Y \leftrightarrow \forall x(x \in X \rightarrow x \in Y)$.

In $L^{(,)}$ one makes the following additional definitions:

$X \times Y = \{z \colon \exists \mathrm{x} \exists \mathrm{y}(x \in X \wedge y \in Y \wedge z = (x, y))\}$,

R is a *relation* $\leftrightarrow \exists X \exists Y \; R \subseteq X \times Y$

R is a *relation on* $X \leftrightarrow R \subseteq X \times X$

$xRy \leftrightarrow (x, y) \in R$

$\mathrm{Field}(R) = \{x \colon \exists y(yRx \vee xRy\}$,

$R_x = \{y \colon x \neq y \wedge yRx\}$,

f is a *function* from X to Y, $f\colon X \to Y \;\leftrightarrow$

$$f \subseteq X \times Y \wedge \forall x{\in}X \exists! y{\in}Y\, (x, y) \in f$$

We also make the following further definitions:

$I = \{x\colon I(x)\}$

$x \mathrel{\varepsilon} y \leftrightarrow x \in y^*$

$[\![x\colon \varphi(x)]\!] = \lambda\{x\colon \varphi(x)\}$

$[\![a]\!] = \lambda\{a\}$

$[\![a_1, \ldots, a_n]\!] = \lambda\{a_1, \ldots, a_n\}$

$a \cup b = [\![x\colon x \mathrel{\varepsilon} a \vee x \mathrel{\varepsilon} b]\!]$

$a \cap b = [\![x\colon x \mathrel{\varepsilon} a \wedge x \mathrel{\varepsilon} b]\!]$

$\bigcup a = [\![x\colon \exists y(y \mathrel{\varepsilon} a \wedge x \mathrel{\varepsilon} y)]\!]$

$0 = \lambda\varnothing$

$a \subseteq b \leftrightarrow \forall x(x \mathrel{\varepsilon} a \to x \mathrel{\varepsilon} b)$

$Pa = [\![x\colon I(x) \wedge x \subseteq a]\!]$

The defined relation ε is the *membership relation on individuals*. It follows easily from the comprehension and labelling axioms that

$$I(x) \to x = [\![y\colon y \mathrel{\varepsilon} x]\!]$$

Since λ is one-to-one on sets, an informal application of Cantor's theorem shows that not every class can be a set, and so not every individual can be an identifier. We now define classes—the *Russell classes* and *the least inclusive class*—for which this can be explicitly demonstrated.

There are two Russell classes, namely,

$$R_1 = \{x: x \notin x^*\}, \quad R_2 = \{x: I(x) \wedge x \notin x^*\};$$

clearly $R_2 \subseteq R_1$. A class X is said to be *inclusive*, and we write $Inc(X)$, if $\forall x[I(x) \wedge x^* \subseteq X \rightarrow x \in X]$.

The *least* inclusive class E is defined by

$$E = \{x: \forall X[Inc(X) \rightarrow x \in X]\}.$$

It is easily shown that R_2 is inclusive, so that $E \subseteq R_2 \subseteq I$. Moreover, we have the

Theorem. *None of* R_1, R_2 *or* E *is a set.*

Proof. Suppose that $S(R_1)$. Then $\lambda(R_1)^* = R_1$ so that

$$\forall x[x \in (\lambda R_1)^* \leftrightarrow x \in R_1 \leftrightarrow x \notin x^*].$$

Hence

$$\lambda R_1 \in (\lambda R_1)^* \leftrightarrow \lambda R_1 \notin (\lambda R_1)^*,$$

a contradiction. The argument for $\neg S(R_2)$ is similar.

