THE CONTINUUM IN SMOOTH INFINITESIMAL ANALYSIS

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Abstract: In this paper an investigation is made of the properties of the continuum in smooth infinitesimal analysis: it is shown that it differs in certain important respects from its counterpart in constructive analysis.

As presented in [1], *smooth infinitesimal analysis*, SIA, is a theory formulated within higher-order intuitionistic logic and based on the following axioms:

Axioms for the continuum, or smooth real line R. These are the usual axioms for a(n) (intuitionistic) field expressed in terms of two operations + and $\cdot$, and two distinguished elements $0, 1$.

Axioms for the strict order relation $<$ on R. These are:

1 $a < b$ and $b < c$ implies $a < c$.

2 $\neg(a < a)$.

3 $a < b$ implies $a + c < b + c$ for any c.

4 $a < b$ and $0 < c$ implies $a \cdot c < b \cdot c$.

5 either $0 < a$ or $a < 1$.

6 $a \neq b$[1] implies $a < b$ or $b < a$.

[1] Here $a \neq b$ stands for $\neg a = b$. It should be pointed out that axiom 6 is omitted in some presentations of SIA, e.g. those in [3] and [4].

P. Schuster, U. Berger and H. Osswald (eds.),
Reuniting the Antipodes – Constructive and Nonstandard Views of the Continuum, 19–24.

The relation $\leq$ on $\mathbf{R}$ is defined by $a \leq b \Leftrightarrow \neg b < a$. The open interval (a, b) and closed interval $[a, b]$ are defined as usual, viz. $(a, b) = \{ x : a < x < b \}$ and $[a, b] = \{ x : a \leq x \leq b \}$; similarly for half-open, half-closed, and unbounded intervals.

Write Δ for the subset $\{ x : x^2 = 0 \}$ of $\mathbf{R}$; we use the letter ε as a variable ranging over Δ. Elements of Δ are called (nilsquare) *infinitesimals* or *microquantities.* Since, clearly, $0 \in \Delta$, Δ may be regarded as an *infinitesimal neighbourhood of* 0. Δ is subject to the

Microaffineness Principle. *For any map* $g : \Delta \to \mathbf{R}$ *there exist unique* $a, b \in \mathbf{R}$ *such that, for all* ε, *we have*

$$g(\varepsilon) = a + b \cdot \varepsilon.$$

Notice that then $a = g(0)$.

From these three axioms it follows that the continuum in SIA differs in certain key respects from its counterpart in *constructive analysis* CA, which is furnished with an elegant axiomatization in [2].

To begin with, the third basic property of the strict ordering relation $<$ in CA, given as axiom R2(3) on p. 102 of [2], and which may be written

$$(*) \qquad \neg(x < y \vee y < x) \to x = y$$

is incompatible with the axioms of SIA. For (*) implies

$$(**) \qquad \forall x \neg(x < 0 \vee 0 < x) \to x = 0.$$

But in SIA we have by Exercise 1.6 and Thm. 1.1 (i) of [1],

$$\forall x \in \Delta\, \neg(x < 0 \vee 0 < x) \wedge \Delta \neq \{0\},$$

which clearly contradicts (**).

Thus in CA the set Δ of infinitesimals would be degenerate (i.e., identical with $\{0\}$), while the nondegeneracy of Δ in SIA is one of its characteristic features.

Next, call a binary relation S on $\mathbf{R}$ *stable* if it satisfies

$$\forall x \forall y (\neg\neg x S y \to x S y).$$

In CA, the equality relation is stable, a fact which again follows from principle R2(3) referred to above. But in SIA it is not stable, for, as shown in Thm. 1.1 (ii) of [1], there we have $\forall x \in \Delta\, \neg\neg x = 0$. If $=$ were stable, it would follow that $\forall x \in \Delta\, x = 0$, in other words, that Δ is degenerate, which is not the case in SIA.

