TYPES, SETS AND CATEGORIES

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This essay is an attempt to sketch the evolution of type theory from its beginnings early in the last century to the present day. Central to the development of the type concept has been its close relationship with set theory to begin with and later its even more intimate relationship with category theory. Since it is effectively impossible to describe these relationships (especially in regard to the latter) with any pretensions to completeness within the space of a comparatively short article, I have elected to offer detailed technical presentations of just a few important instances.

1 THE ORIGINS OF TYPE THEORY

The roots of type theory lie in set theory, to be precise, in Bertrand Russell's efforts to resolve the paradoxes besetting set theory at the end of the 19^{th} century. In analyzing these paradoxes Russell had come to find the set, or class, concept itself philosophically perplexing, and the theory of types can be seen as the outcome of his struggle to resolve these perplexities. But at first he seems to have regarded type theory as little more than a *faute de mieux.*

In June 1901 Russell learned of the set-theoretic paradox, known to Cantor by 1899, that issued from applying the latter's theorem concerning power sets to the class V of all sets. According to that theorem the class of all subclasses of V would, impossibly, have to possess a cardinality exceeding that of V. This stimulated Russell to formulate the paradox that came to bear his name (although it was independently discovered by Zermelo at about the same time) concerning the class of all classes not containing themselves (or predicates not predicable of themselves). Realizing that his paradox applied also to the logical system previously elaborated by Frege in his *Grundgesetze*, Russell communicated it to Frege in June 1902, occasioning, in the latter's own words, the "greatest surprise, and I would almost say, consternation, since it has shaken the basis on which I intended to build arithmetic". Frege responded by attempting to patch up the postulate of his system (Basic Law V) subject to the paradox, fashioning a repair that Russell was initially inclined to endorse (see the final note to Appendix A of *The Principles of Mathematics*)[1]). Frege's repair in fact proved inadequate; but in any case Russell had already begun to search for a solution to the paradox as it applied to classes. This led to his proposing the theory, or doctrine, of types.

[1] Russell [1903, 1].

It is of interest to see how Russell's doctrine of types emerges from his analysis of classes in the *Principles.* He puts forward the following definition of class (p. 515):

> A class is an object uniquely determined by a propositional function, and determined equally by any equivalent propositional function.

Here Russell is, in essence, adopting an extensional view of classes.[2] He then considers the following possible characterizations of the class notion:

> A class may be identified with (α) the predicate, (β) the class concept,(γ) the concept of the class, (δ) Frege's range, (ε) the numerical conjunction of the terms of the class, (ζ) the whole composed of the terms of the class.

To understand what Russell intends by the first three of these, let us follow him in considering the term *man.* Here (α) then corresponds to the predicate *...is human*, (β) to the function of *man* in propositions such as *Socrates is a man*, and (γ) to the concept of the *class of men.* Clearly each of these characterizations is intensional in nature, so that none of them can be taken as a satisfactory determination of the notion of the essentially extensional notion of class.

Under (δ), "Frege's range" refers to the device introduced by Frege in the *Grundgesetze* by which each function is assigned an object called its *range* in such a way that two functions are assigned the same range precisely when they have identical values for all values of their arguments. This is Basic Law V, which Russell had already shown to be inconsistent. In view of this Russell is led to observe "(δ) suffers from a doubt as to their being such an entity, and also from the fact that, if ranges are terms, the contradiction is inevitable." Once again, then, an unsatisfactory determination.

Under (ε) we have the *class-as-many.* Russell regards this notion as "logically unobjectionable" since he thinks that the existence of classes-as-many is guaranteed ontologically, and so at the very least classes-as-many can always be assumed to exist without falling into contradiction. But classes-as-many suffer from the drawback that they are not single entities, except when the class has only one member.

Finally, category (ζ) is the *class-as-one.* The problem here is that Russell's paradox shows that classes-as–one do not always exist.