Finally suppose that $S(E)$; then, writing $e = \lambda E$, we have $e^* = E$. Now

$$Inc(X) \rightarrow E \subseteq X \rightarrow e^* \subseteq X \rightarrow e \in X.$$

Therefore $e \in E = e^*$. But this contradicts the observation above that $E \subseteq R_2$. It follows that $\neg S(E)$. ■

In fact E is the *well-founded* universe, and can be shown to be, with the operation of unions and singleton (and assuming the singleton and union axioms), the initial Zermelo-Fraenkel algebra in the sense of Joyal and Moerdijk.

If we define an individual a to be *transitive* —Trans(a)— provided $\forall x\ \forall y\ [x\ \varepsilon\ a \wedge y\ \varepsilon\ x \rightarrow y\ \varepsilon\ a]$, and a class X to be *transitively inclusive* – Inctran(X)—provided that $\forall x[I(x) \wedge \text{Trans}(x) \wedge x^* \subseteq X \rightarrow x \in X]$, then the class ORD of von Neumann ordinals may be defined as the least transitively inclusive class, that is,

$$\text{ORD} = \{x\colon \forall X[\text{Inctrans}\ (X) \rightarrow x \in X]\}.$$

Again, Ord, together with the operation of union and ordinal successor, may be shown to be the initial ZF algebra with an inflationary successor.

All these arguments may be carried out constructively.

2. EXTENSIONS OF M.

We next consider extensions of **M** in which the existence of sets, or, equivalently, of identifiers, is ensured.

The first and way of extending **M** so as to ensure set existence is simply to introduce the *Zermelo axioms* which we will take in the form of postulating the existence of suitable identifiers. This leads to the system **ZM**, which is obtained by adding to **M** the axioms

Empty set $\quad I(0)$

Singletons $\quad \forall x[I(x) \rightarrow I([\![x]\!])]$

Union $\forall x[I(x) \rightarrow I(\bigcup x)]$

Power set $\forall x[I(x) \rightarrow I(Px)]$

Separation $\forall x[I(x) \rightarrow I([\![x\colon \varphi(x)]\!]]$

Infinity $\exists u[I(u) \wedge 0 \ \varepsilon\ u \wedge \forall x \in u\ x \cup [\![x]\!] \varepsilon\ u]$

The theory **ZFM** is obtained by adding to **ZM** the axiom scheme of

Replacement $\forall u[I(u) \wedge \forall x \varepsilon u \exists ! y[I(y) \wedge \varphi(x, y)] \rightarrow$
$\exists v[I(v) \wedge \forall y[y \ \varepsilon\ v \leftrightarrow I(y) \wedge \varphi(x, y)]]]$

The theories **ZM*, ZFM*** are obtained by adding respectively to **ZM, ZFM** the

Axiom of Foundation $\forall u[I(u) \wedge u \neq 0 \rightarrow \exists x \varepsilon u[I(x) \wedge x \cap u = 0]$.

It is not difficult to show that each of the theories **ZM, ZFM, ZM*, ZFM*** is equiconsistent with its respective counterpart **Z** (Zermelo set theory), **ZF** (Zermelo-Fraenkel set theory), **Z*** (**Z** with foundation), **ZF*** (**ZF** with foundation). Consider, for example, **ZM**. We can interpret the language L_Z of **Z** in the language L of **M** by construing the individual variables of L_Z as ranging over the class $\{x\colon I(x)\}$ and the membership relation of L_Z as ε. Clearly the interpretations of the axioms of **Z** then become provable in **ZM.**

Conversely, suppose given a model M = (M, E) of **Z** in the usual set-theoretic sense. Choose some object k such that $k \notin M$ and let $M^+ = M \cup \{k\}$. We obtain a model of **ZM** by interpreting

Individuals of **M** .... as .... elements of M^+

Classes of **M** .... subsets of M^+

$\in$ .... set membership in the usual sense

$I(x)$ .... $x \in M$

$S(X)$ .... $X = \{x: xEu\} = u$^ for some $u \in M$

* .... the map $u \mapsto u$^ for $u \in M$, $k \mapsto M$

λ .... the map $PM^+ \to M^+$ defined by

$$\lambda(X) = \begin{cases} u & \text{if } X = u\text{\^{} for some } u \in M \\ k & \text{if not} \end{cases}$$

$\{x: \varphi(x)\}$ .... set of members of M satisfying the interpretation of φ

We note that, if M is a standard model, then u^ = u for $u \in M$, and so we recapture the classical picture in which the labelling and colabelling maps are both the identity on sets.

