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ON THE STRENGTH OF THE SIKORSKI EXTENSION THEOREM FOR BOOLEAN ALGEBRAS

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§1. **Introduction.** The *Sikorski Extension Theorem* [6] states that, for any Boolean algebra A and any complete Boolean algebra B, any homomorphism of a subalgebra of A into B can be extended to the whole of A. That is,

> **Inj**: *Any complete Boolean algebra is injective* (in the category of Boolean algebras).

The proof of **Inj** uses the axiom of choice (**AC**); thus the implication **AC** → **Inj** can be proved in Zermelo-Fraenkel set theory (**ZF**). On the other hand, the *Boolean prime ideal theorem*

> **BPI**: *Every Boolean algebra contains a prime ideal* (*or*, *equivalently*, *an ultrafilter*)

may be equivalently stated as:

The two element Boolean algebra 2 *is injective*,

and so the implication **Inj** → **BPI** can be proved in **ZF**.

In [3], Luxemburg surmises that this last implication cannot be reversed in **ZF**. It is the main purpose of this paper to show that this surmise is correct. We shall do this by showing that **Inj** implies that **BPI** holds in every Boolean extension of the universe of sets, and then invoking a recent result of Monro [5] to the effect that **BPI** does *not* yield this conclusion.

§2. **Preliminaries.** We work in **ZF**; thus the axiom of choice is *not* assumed. We shall suppose some familiarity with Boolean-valued models of set theory as presented, e.g. in [1]. We employ the standard notations. If B is a complete Boolean algebra, $V^{(B)}$ is the Boolean-valued universe constructed from B. There is a canonical embedding $x \mapsto \hat{x}$ from the real universe V of sets into $V^{(B)}$. If σ is a sentence of the language of set theory (possibly containing names for elements of $V^{(B)}$), we write $[\![\sigma]\!]^B$ (or just $[\![\sigma]\!]$) for the Boolean value of σ calculated in $V^{(B)}$, and $V^{(B)} \models \sigma$ for $[\![\sigma]\!]^B = 1_B$, the top element of B. The object $U_B \in V^{(B)}$ defined by $U_B = \{\langle \hat{x}, x\rangle : x \in B\}$ is called the *canonical* (generic) *ultrafilter* in $\hat{B}$; as is well known, we have

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$V^{(B)} \models U_B$ *is an ultrafilter in the Boolean algebra* $\hat{B}$.

Note also that, if A is any Boolean algebra, then $V^{(B)} \models \hat{A}$ *is a Boolean algebra.*

Since we are not assuming the axiom of choice, we must now consider a number of delicate points about $V^{(B)}$ which can be proved to hold without its use. Chief among these is the following special case of the Maximum Principle (1.27 of [1]).

2.1. LEMMA (proved in **ZF**). *If* $V^{(B)} \models \exists! x\phi(x)$, *then there is* $u \in V^{(B)}$ *such that* $V^{(B)} \models \phi(u)$. *(Here* $\phi(x)$ *has just the free variable* x, *but may contain names for elements of* $V^{(B)}$.*)*

PROOF. First note that since $V^{(B)} \models \exists! x\phi(x)$, we have

$$(*) \qquad [\![\phi(u)]\!] \wedge [\![\phi(v)]\!] \leq [\![u = v]\!]$$

for any $u, v \in V^{(B)}$. Now we have

$$1 = [\![\exists x\phi(x)]\!] = \bigvee_{x \in V^{(B)}} [\![\phi(x)]\!]$$

and using the axioms of replacement and regularity we can find a set $\{u_i : i \in I\} \subseteq V^{(B)}$ such that

$$1 = \bigvee_{x \in V^{(B)}} [\![\phi(x)]\!] = \bigvee_{i \in I} [\![\phi(u_i)]\!].$$

If we now define $u \in V^{(B)}$ by $\mathrm{dom}(u) = \bigcup_{i \in I} \mathrm{dom}(u_i)$, and for $z \in \mathrm{dom}(u)$,

$$u(z) = \bigvee_{i \in I} [\![\phi(u_i) \wedge z \in u_i]\!],$$

then, using (*), we easily show, as in the proof of 1.25 of [1], that $[\![\phi(u_i)]\!] \leq [\![u = u_i]\!]$ for each $i \in I$. It follows that

$$[\![\phi(u)]\!] \geq \bigvee_{i \in I} [\![\phi(u_i)]\!] \wedge [\![u = u_i]\!] = \bigvee_{i \in I} [\![\phi(u_i)]\!] = 1. \quad \blacksquare$$