[2]But interestingly Russell earlier remarks (p. 513) that he was led to an extensional view of classes "against [his] inclination", and only because of "the necessity of discovering some entity determinate for a given propositional function, and the same for any equivalent propositional function." This would seem to be essentially the same reason Frege was led to introduce ranges, or, more pertinently, their special case extensions of concepts. Russell goes on to say that the only entities he has been able to discover which meet the requirements are, first, the class as one of all objects making the given propositional function true, and, second, the class (also as one) of all propositional functions equivalent to the given function. One notes that the first expedient involves a type reduction, and the second a type augmentation.

Russell is accordingly left with no satisfactory way of defining classes. Yet, as he says, "without a single object to represent an extension, Mathematics crumbles." In the end he is forced to adopt (ε), the class-as-many, as the definition of a class. This requires abandoning the doctrine that the subject of a proposition is always a single term; instead, it can be essentially many terms. Once this is accepted, one sees that the plural nature of classes does not prevent them from being "counted as though each were a genuine unity".

But in that case, Russell observes, it then becomes necessary "to distinguish (1) terms, (2) classes, (3) classes of classes, and so on *ad infinitum*". Moreover, these collections will be disjoint, and to able to assert $x \in u$ requires that the collection to which x belongs should be of a degree one lower than that to which u belongs. This expedient leads to a resolution of the paradox, since $x \in x$ has now been rendered a meaningless proposition. The hierarchy of collections (1), (2), (3), ... is the germ of the doctrine of types.

In Appendix B to the *Principles*, entitled *The Doctrine of Types*, Russell states that he is putting forward the doctrine "tentatively, as affording a possible solution of the contradiction"; but it "requires, in all probability, to be transformed into some subtler shape before it can answer all difficulties." He proceeds to explain exactly what he means by a type:

> Every propositional function $\varphi(x)$ ... has, in addition to its range of truth, a range of significance, i.e. a range within which x must lie if $\varphi(x)$ is to be a proposition at all, whether true or false. This is the first point in the theory of types; the second point is that ranges of significance form types, i.e. if x belongs to the range of significance of $\varphi(x)$, then there is a class of objects, the type of x, all of which must also belong to the range of significance of $\varphi(x)$, however φ may be varied; and the range of significance is always either a single type or a sum of several whole types.

As observed above, Russell believed the doctrine of types to be adequate for resolving the paradox for classes. But according to his definition of type, propositions themselves can serve as the constituting objects for types. Russell proceeds to formulate a paradox for propositions similar to that for classes, which he does not think is resolved by the doctrine of types as he has formulated it. This paradox may be stated in the following way. For any class P of propositions, write $\sqcap P$ for the proposition *every proposition in P is true*. Russell makes the key (but as we shall see below, questionable) assumption that, for classes P, Q of propositions,

$$P \neq Q \Rightarrow \sqcap P \neq \sqcap Q, \qquad (*)$$

or equivalently (in classical logic, at least)

$$\sqcap P = \sqcap Q \Rightarrow P = Q. \qquad (**)$$

Now let Q be the class of propositions p for which there is a class P of propositions such that $p = \sqcap P$ and $p \notin P$. If one now writes q for $\sqcap Q$, then, using (**), $q \in Q \Leftrightarrow q \notin Q$. Faced with this conclusion Russell then remarks

> It is possible of course to hold that propositions themselves are of various types and that their logical products must have propositions of only one type as factors. But this suggestion seems harsh and highly artificial.

A number of years were to pass before Russell came finally to adopt this "harsh and highly artificial" suggestion in the form of the ramified theory of types.

Russell remarks that if the identity relation is replaced by the relation of *logical equivalence,* then there is no reason to uphold implication (*), and the contradiction disappears. But this escape is blocked, according to Russell, because

> it is quite self-evident that equivalent propositional functions are often not identical.

At this point, then, Russell could see no way out of the contradiction.