The same procedure works for the remaining theories.

In a similar manner **M** can be extended to yield theories equiconsistent, respectively, with Gödel-Bernays set theory and Morse-Kelley set theory **MK.**

We outline the procedure for the latter. The theory **MKM** in L is obtained by adding to **ZFM** the following axioms:

Axiom of faithful colabelling $\forall x\, \lambda(x^*) = x$.

Axiom of individual extensionality $\forall u \forall v[u = v \leftrightarrow \forall x[x \,\varepsilon\, u \leftrightarrow x \,\varepsilon\, v]]$.

Axiom of membership $\forall u[I(u) \leftrightarrow \exists v\, u \,\varepsilon\, v]$.

Axiom of individual comprehension $\exists v \forall x[I(x) \to [x \,\varepsilon\, v \leftrightarrow \varphi(x)]$,
where φ is any formula containing only *individual* variables.

The language L_{MK} of **MK** can be interpreted in L by construing the (class) variables of L_{MK} as ranging over the class V of L, and construing the membership relation of L_{MK} as ε. It is now easily shown that the interpretation of the axioms of **MK** become provable in **MKM.** Conversely, suppose given a model (M, E) of **MK** in the usual set-theoretic sense. Choose an object $k \notin M$, let $M^+ = M \cup \{k\}$ and $M' = \{x \in M: \exists u\; xEu\}$. Then one obtains a model of **MKM** by interpreting

Individuals of **M** .... as .... elements of M

Classes of **M** .... subsets of M^+

$\in$.. set membership in the usual sense

$I(x)$ .... $x \in M'$

$S(X)$ .... $X = \{x: xEu\} = u^\wedge$ for some $u \in M'$

$*$ .... the map $u \mapsto u\hat{}$ for $u \in M$, $k \mapsto M$

λ .... the map $PM^+ \to M^+$ defined by

$$\lambda(X) = \begin{cases} u & \text{if } X = u\hat{} \text{ for some } u \in M \\ k & \text{if not} \end{cases}$$

$\{x: \varphi(x)\}$ .... set of members of M satisfying the interpretation of φ

A more interesting—and quite natural—possibility is to admit as sets in **M** all those classes whose defining properties are *independent of the labelling apparatus.* Thus, calling a property *pure* if it contains no occurrence of the symbols I, S, λ or $*$, we introduce the

Axiom Scheme of Purity (P)

$$S(\{x: \varphi(x)\}) \quad \text{for all pure } \varphi.$$

This may be equivalently stated (in the presence of the labelling axioms) as $I([\![x: \varphi(x)]\!])$ for all pure φ. We denote by **MP** the theory obtained by adjoining **P** to **M**.

We observe that in **MP** the empty class, any finite or cofinite class, and the universal class V are all sets. It follows from this last fact that the axiom of separation is refutable in MP.

MP is interpretable in (the standard model of) *second-order arithmetic,* where by the latter we mean the second order theory **SOA** based on Peano's axioms for the natural numbers with the axiom of induction in the form

$$\forall X[0 \in X \wedge \forall x[x \in X \rightarrow x + 1 \in X] \rightarrow \forall x\, x \in X],$$

together with the axioms of extensionality

$$\forall X\, \forall Y[\forall x[x \in X \leftrightarrow x \in Y] \rightarrow X = Y]$$

and comprehension

$$\exists X \forall x[x \in X \leftrightarrow \varphi(x)],$$

where $\varphi(x)$ is an arbitrary formula not containing X.