The next point is that without loss of generality we may assume that $V^{(B)}$ is *separated*, i.e. for any $x, y \in V^{(B)}$ we have $V^{(B)} \models x = y$ iff $x = y$. To justify this, we define the equivalence relation $\sim$ on $V^{(B)}$ by $x \sim y$ iff $V^{(B)} \models x = y$, and then employ "Scott's trick" of replacing each $x \in V^{(B)}$ by the set of objects y in $V^{(B)}$ of lowest rank such that $x \sim y$. This procedure turns $V^{(B)}$ into a separated structure.

Finally, we shall need the following ideas derived from [7]. If $V^{(B)} \models \langle A, \leq_A \rangle$ *is a Boolean algebra*, define

$$A \otimes B = \{x \in V^{(B)} : [\![x \in A]\!]^B = 1_B\}.$$

(Since $V^{(B)}$ is now separated, $A \otimes B$ is easily shown to be a set.) Define $\leq$ (sometimes written $\leq_{A \otimes B}$) on $A \otimes B$ by

$$x \leq y \leftrightarrow [\![x \leq_A y]\!]^B = 1_B$$

for $x, y \in A \otimes B$. Using Lemma 2.1, it is readily shown (in **ZF**) that $\langle A \otimes B, \leq \rangle$ is a Boolean algebra in which, for $x, y \in A \otimes B$,

$x \wedge y$ is the unique $z \in A \otimes B$ such that

$$[\![z = \inf\{x, y\} \textit{ in } A]\!]^B = 1_B,$$

$x \vee y$ is the unique $z \in A \otimes B$ such that

$$[\![z = \sup\{x, y\} \text{ in } A]\!]^B = 1_B,$$

x^* is the unique $z \in A \otimes B$ such that

$$[\![z = x^* \text{ in } A]\!]^B = 1_B.$$

(Here x^* denotes the Boolean complement of x.) Moreover, B is embeddable in $A \otimes B$ via the map e defined by setting, for each $b \in B$, $e(b) =$ unique $x \in A \otimes B$ for which

$$(2.2) \qquad [\![x = 1_A]\!]^B = b, \qquad [\![x = 0_A]\!]^B = b^*.$$

(This definition uses the mixing lemma in $V^{(B)}$ (1.25 of [1]), whose proof does not require **AC**.) *We shall use the embedding e to identify B with its image in $A \otimes B$, so that B becomes a subalgebra of $A \otimes B$.* From (2.2) it follows that, for $b \in B$,

$$(2.3) \qquad [\![b = 1_A]\!]^B = b, \qquad [\![b = 0_A]\!]^B = b^*$$

and

$$(2.4) \qquad [\![b = 0_A \vee b = 1_A]\!]^B = 1_B.$$

§3. The main result. Given Boolean algebras A and B, we write $B \preccurlyeq A$ for *B is a subalgebra of A*. B is called an *absolute subretract* if for any Boolean algebra A, whenever $B \preccurlyeq A$ there is an (epi) morphism $h: A \to B$ which is the identity on B. We can now prove the following result.

3.1. THEOREM. *Let B be a complete Boolean algebra. Then the following conditions are provably equivalent in* **ZF**.

(i) *B is injective*;

(ii) *B is an absolute subretract*;

(iii) *for any Boolean algebra A such that $B \preccurlyeq A$, there is $U \in V^{(B)}$ such that $V^{(B)} \models U$ is an ultrafilter in $\hat{A}$ and $U_B \subseteq U$*;

(iv) *for any $C \in V^{(B)}$ such that $V^{(B)} \models C$ is a Boolean algebra, there is $U \in V^{(B)}$ such that $V^{(B)} \models U$ is an ultrafilter in C.*

PROOF. (i) → (ii) is obvious.

(ii) → (iii). Assume (ii), let $B \preccurlyeq A$, and let $h: A \to B$ be a homomorphism which is the identity on B. If we put $U = \{\langle \hat{a}, h(a)\rangle : a \in A\}$ then it is easily verified that $V^{(B)} \models$ *U is an ultrafilter in $\hat{A}$*. Moreover, we have, for $b \in B$, $[\![\hat{b} \in U_B]\!] = b = h(b) = [\![\hat{b} \in U]\!]$ whence $V^{(B)} \models U_B \subseteq U$. Hence (iii).