Now in deriving the contradiction it is (*) (or (**)) that is the culprit. Writing **Prop** for the class of all propositions, each asserts that the map $P \mapsto \sqcap P$ is one-one from the class of subclasses of **Prop** to **Prop**. And this, as is now well-known (and likely also then to Russell himself), violates Cantor's theorem. In Boolos [1997] it is shown how to prove Cantor's theorem through the provision of explicit counterexamples.[3] We will adapt his argument to produce a counterexample to (**). By Boolos's argument there exists a subclass M of **Prop** and a strict well-ordering $\prec$ of M such that (1) $\sqcap M \in M$ and, (2) for $p \in M, p = \sqcap\{q \in \mathbf{Prop} : q \prec p\}$.[4] Now let $N = \{q : q \prec \sqcap M\}$. Then $\sqcap M \notin N$ and so it follows from (1) that $M \neq N$. But, by (1) and (2), $\sqcap M = \sqcap N$. The classes of propositions M and N thus together constitute a counterexample to (**).

It is tempting to speculate that, had Russell known of this argument, he might have been led to abandon (*) and so perhaps have been more inclined to accept the adequacy of his early theory of types.

Since Russell was by no means convinced that his initial theory of types would prove adequate for resolving the logical paradoxes that had arisen in the foundations of mathematics, he developed a number of other theories in attempting to meet the challenge. But he soon became aware of the inadequacies of these and returned once more to the theory of types. The refined theory of types which resulted from his labours was first outlined in a paper published in 1908: *Mathematical logic as based on the theory of types.*[5] Russell prefaces his paper by remarking that while the theory he is about to present "recommends itself ... in the first instance, by the ability to solve certain contradictions," it "seems not wholly dependent on this indirect recommendation." For it has also "a certain consonance with common sense which makes it inherently credible." It is the the-

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[3] Boolos's argument is itself an adaptation of the first argument Zermelo used to derive the well-ordering theorem from the axiom of choice.

[4] If we express the assertion $q \prec p$ by saying that q is *below* p, then (2) may be expressed as: for any proposition $p \in M, p$ is identical with the proposition *every proposition below* p *is true.*

[5] Russell [1908].

ory's latter feature which, in part, led Russell to put it forward as a foundation for mathematical logic — and so also for mathematics.

Russell offers an account of the various contradictions with which he is concerned, finding that they all have in common "the assumption of a totality such that, if it were legitimate, it would at once be enlarged by new members defined in terms of itself". This leads to the enunciation of the rule which he calls the *vicious circle principle*, to wit:

> Whatever involves all of a collection must not be one of a collection, or, conversely: If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total.

Russell recognizes that this principle, being purely negative, "suffices to show that many theories are wrong, but it does not show how the errors are to be rectified." Russell sees a positive theory emerging from an analysis of the use of the term "all", and in particular from drawing a careful distinction between it and the term "any". While restrictions must, on pain of contradiction, be imposed on the use of "all", no such restrictions need be placed on "any". Thus "all" corresponds to the bound variable of universal quantification, which ranges over a type — the *type* of that variable — while "any" corresponds to a free variable for which the name of anything can be substituted, irrespective of type.

Russell then proceeds to introduce a theory of types for propositions — an expedient which, it will be recalled, he had shunned in 1903 as being "harsh and artificial". The entities populating Russell's typed logical universe are in fact of three kinds: *individuals, propositions,* and *(propositional) functions.* He defines a *type* as "the range of significance of a propositional function, that is, as the collection of arguments for which the said function has values." Individuals are decreed to occupy the first, or lowest type. Types are assigned to propositions through the device of *orders.* Thus first-order propositions are those whose quantified variables (if any) are individuals: these form the second type. Second-order propositions or functions are then those whose quantified variables are of at most second type, and so on.