An interpretation of **MP** in the standard model ($\mathbb{N}$, $P\mathbb{N}$, $s, \in$) of **SOA** may now be set up in the following way. Let A be the collection of analytic (second-order definable) subsets of $\mathbb{N}$, and let κ: A $\rightarrow \mathbb{N}$ be a one-one coding map on a whose range does not exhaust the whole of $\mathbb{N}$; fix $n_0 \in \mathbb{N}$ – range(κ) and $U_0 \in P\mathbb{N}$ - A. Now a model of **MP** is obtained by interpreting

Individuals of **M** .... as .... elements of $\mathbb{N}$

Classes of **M** .... subsets of $\mathbb{N}$

$\in$... $\in$

$I(x)$ .... $x \in$ range(κ)

$S(X)$ .... $X \in$ A

λ .... the map $P\mathbb{N} \to \mathbb{N}$ defined by

$$\lambda(X) = \kappa(X) \text{ if } \zeta \in A$$
$$= n_0 \text{ if not}$$

* .... the map $\mathbb{N} \to P\mathbb{N}$ defined by

$$n^* = \kappa^{-1}(n) \text{ if } n \in \text{range}(\kappa)$$
$$= U_0 \text{ if not}$$

$\{x: \varphi(x)\}$ .... set of members of $\mathbb{N}$ satisfying the interpretation of φ

Conversely, we may strengthen **MP** to a theory **MP$^+$** which is of the same proof-theoretic strength as second-order arithmetic. To the language L of **M** add a new unary function symbol s (‘successor”) of signature Individuals $\to$ Individuals. Enlarge the concept of property to include formulas in which the symbol s occurs. The theory **MP$^+$** is obtained by adding to **MP** the Peano axioms stated in L in the natural way, namely

$\forall x\, 0 \neq sx$,

$\forall x \forall y[sx = sy \to x = y]$,

$\forall X[0 \in X \wedge \forall x[x \in X \to sx \in X] \to \forall x\, x \in X]$.

The interpretation of **MP** in the standard model of **SOA** given above extends readily to **MP$^+$**. And clearly **SOA** is a subtheory of **MP$^+$**. Thus **MP$^+$** may be considered a version of second-order arithmetic in which classes correspond to arbitrary sets of integers, sets to analytic sets, and identifiers to code numbers of these latter.

Another extension of **M** is related to *Ackermann set theory*. As we recall from [3] and [4], this is the theory **A** formulated in a first-order language $L_{\mathbf{A}}$ with a unary predicate symbol M and a binary predicate symbol $\in$. The variables of $L_{\mathbf{A}}$ are called *class variables* and will be denoted by capital letters. $M(X)$ is read "X is a *set*"; lower-case variables will be used for sets.

The axioms of **A** are:

Extensionality $\forall X(X \in A \leftrightarrow X \in B) \rightarrow A = B$

Class Comprehension $\exists X \forall x[x \in X \leftrightarrow \varphi(x)]$, where φ is any formula

Completeness $\forall X \forall x[X \in x \vee X \subseteq x \rightarrow M(X)]$

Set Comprehension $\forall x_1 ... \forall x_n[\forall X[\alpha(X) \rightarrow M(X)] \rightarrow \exists w \forall X[X \in w \leftrightarrow \alpha(X)]]$,
where α is any formula having just the free variables $X, x_1, ..., x_n$ and in which M *does not occur*

The theory **A*** is obtained by adding to **A** the *axiom of foundation* for sets. It is known ([4]) that **ZF** and **A*** are equivalent theories in the sense that the sentences involving just set variables provable in **A*** coincide with the theorems of **ZF**.

A *natural model* of **A** or **A*** is a model of the form $(R_\beta, R_\alpha, \in)$ in which R_β is the set of sets of rank $< \beta$, and R_α, with $\alpha < \beta$, is the interpretation of the predicate M. It is known ([4]) that in such a model α is not definable in the structure $(R_\beta, \in)$, so that, in particular, $\beta \neq \alpha + n$ for any natural number n. This observation will be used tacitly below.