Axiom 6 of SIA, together with the transitivity and irreflexivity of $<$, implies that $<$ is stable. This may be seen as follows. Suppose $\neg\neg a < b$. Then certainly $a \neq b$, since $a = b \rightarrow \neg a < b$ by irreflexivity. Therefore $a < b$ or $b < a$. The second disjunct together with $\neg\neg a < b$ and transitivity gives $\neg\neg a < a$, which contradicts $\neg a < a$. Accordingly we are left with $a < b$. As can be deduced from assertion 8 on p. 103 of [2], the stability of $<$ implies *Markov's principle*, which is not affirmed in CA.[2]

A subset $A \subseteq \mathbf{R}$ is *indecomposable* if it admits only trivial partitionings, that is, if $A = U \cup V$ and $U \cap V = \emptyset$, then $U = \emptyset$ or $V = \emptyset$. Clearly A is indecomposable iff any map $f : A \rightarrow 2 = \{0, 1\}$ is constant.

In SIA one also assumes the

Constancy Principle. *If $A \subseteq \mathbf{R}$ is any closed interval on* **R**, *or* **R** *itself, and $f : A \rightarrow \mathbf{R}$ satisfies $f(a + \varepsilon) = f(a)$ for all $a \in A$ and $\varepsilon \in \Delta$, then f is constant.*

As shown in Thm. 2.1 of [1], it follows in SIA from the Constancy Principle that **R** *itself and each of its closed intervals is indecomposable.* From this we can deduce that in SIA *all intervals in* **R** *are indecomposable.* To do this we employ the following

Lemma. Suppose that A is an inhabited subset of **R** satisfying

(*) for any $x, y \in A$ there is an indecomposable set B such that

$$\{x, y\} \subseteq B \subseteq A.$$

Then A is indecomposable.

Proof. Suppose A satisfies (*) and $A = U \cup V$ with $U \cap V = \emptyset$. Since A is inhabited, we may choose $a \in A$. Then $a \in U$ or $a \in V$. Suppose $a \in U$; then if $y \in V$ there is an indecomposable B for which $\{a, y\} \subseteq B \subseteq A = U \cup V$. It follows that $B = (B \cap U) \cup (B \cap V)$, whence $B \cap U = \emptyset$ or $B \cap V = \emptyset$. The former possibility is ruled out by the fact that $a \in B \cap U$, so $B \cap V = \emptyset$, contradicting $y \in B \cap V$. Therefore $y \in V$ is impossible; since this is the case for arbitrary y, we conclude

[2] In versions of SIA that omit axiom 6 neither the stability of $<$, nor Markov's principle, can be derived.

that $V = \emptyset$. Similarly, if $a \in V$, then $U = \emptyset$, so that A is indecomposable as claimed.

We use this lemma to show that the open interval $(0,1)$ is indecomposable; similar arguments work for arbitrary intervals. In fact, if $\{x, y\} \subseteq (0,1)$, it is easy to verify that

$$\{x, y\} \subseteq [\tfrac{xy}{x+y}, \tfrac{1-xy}{2-x-y}] \subseteq (0,1).$$

Thus, in view of the indecomposability of closed intervals, $(0,1)$ satisfies condition (*) of the lemma, and so is indecomposable.

Aside from certain infinitesimal subsets to be discussed below, in SIA indecomposable subsets of $\mathbf{R}$ correspond to connected subsets of $\mathbf{R}$ in classical analysis, that is, to intervals. In particular, any puncturing of $\mathbf{R}$ is *decomposable*, for it follows immediately from Axiom 6 that

$$\mathbf{R} - \{a\} = \{\, x : x > a \,\} \cup \{\, x : x < a \,\}.$$

Similarly, the set $\mathbf{R} - \mathbf{Q}$ of irrational numbers is decomposable as

$$\mathbf{R} - \mathbf{Q} = [\{\, x : x > 0 \,\} - \mathbf{Q}] \cup [\{\, x : x < 0 \,\} - \mathbf{Q}\}.$$

This is in sharp contrast with the situation in *intuitionistic analysis* IA, that is, CA augmented by Kripke's scheme, the continuity principle, and bar induction. For it is shown in [5] that in IA not only is any puncturing of $\mathbf{R}$ indecomposable, but that this is even the case for the set of irrational numbers (further indecomposabilty results for IA may be found in [6].) This would seem to indicate that in some sense the continuum in SIA is considerably less "syrupy"[3] than its counterpart in IA.