From the hierarchy of propositions Russell derives a hierarchy of propositional functions. Now the hierarchy of propositions may be considered one-dimensional in that just a single number is required to specify its position in the hierarchy. The hierarchy of propositional functions, however, is multidimensional since the type of a involves not just its order as a proposition, but also the orders of its arguments, or free variables. It is this feature of Russell's theory of types which led to its being called the *ramified* theory of types.

As Quine[6] has pointed out, the logical universe presented by Russell has a number of ambiguous features. For instance, in Russell's formulation, individuals are objects, but propositions and propositional functions are notations, dependent

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[6] In his commentary on Russell's paper in Van Heijenoort [1967].

on the vicissitudes of presentation. That being the case, it is far from clear whether Russell is "assigning types to his objects or to his notations."

Russell next addresses the question of whether his theory is adequate for the development of mathematics. He gives examples from elementary mathematics to show that the answer is no unless some method is found "of reducing the order of a propositional function without affecting the truth or falsehood of its values". To achieve this he introduces the *Axiom of Reducibility,* which asserts that every function is coextensive with a predicative function (in the same arguments). Here by a *predicative* function is meant one in which the types of quantified variables do not exceed the types of the arguments. (All others are termed *impredicative.*) Russell justifies this postulate on the grounds that it has all the useful consequences flowing from the class concept, without having to go so far as to admit the actual presence of classes.

As Quine[7] observes, this expedient is "oddly devious". If every propositional function is coextensive with a predicative one, then attention could have been confined to these latter from the beginning. In other words, the "ramification" of types is rendered superfluous; the types of propositional functions could have been made dependent solely on their arguments. Indeed, "The axiom of reducibility is self-effacing: if it is true, the ramification it was meant to cope with was pointless to begin with."[8]

In the same year (1908) that Russell published his paper on logical types, Ernst Zermelo published his paper *Investigations in the foundations of set theory I.*[9] In it he presents the first axiomatization of set theory. While both Russell's and Zermelo's theories were intended to furnish a foundation for mathematics — a framework, that is, within which the usual development of mathematics could proceed free of contradiction — there are a number of differences between the two approaches. Russell had always maintained that mathematics is, at bottom, logic, and his system naturally reflects that conviction. For Russell, once the correct logical framework had been found, mathematics would then follow; hence his concern with the logical paradoxes, and his consequent elaboration of the theory of types as a strategy for circumventing them. Once this had been taken care of, the development of mathematics could proceed along the lines mapped out by the great 19^{th} century German analysts Weierstrass, Dedekind, and Cantor, whose work had so impressed Russell.

In formulating his axiom system for set theory Zermelo was animated by rather different concerns. In 1904 he published his proof that every set could be well-ordered. The proof excited considerable controversy, because of its use of the axiom of choice and the vagueness of the set-theoretic background. This led Zermelo to publish a new, improved proof (again based on the axiom of choice, but in a new formulation) of the theorem in 1908, which was itself to be underpinned by the sharpened, axiomatized notion of set presented in *Investigations.* While designed

[7] *Op. cit.*
[8] *Op. cit.*
[9] Zermelo [1908].

to avoid the set-theoretic paradoxes, the primary purpose of Zermelo's system was to provide the working mathematician with a reliable foundational toolkit.

While for Russell sets or classes were essentially logical entities, Zermelo treated them as mathematical objects subject to certain axiomatic conditions. The most interesting of these latter is the third, the *axiom of separation.* This axiom is the counterpart, for sets considered as mathematical objects, of the principle governing their introduction as logical entities, namely, as extensions of propositional functions. Zermelo formulates his axiom by introducing the notion of *definite property*, and then stipulating that a definite property separates a subset from a previously given set. Sadly, Zermelo's notion of definite property is itself far from being definite, since in its formulation Zermelo invokes "the universally valid laws of logic", but neglects to specify what these are. This shortcoming was later rectified, in various ways, by Weyl, Fraenkel, Skolem and von Neumann. Zermelo's system, as modified and extended by the latter three as well as Gödel and Bernays, in due course became the standard foundation of mathematics, a role it still plays today.