The extension of **M** corresponding most closely to **A** is the theory **MA**, which is obtained by adding the following axioms to **M**:

Faithful colabelling.

Heredity (H_1) $\forall x \forall y[I(y) \wedge x \,\varepsilon\, y \to I(x)]$

Heredity (H_2) $\forall X \forall Y[S(Y) \wedge X \subseteq Y \to S(X)]$

Simple Set Comprehension (SSC)

$$\forall x_1 \ldots \forall x_n[[I(x_1) \wedge \ldots \wedge I(x_n) \wedge \forall x[\varphi(x) \to I(x)]] \to S(\{x\colon \varphi(x)\})],$$

where φ is a *simple* formula, that is, has only individual free variables and no occurrences of *S, I* or λ.

The theory **MA*** is obtained from by adding the axiom of foundation to **MA**.

Any natural model $(R_\beta, R_\alpha, \in)$ of **A** can be converted into a model of **MA***

by interpreting

Individuals of **M** .... as .... elements of $R_{\alpha+1}$

Classes of **M** .... elements of R_β

$\in$ .... set membership in the usual sense

$I(x)$ .... $x \in R_\alpha$

$S(X)$ .... $X \in R_\alpha$

λ .... the map $R_\beta \to R_{\alpha+1}$ defined by

$$\lambda(u) = \begin{cases} u & \text{if } u \in R_{\alpha+1} \\ \varnothing & \text{if not} \end{cases}$$

* .... insertion map $R_{\alpha+1} \hookrightarrow R_\beta$,

$\{x\colon \varphi(x)\}$ .... set of elements of $R_{\alpha+1}$ satisfying the interpretation of φ.

Conversely, **A** (or **A***) is correctly interpretable in **MA** (or **MA***) by interpreting the class variables of $L_{\mathbf{A}}$ as class variables of L, M as S and $\in$ as the relation η, where $X \eta Y$ is defined to mean $\lambda X \in Y$. Thus **MA** and **A** are very close to being equiconsistent theories.

The (additional) axioms of **ZM** are derivable in **MA** in a way similar to that in which the axioms of **Z** are proved in **A.** We give a couple of examples.

Union. Assume $I(a)$. The formula $\varphi(x, a) \equiv \forall y \in a^* \, x \in y^*$ is simple and the implication $\forall x\varphi(x, a) \rightarrow I(x)$ follows from $\mathbf{H_1}$. So **SSC** gives $S(\{x: \varphi(x, a)\})$, whence $I(⟦x: \varphi(x, a)⟧)$ by Labelling. But clearly $⟦x: \varphi(x, a)⟧ = \bigcup a$.

Infinity. Let $\varphi(X)$ be the formula

$$\exists u{\in}X \forall x(x \notin u^*) \wedge \forall x{\in}X \exists u{\in}X\, \forall y(y \in u^* \leftrightarrow y = x \vee y \in x^*),$$

and $\alpha(x)$ the simple formula $\forall X[\varphi(X) \rightarrow x \in X]$. It follows from the empty set, singleton, and union axioms (the first two of which are presumed to have already been verified in a way similar to that of the third above) that $\varphi(I)$ holds. From this we deduce that $\forall x[\alpha(x) \rightarrow I(x)]$, and accordingly $S(\{x: \alpha(x)\})$ by **SSC.** Writing $A = \{x: \alpha(x)\}$, it quickly follows that

$$0 \in A \wedge \forall x{\in}A(x \cup ⟦x⟧ \in A).$$

Therefore, writing $a = \lambda A$, Labelling gives $I(a)$ and

$$0 \,\varepsilon\, a \wedge \forall x \varepsilon a(x \cup ⟦x⟧ \,\varepsilon\, a).$$

Infinity follows.