It can be shown that the various "infinitesimal" subsets of $\mathbf{R}$ introduced in [1] are indecomposable. For example, the indecomposability of Δ can be established as follows. Suppose $f : \Delta \to \{0,1\}$. Then by Microaffineness there are unique $a, b \in \mathbf{R}$ such that $f(\varepsilon) = a + b \cdot \varepsilon$ for all ε. Now $a = f(0) = 0$ or 1; if $a = 0$, then $b \cdot \varepsilon = f(\varepsilon) = 0$ or 1, and clearly $b \cdot \varepsilon \neq 1$. So in this case $f(\varepsilon) = 0$ for all ε. If on the other hand $a = 1$, then $1 + b \cdot \varepsilon = f(\varepsilon) = 0$ or 1; but $1 + b \cdot \varepsilon = 0$ would imply $b \cdot \varepsilon = -1$ which is again impossible. So in this case $f(\varepsilon) = 1$ for all ε. Therefore f is constant and Δ indecomposable.

[3] It should be emphasized that this phenomenon is a consequence of axiom 6: it cannot necessarily be affirmed in versions of SIA not including this axiom.

In SIA *nilpotent infinitesimals* are defined to be the members of the sets

$$\Delta_k = \{ x \in \mathbf{R} : x^{k+1} = 0 \},$$

for $k = 1, 2, \ldots$, each of which may be considered an infinitesimal neighbourhood of 0. These are subject to the

Micropolynomiality Principle. *For any $k \geq 1$ and any $g : \Delta_k \to \mathbf{R}$, there exist unique $a, b_1, \ldots, b_k \in \mathbf{R}$ such that for all $\delta \in \Delta_k$ we have*

$$g(\delta) = a + b_1\delta + b_2\delta^2 + \cdots + b_k\delta^k.$$

Micropolynomiality implies that no Δ_k coincides with $\{0\}$.

An argument similar to that establishing the indecomposability of Δ does the same for each Δ_k. Thus let $f : \Delta_k \to \{0, 1\}$; Micropolynomiality implies the existence of $a, b_1, \ldots, b_k \in \mathbf{R}$ such that $f(\delta) = a + \zeta(\delta)$, where $\zeta(\delta) = b_1\delta + b_2\delta^2 + \cdots + b_k\delta^k$. Notice that $\zeta(\delta) \in \Delta_k$, that is, $\zeta(\delta)$ is nilpotent. Now $a = f(0) = 0$ or 1; if $a = 0$ then $\zeta(\delta) = f(\delta) = 0$ or 1, but since $\zeta(\delta)$ is nilpotent it cannot =1. Accordingly in this case $f(\delta) = 0$ for all $\delta \in \Delta_k$. If on the other hand $a = 1$, then $1 + \zeta(\delta) = f(\delta) = 0$ or 1, but $1 + \zeta(\delta) = 0$ would imply $\zeta(\delta) = -1$ which is again impossible. Accordingly f is constant and Δ_k indecomposable.

The union $\mathbf{D}$ of all the Δ_k is the *set of nilpotent infinitesimals*, another infinitesimal neighbourhood of 0. The indecomposability of $\mathbf{D}$ follows immediately by applying the Lemma above.

The next infinitesimal neighbourhood of 0 is the closed interval $[0, 0]$, which, as a closed interval, is indecomposable. It is easily shown that $[0, 0]$ includes $\mathbf{D}$, so that it does not coincide with $\{0\}$.

It is also easily shown, using axioms 2 and 6, that $[0, 0]$ coincides with the set

$$\mathbf{I} = \{ x \in \mathbf{R} : \neg\neg x = 0 \}.$$

So $\mathbf{I}$ is indecomposable. (In fact the indecomposability of $\mathbf{I}$ can be proved independently of axioms 1–6 through the general observation that, if A is indecomposable, then so is the set $A^\star = \{ x : \neg\neg x \in A \}$.)

Finally, we observe that the sequence of infinitesimal neighbourhoods of 0 generates a strictly ascending sequence of decomposable subsets containing $\mathbf{R} - \{0\}$, namely:

$$\mathbf{R} - \{0\} \subset (\mathbf{R} - \{0\}) \cup \{0\} \subset (\mathbf{R} - \{0\}) \cup \Delta_1 \subset (\mathbf{R} - \{0\}) \cup \Delta_2 \subset \ldots$$
$$(\mathbf{R} - \{0\}) \cup \mathbf{D} \subset (\mathbf{R} - \{0\}) \cup [0, 0].$$

Note

It was hearing Dirk van Dalen's stimulating talk at the conference that got me thinking about the topic discussed in this paper. I would like to thank Peter Schuster for encouraging me to write it, and the referee for helpful suggestions.

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