Type theory has, by contrast, undergone a more tortuous development and has met with a more complex reception. Its initial phase culminated in Whitehead and Russell's monumental trilogy *Principia Mathematica*[10] of 1910–13. This is a fully elaborated treatment of the framework introduced by Russell in his 1908 paper. It includes both an axiom of infinity and an axiom of reducibility.

Classes play a very minor role in *Principia.* By the time *Principia* was written, Russell had come to regard classes as dubious logical entities and so sought to avoid having to postulate their existence outright. Russell and Whitehead see the introduction of classes as arising from the necessity to deal with *extensions*:

> It is an old dispute whether formal logic should concern itself mainly with intensions or with extensions. In general, logicians whose training was mainly philosophical have decided for intensions, while those whose training was mainly mathematical have decided for extensions. The facts seem to be that, while mathematical logic requires extensions, philosophical logic refuses to supply anything except intensions. Our theory of classes recognizes and reconciles these two apparently opposite facts, by showing that an extension (which is the same as a class) is an incomplete symbol, whose use always acquires its meaning through a reference to an intension.

In order to be able to handle extensions, and yet at the same time avoiding any untoward ontological commitments, in *Principia* classes are introduced, then, as *incomplete symbols*, that is, symbols whose "*uses* are defined, but [which] themselves are not assumed to mean anything at all". Thus, "classes...are merely symbolic or linguistic conveniences, not genuine objects as their members are if they are individuals". In other words, classes are mere shadows.

[10] Whitehead and Russell [1910–13].

The reconditeness[11] of *Principia* and its focus on philosophical niceties far removed from the concerns of working mathematicians hardly commended it to them as a practical foundation for their subject. Zermelo's axiomatic approach to classes, by contrast, seemed simpler and closer to the actual practice of mathematics.

Nevertheless, the concept of the ramified hierarchy has been an important, if usually unremarked, influence in mathematical logic. For instance, in the 1930s Gödel discovered that, if the ramification of properties over the natural numbers is extended into the transfinite, the process breaks down at level ω_1, the first uncountable ordinal. (This may be considered a provable form of the Axiom of Reducibility.) This led him to introduce the hierarchy of constructible sets which he employed to prove the relative consistency of the continuum hypothesis. Here we have a significant, if indirect, application of type-theoretic ideas to set theory.

2 CRITIQUING RAMIFIED TYPES

A number of mathematicians/logicians, notably H. Weyl, L. Chwistek and F. P. Ramsey, attempted to remedy what they had identified as technical deficiencies in *Principia.* In his *Das Kontinuum* of 1918 Weyl presents a predicative formulation of mathematical analysis — not, as Russell and Whitehead had attempted, by introducing a hierarchy of logically ramified types, which Weyl seems to have regarded as too complicated — but rather by confining the comprehension principle to formulas whose bound variables range over just the initial given entities (numbers). Thus he restricts analysis to what can be done in terms of natural numbers with the aid of three basic logical operations, together with the operation of substitution and the process of "iteration", i.e., primitive recursion. Weyl recognized that the effect of this restriction would be to render unprovable many of the central results of classical analysis[12] — e.g., Dirichlet's principle that any bounded set of real numbers has a least upper bound — but he was prepared to accept this as part of the price that must be paid for the security of mathematics.

In 1925 Chwistek formulated what he called the theory of *constructive types.*[13] This is essentially the system of *Principia* purged of all existential propositions, including the axiom of reducibility. It has the property that each of its symbols

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[11] In this connection it is worth quoting the following extract from a review of *Principia* in a 1911 number of the London magazine *The Spectator:*

> It is easy to picture the dismay of the innocent person who out of curiosity looks into the later part of the book. He would come upon whole pages without a single word of *English below* the headline; he would see, instead, scattered in wild profusion, disconnected Greek and Roman letters of every size interspersed with brackets and dots and inverted commas, with arrows and exclamation marks standing on their heads, and with even more fantastic signs for which he would with difficulty so much as find names.