3.TYPE-REDUCING CORRESPONDENCES IN M

In [2] a system F of many-sorted first-order logic is introduced with the purpose of providing a common framework for presenting Frege's construction of the natural numbers and Zermelo's proof of the axiom of choice from the well-ordering theorem. In F a pivotal role is played by the notion of *type-reducing correspondence*, that is, a correspondence between second-level and first-level entities. Now the term λ, as a map from classes to individuals, is a type-reducing correspondence in this sense, and so the results derived for F can be adapted to our present framework. One such result is the *Zermelo-Bourbaki Lemma* (Lemma 2.1 0f [2]), which asserts that each type-reducing correspondence gives rise to a well-ordering defined on a collection of entities of lowest type. The statement and proof of the result is easily adapted to $\mathbf{M}^{(\iota)}$ where it takes the form of the

Zermelo-Bourbaki Lemma in $\mathbf{M}^{(\iota)}$. *Given any term τ taking classes to individuals, and any predicate Q defined on classes, there is a relation R such that R is a well-ordering and, writing M =* Field(R),

$$\forall x[x \in M \to Q(R_x) \wedge \tau(R_x) = x]$$

$$Q(M) \to \tau(M) \in M.$$

Here the term "well-ordering" has its usual meaning, readily formulated in $\mathrm{L}^{(\cdot)}$.

When τ is the labelling term λ, one finds, as in '4 of [2], that M is the class ORD of von Neumann ordinals (defined in $\mathrm{L}^{(\cdot)}$ in the usual way), R is the membership relation ε on individuals, and that ORD is not a set. The arguments of '4 of [2] also adapt easily to show that ORD is nonempty iff the empty set axiom holds, that the axioms of union and singletons jointly ensure that ORD is unbounded, and that these, together with the axiom of infinity, imply that the natural numbers defined as a subclass of ORD in the usual way constitute a set.

In [2] other type-reducing correspondences are introduced which can also be considered in $\mathbf{M}^{(\cdot)}$. For example, suppose to $\mathrm{L}^{(\cdot)}$ we add a type-reducing correspondence σ and to $\mathbf{M}^{(\cdot)}$ the axiom

$$\forall X[X \neq \varnothing \to \sigma(X) \in X]:$$

σ is accordingly a *choice term.* Then, as shown in Corollary 2.2 of [2], the corresponding well-ordering has field V, the universal class.

Another possibility is to add a type-reducing correspondence ν together with the axiom

$$\forall X \forall Y[\nu(X) = \nu(Y) \leftrightarrow X \approx Y].$$

Here $\approx$ stands for "X and Y are *equinumerous*", that is, "there is a bijection between X and Y", a notion readily formulated in $L^{(,)}$. In this extension of $\mathbf{M}^{(,)}$ one can prove, as in '3 of [2],

Frege's Theorem. *There is a Dedekind-infinite class.*

And from this, one goes on to prove the existence of the natural numbers. It should be noted, however, that the Dedekind-infinite class whose existence is asserted here (nor the associated class of natural numbers) does not necessarily have to be a *set.* In fact, the theorem can still hold when no sets or identifiers exist at all.

We mention here that the definability of the equinumerosity relation $\approx$ in $L^{(,)}$ yields another way of defining the set predicate S. For, as observed in '5 of [2], we may introduce the following version of the *von Neumann maximization principle*, namely

VN $$\forall X[S(X) \leftrightarrow \neg X \approx V].$$

From this one derives—as in '5 of [2]—

- The *well-ordering principle*: V can be well-ordered.

- The *second heredity axiom* $\mathbf{H_2}$.
- The *weak axiom of replacement for classes*: $S(X) \wedge X \approx Y \rightarrow S(Y)$.

The consistency of **VN** with $\mathbf{M}^{()}$ is readily established through a straightforward adaptation of the model given in '5 of [2].