(iii) → (iv). Assume (iii) and let $C \in V^{(B)}$ satisfy $V^{(B)} \models$ *C is a Boolean algebra.* Then $C \otimes B$ is a Boolean algebra and $B \preccurlyeq C \otimes B$. It follows that $V^{(B)} \models \hat{B}$ *and $(C \otimes B)^\wedge$ are Boolean algebras and $\hat{B} \preccurlyeq (C \otimes B)^\wedge$*. Now, working in $V^{(B)}$, let F be the filter in $(C \otimes B)^\wedge$ generated by the canonical ultrafilter U_B; that is, in $V^{(B)}$, $F = \{x \in (C \otimes B)^\wedge : \exists y \in U_B \cdot y \leq_{(C \otimes B)^\wedge} x\}$. We claim that in $V^{(B)}$, *C is isomorphic to the quotient algebra $(C \otimes B)^\wedge/F$*. To see this, define $h \in V^{(B)}$ by $h = \{\langle \check{x}, x\rangle^{(B)} : x \in C \otimes B\} \times \{1_B\}$. It is easy to verify that, in $V^{(B)}$, *h is a homomorphism of $(C \otimes B)^\wedge$ onto C*. To show that $C \cong (C \otimes B)^\wedge/F$ in $V^{(B)}$, it suffices to show that $V^{(B)} \models F = h^{-1}(1_C)$. To prove this, we observe that, for $x \in C \otimes B$,

$$\begin{aligned}[\![\hat{x} \in F]\!] &= [\![\exists y \in U_B \cdot y \leq_{(C\otimes B)^\frown} \hat{x}]\!] \\ &= \bigvee_{b\in B} b \wedge [\![\hat{b} \leq_{(C\otimes B)^\frown} \hat{x}]\!] \\ &= \bigvee\{b \in B\colon b \leq_{C\otimes B} x\}.\end{aligned}$$

Also, for $b \in B$, we have

$$\begin{aligned} b \leq [\![x = 1_C]\!] &\leftrightarrow [\![b = 1_C]\!] \leq [\![x = 1_C]\!] \quad \text{(by 2.3)} \\ &\leftrightarrow V^{(B)} \models b = 1_C \to x = 1_C \\ &\leftrightarrow V^{(B)} \models b \leq_C x \quad \text{(by 2.4)} \\ &\leftrightarrow b \leq_{C\otimes B} x.\end{aligned}$$

Hence $[\![h(\hat{x}) = 1_C]\!] = [\![x = 1_C]\!] = \bigvee\{b\in B\colon b \leq_{C\otimes B} x\} = [\![\hat{x} \in F]\!]$ by the above, proving the claim.

Now by (iii) there is $U \in V^{(B)}$ such that $V^{(B)} \models$ *U is an ultrafilter in $(C \otimes B)^\frown$ containing U_B*. Then $V^{(B)} \models F \subseteq U$ and so $V^{(B)} \models$ *$h[U]$ is an ultrafilter in $(C \otimes B)^\frown/F \cong C$*. This gives (iv).

(iv) → (i). Assume (iv) and let h be a homomorphism of a subalgebra C of a Boolean algebra A into B. Put $U = \{\langle \hat{x}, h(x)\rangle\colon x \in C\}$; then, as before, $V^{(B)} \models$ *U is an ultrafilter in $\hat{C}$*. Working in $V^{(B)}$, let F be the filter in $\hat{A}$ generated by U, and let h be the canonical epimorphism of $\hat{A}$ onto $\hat{A}/F$. Using (iv), let $U \in V^{(B)}$ be an ultrafilter in $\hat{A}/F$. Then, in $V^{(B)}$, *$U' = h^{-1}[U]$ is an ultrafilter in $\hat{A}$ extending F*. (Note that, in claiming $h^{-1}[U]$ as an explicit object of $V^{(B)}$, we are tacitly using Lemma 1.1.) If we now define $g\colon A \to B$ by $g(a) = [\![\hat{a} \in U']\!]^B$, then it is readily verified that g is a homomorphism of A into B extending h. ∎

Remarks. (1) The equivalence of (i) and (ii) was originally proved in [3] by a method entirely different from the one employed here, and (iv) → (i) is essentially proved in [4]. It is (i) → (iv) which appears to be new; as we shall see, it is crucial for our purposes.