[12] This would also have been the case for a ramified type theory lacking the axiom of reducibility.

[13] Chwistek [1925].

can be obtained by the application of a finite number of operations applied to a set of initial symbols. Chwistek's system is in fact no stronger than Weyl's.

The year 1926 saw the publication of Ramsey's important paper *The Foundations of Mathematics*.[14] The core of Ramsey's paper is a critique of the logical system of *Principia Mathematica*, together with a proposal for setting it right. Ramsey identifies three "great defects in *Principia.* The first, that in it only definable classes are admitted, in violation of "the extensional attitude of modern mathematics, an essential part of which is "the possibility of indefinable classes and relations in extension." He continues

> The mistake is made not by having a primitive proposition asserting that all classes are definable, but by giving a definition of class which applies only to definable classes, so that all mathematical propositions about some or all classes are misinterpreted. This misinterpretation is not merely objectionable on its own account in a general way, but is especially pernicious in connection with the Multiplicative Axiom, which is a tautology when properly interpreted, but when misinterpreted after the fashion of *Principia Mathematica* becomes a significant empirical proposition, which there is no reason to suppose true.

By the multiplicative axiom Ramsey means the version of the axiom of choice asserting that, for any nonempty family $K$of disjoint nonempty sets, there is a set having exactly one member in common with each set in K. Ramsey says that according to the extensional view he takes of classes, "the Multiplicative Axiom seems... the most evident tautology." It only becomes really doubtful when "the class whose existence it asserts must be one definable by a propositional function of the sort which occurs in *Principia*". Ramsey shows that the multiplicative axiom (more precisely, its logical equivalent that the cardinalities of any pair of sets are comparable) can in fact be falsified in a *Principia*-like system in which there are very few atomic propositional functions.

Ramsey's chief objection against the theory of types presented in *Principia* is the fact that, in order to furnish an adequate foundation for mathematics, it requires the introduction of the "illegitimate" Axiom of Reducibility. Ramsey attributes this to the fact that the authors of *Principia* have failed to observe that the contradictions whose avoidance is one of the work's chief purposes actually fall into two "fundamentally distinct" groups. The first of these, of which Russell's paradox is representative, "consists of contradictions which, were no provision made against them, would occur in a logical or mathematical system itself." They "involve only logical or mathematical terms such as class and number, and show that there must be something wrong with our logic or mathematics." On the other hand the contradictions in the second group, of which the Liar Paradox and the "heterological" paradox are representative, "are not purely logical, and cannot be stated in logical terms alone; for they all contain some reference to thought, language, or symbolism, which are not formal but empirical terms." So "they may

[14] Ramsey [1926].

be due not to faulty logic or mathematics, but to faulty ideas concerning thought and language."

Ramsey points out that the theory of types really consists of two distinct parts "directed respectively against the two groups of contradictions", unified only by "being both deduced in a somewhat sloppy way from the 'vicious circle principle'." But it is essential, he says, to consider them separately. The contradictions in the first group can be eliminated by noting that a propositional function cannot meaningfully take itself as argument, which leads to a division of "functions and classes into a hierarchy of types according to their possible arguments." As a result, there are functions of individuals, functions of functions of individuals, etc. And then "the assertion that a class is a member of itself is neither true nor false, but meaningless." Ramsey regards this part of the theory of types — the part which would later evolve into the simple theory of types — as "unquestionably correct."

As for the second part of the theory, which is designed to contend with the second group of contradictions, it "requires further distinctions between the different functions which take the same arguments." These distinctions are reflected in the orders of *Principia*, the classification of functions according to the level of their bound variables. Ramsey agrees that the contradictions in the second group are circumvented by the introduction of orders, but this resolution "lands us in an almost equally serious difficulty, for it invalidates many important mathematical arguments which appear to contain exactly the same fallacy as the contradictions." To overcome this deficiency, the authors of *Principia* introduce the Axiom of Reducibility, a postulate which in Ramsey's view "there is no reason to suppose true", whose truth, indeed, would be merely "a happy accident and not a logical necessity." "Such an axiom," Ramsey asserts, "has no place in mathematics, and anything which cannot be proved without using it cannot be regarded as proved at all."