4. NONWELLFOUNDED SET THEORIES IN M

As presented in [1], a *nonwellfounded set theory* is a consistent extension of **ZF** in which the axiom of foundation is refutable. Within the framework of this paper the most suitable theory for developing nonwellfounded set theories is $\mathbf{M}^{()}$ because there we have available the notion of function between classes. The definitions of [1] are easily adapted to $\mathbf{M}^{()}$.

Thus we define an *accessible pointed graph* (*APG*) to be a triple E = (E, R, e) in which E is a set, R is a binary relation on E, and $e \in E$, such that for any $x \in E$ there is a finite subclass $\{x_0,..., x_n\}$ of E such that $x_0 = x$, $x_n = e$ and x_iRx_{i+1} for $i = 0, ..., n - 1$. A *decoration* of E is a function $d: E \rightarrow I$ such that, for all $x \in E$,

(*) $$d(x) = [\![d(y): yRx]\!].$$

In this case E is called a *picture* of the identifier $d(e)$.

The **Antifoundation Axiom** is the assertion

AFA: *every APG is the picture of a unique identifier.*

This may be regarded as asserting that the relation ε of membership on identifiers is *universal* in a certain sense. For by labelling (*) can be written

$$d(x)^* = \{d(y): yRx\}$$

or

$$yRx \rightarrow d(y) \in d(x)^* \leftrightarrow d(y)\ \varepsilon\ d(x).$$

In other words, when the elements of E are replaced by their images under d, the relation R is transformed into ε. **AFA** asserts that *any* such relation can be transformed into ε in this way.

In [1] the concept of *complete system* is introduced, and such systems shown to be models of **ZF** together with the antifoundation axiom. As we shall see, they can be converted into models of $\mathbf{M}^{(\cdot)}$ **+ AFA.**

Let U = (U, ∈) be a transitive model of **ZF** which we will regard as fixed. By a U-*system* we mean a pair C = (C, J) with $C \subseteq U$, and J a binary relation on C such that, for each $a \in C$, the set $a_C = \{x: xJa\}$ is a member of U. C is *full* if, for any $x \subseteq C$ such that $x \in U$, there is a unique $a \in C$ for which $x = a_C$. C is *complete* if each accessible pointed graph E = (E, R, e) ∈ U has a unique C-decoration, that is, there is a unique map $f: E \rightarrow C$ in U such that $f(x)_C = \{f(y): yRx\}$. It is shown in [1] that every complete system is both full and a model of **ZF + AFA.** It is also shown, in essence, that a complete system can be constructed from any standard model of **ZFC** (= **ZF** + the axiom of choice) so that **ZF + AFA** is consistent.

We show finally that each complete system can be converted into a model of $\mathrm{M}^{(\cdot)}$ + **AFA** so that this theory is also consistent. In fact, given a standard model U = $(U, \in)$ of **ZF**, and a complete U-system C = (C, J), define an interpretation of $\mathrm{L}^{(\cdot)}$ as follows. Choose some $a_0 \notin C$, let $C' = C \cup \{a_0\}$ and interpret

Individuals of **M** .... as .... elements of C'

Classes of **M** .... subsets of C

$\in$... set membership in the usual sense

$I(x)$ .... $x \in C$

$S(X)$ .... $X \in P(C) \cap U$

λ .... the map $P(C') \to C'$ defined by

$$\lambda(u) = \begin{cases} a & \text{if } u = a_C \text{ for some } a \in C \\ a_0 & \text{if not} \end{cases}$$

* .... map $C' \to P(C')$ defined by $a^* = a_C$ if $a \in C$, $a_0^* = \{a_0\}$

$\{x: \varphi(x)\}$ .... set of elements of U' satisfying the interpretation of φ.

It is now not hard to check that all the axioms of $\mathrm{M}^{(\cdot)}$ + **AFA** are true under this interpretation. Note that in this interpretation the “objective” membership relation is unaffected; it is the choice of the colabelling map which ensures the truth of the antifoundation axiom.

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