(2) Notice that, since the full Maximum Principle is not available, condition (iv) of 3.1 is ostensibly *stronger* than the condition $V^{(B)} \models$ **BPI**. In this connection one may ask whether **Inj** is equivalent to the statement: "For all complete Boolean algebras B, $V^{(B)} \models$ **BPI**". I do not know the answer to this question.

We have as a consequence the main result of the paper.

3.2. Corollary. **Inj** *is not provable from* **BPI** *in* **ZF** (*assuming the consistency of the latter*).

Proof. We first employ 3.1 to show that, if B is injective, then $V^{(B)} \models$ **BPI**. For suppose $C \in V^{(B)}$; let $b = [\![C$ is a Boolean algebra$]\!]^B$ and, using the mixing lemma in $V^{(B)}$, let $C' \in V^{(B)}$ be such that

$$[\![C = C']\!]^B = b, \qquad [\![C' = \hat{2}]\!]^B = b^*.$$

Then $V^{(B)} \models$ *C' is a Boolean algebra* and so, by (i) → (iv) of 3.1, there is $U \in V^{(B)}$ such that $V^{(B)} \models$ *U is an ultrafilter in C'*. Clearly

$$\begin{aligned} b = [\![C' = C]\!]^B &\leq [\![U \textit{ is an ultrafilter in } C']\!]^B \wedge [\![C' = C]\!]^B \\ &\leq [\![U \textit{ is an ultrafilter in } C]\!]^B \\ &\leq [\![\exists X.\ X \textit{ is an ultrafilter in } C]\!]^B.\end{aligned}$$

It follows that $V^{(B)} \models$ **BPI**.

Now let M be the Halpern-Levy model of **ZF** in which **BPI** holds but **AC** fails (see, e.g., [2]). Monro [5] has constructed a complete Boolean algebra B in M such that, in M, $[\![\mathbf{BPI}]\!]^B = 0_B$. It follows from the above that, in M, B is not injective, and so **Inj** fails in M. The result follows. ∎

§4. **Some final observations.** Let us call a sentence σ of the language of set theory *persistent* if we can prove in **ZF** that, if σ holds, it continues to hold in every Boolean extension of V. Of course, **AC** is persistent. Monro [5] shows, on the other hand, that several consequences of **AC**, in particular **BPI** and the ordering principle, are not persistent. In contrast, we have

4.1. THEOREM (**ZF**). **Inj** *is persistent.*

PROOF. Suppose **Inj** holds, and let B be a complete Boolean algebra. We need to show that $V^{(B)} \models$ **Inj**, and by 3.1 it suffices to show that, in $V^{(B)}$, *every complete Boolean algebra is an absolute subretract*. And for this to be the case it suffices to show that, if A, C are any elements of $V^{(B)}$ such that $V^{(B)} \models A$ *and* C *are Boolean algebras,* C *is complete, and* $C \preccurlyeq A$, then there is $h \in V^{(B)}$ such that $V^{(B)} \models h$ *is a homomorphism of* A *onto* C *which is the identity on* C.

Now $B \preccurlyeq C \otimes B \preccurlyeq A \otimes B$, and, by 5.2.1 of [7] (whose proof does not require **AC**), $C \otimes B$ is complete. Since **Inj** is assumed to hold, there is a homomorphism $g: A \otimes B \to C \otimes B$ which is the identity on $C \otimes B$, and hence also on B. We have

$$V^{(B)} \models \hat{B} \preccurlyeq (C \otimes B)\hat{} \preccurlyeq (A \otimes B)\hat{} \text{ and } \hat{g} \text{ is a homomorphism of } (A \otimes B)\hat{} \text{ onto } (C \otimes B)\hat{}.$$

Also, if F, F' are the filters generated by the canonical ultrafilter U_B in $(C \otimes B)\hat{}$, $(A \otimes B)\hat{}$ respectively, then, by the proof of (iii) → (iv) of 3.1 we have

$$V^{(B)} \models C \cong (C \otimes B)\hat{}/F \text{ and } A \cong (A \otimes B)\hat{}/F.$$

It now follows easily from this and the fact that g is the identity on $C \otimes B$ that in $V^{(B)}$, $\hat{g}$ *induces a homomorphism of* A *onto* C *which is the identity on* C. This completes the proof. ∎

It is tempting to conjecture on the basis of this result and 3.2 that **Inj** is actually *equivalent* to **AC**. I have not, however, been able to settle this question.

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