Ramsey then considers the question of why the Axiom of Reducibility fails to reproduce the contradictions (those in the second group) the ramified theory of types was explicitly designed to avoid. For in asserting that any function is coextensive with a predicative function the Axiom of Reducibility "may appear to lose again whatever was gained by making the distinction." Ramsey says that the reason why these contradictions fail to reappear is their "peculiar nature". For they "are not purely mathematical, but all involve the ideas of thought or meaning", in a word, they are *intensional.* If any purely mathematical contradictions were to arise through the conflating of arbitrary with predicative functions, or from the identification of intensionally different, but extensionally equivalent functions, then these contradictions would, says Ramsey, be "reinstated by the Axiom of Reducibility, owing to the extensional nature of mathematics, in which [extensionally] equivalent functions are interchangeable." The fact that no such contradictions have been shown to arise underscores the intensional nature of the contradictions in the second group, making it "even more probable that they have a psychological or epistemological, and not a purely logical or mathematical solution." Ramsey's conclusion is that "there is something wrong with the account of the matter given in *Principia.*"

Ramsey's attitude towards the foundations of mathematics and logic was robustly extensional, and accordingly realist. This realist attitude insulated him from the worries about impredicative definition that had exercised Russell (and Weyl). In Ramsey's view, the distinction of orders of functions is just a complication imposed by the structure of our language and not, unlike the hierarchy of types, something inherent in the way things truly are. Ramsey's suggestion in *The Foundations of Mathematics* for repairing the defects in *Principia* reflects his realist orientation. In essence, his proposal is to render the whole apparatus of orders superfluous through the simple expedient of eliminating quantifiers in definitions. So a universal quantifier is then regarded as indicating a conjunction, and an existential quantifier a disjunction, of the collection of propositions obtained from a given propositional function by the substitution of a name of any one of the members of the collection of objects, itself conceived as existing independently of any definition we might envisage, which constitutes the propositional function's range of significance.[15] It does not matter whether it may be impossible in practice to write out the resulting expressions in full. Once the apparatus of orders is abandoned, the simple theory of types remains, and Ramsey was convinced that this would provide an adequate foundation for mathematics.

3 CHURCH'S VERSION OF THE SIMPLE THEORY OF TYPES

When the ramified structure of the type hierarchy is abandoned, what remains is, as we have seen, a *simple* theory of types, similar to that which Russell had originally proposed for classes-as-many.

Neither Chwistek nor Ramsey produced a detailed formulation of the simple theory of types, although it seems clear that either could have achieved this had he so wished. In fact the first fully worked out formulation is that of Carnap [1929]; later formulations include Tarski [1931] and Gödel [1931].

The form of the simple theory of types which has proved definitive was put forward by Church [1940]. Church's system is based on functions instead of relations or classes, and incorporates certain features of the λ-calculus which had been previously developed by him. Church's system has had a profound influence on computer science, particularly in the areas of automated theorem proving, computational logic, and formal methods. Today the field of type theory (in computer science at least) is largely the study of Church-style systems based on functions with λ-notation.

A straightforward version of Church's simple type theory is the following system T.[16]

[15] On this reckoning, then, the statement ***Citizen Kane*** *has all the qualities that make a great film* would be taken as an abbreviation for something like ***Citizen Kane*** *is a film, brilliantly directed, superbly photographed, outstandingly performed, excellently scripted, etc.*

[16] Based on the presentation by Farmer [2006]; here we give a version free of the background

T is equipped with the following *types* and *terms.*

Types: $\mathbf{I}$, the type of individuals
$\mathbf{\Omega}$, the type of propositions or truth values
A collection of base types
Function types $(\mathbf{A} \to \mathbf{B})$, for any types $\mathbf{A}, \mathbf{B}$ (we omit the parentheses whenever possible)

Terms: Terms are introduced according to the following rules, so that each term t is assigned a type $\mathbf{A}$, written t: $\mathbf{A}$.
Variables $x_A, y_A, z_A, \ldots$ of each type $\mathbf{A}$ (we omit the subscript whenever possible)
Individuals a, b, c, ... of type $\mathbf{I}$
Function application. If t: $\mathbf{A}$ and f: $\mathbf{A} \to \mathbf{B}$, then $f(t) : \mathbf{B}$
Function abstraction. If t: $\mathbf{B}$, then $(\lambda x_A.t) : \mathbf{A} \to \mathbf{B}$
Equality. If $t : \mathbf{A}, u : \mathbf{A}$, then $t = u : \Omega$.

Terms of type Ω are called *propositions* or *formulas.*

Notice that T lacks logical operators. It was observed by Henkin [1963] that they can in fact be defined in a system like T. Let ω, ω' be variables of type Ω, and let α, β be propositions. Then we write

$\top$	for	$(\lambda\omega.\omega) = (\lambda\omega.\omega)$
$\bot$	for	$(\lambda\omega.\top) = (\lambda\omega.\omega)$
$\neg\alpha$	for	$\alpha = \bot$
$\alpha \Leftrightarrow \beta$	for	$\alpha = \beta$
$\alpha \wedge \beta$	for	$\lambda f.f(\top)(\top) = \lambda f.f(\alpha)(\beta),$ with $f : \Omega \to (\Omega \to \Omega)$
$\alpha \Rightarrow \beta$	for	$\alpha \wedge \beta \Leftrightarrow \alpha$
$\forall x.\alpha$	for	$\lambda x.\alpha = \lambda x.\top$
$\alpha \vee \beta$	for	$\forall\omega[[(\alpha \Rightarrow \omega) \wedge (\beta \Rightarrow \omega)] \Rightarrow \omega]$
$\exists x.\alpha$	for	$\forall\omega[[\forall x(\alpha \Rightarrow \omega) \Rightarrow \omega]$

A simple type theory like T, employing just function application, function abstraction, and equality, can serve as a foundation for mathematics in the sense that the usual mathematical structures, e.g. the natural numbers and the reals, admit straightforward formulations within it. And with appropriate axioms (including an axiom of infinity to the effect that the type of individuals is infinite), it can be shown that simple type theory has the same proof-theoretic strength as bounded Zermelo set theory.

Church's formulation of the simple theory of types is evidently far from being a *faute de mieux*; it is, rather, the theory that naturally ensues from an analysis of the function concept. This is well summed up by Robin Gandy:

> The simple theory of types provides a straightforward, reasonably secure foundation for the greater part of classical mathematics. This is

assumption of classical logic.

> why a number of authors (Carnap, Gödel, Tarski, Church, Turing) gave a precise formulation of it, and used it as a basis for metamathematical investigations. The theory is straightforward because it embodies two principles which (at least before the advent of modern abstract concepts) were part of the mathematicians' normal code of practice. Namely that a variable always has a precisely delimited range, and that a distinction must always be made between a function a and its arguments. [Gandy, 1977, p. 173]

4 TYPES VS. SETS

What is the difference between a type and a set? A first thought, naturally, is that types resemble the syntactic or grammatical categories forming the basis of language in that the specification of types furnish the conditions for expressions to be well-formed, or meaningful. Everybody knows (instinctively at least) that the linguistic phrase "___*sees*___" becomes grammatical only when the first blank is filled in with a noun or a noun phrase and the second with a noun, a noun phrase, or an adverb.

The typing concept also arises in concrete situations. Consider, for example, an automotive toolkit. Here one is provided with nuts, bolts and wrench bits to attach nuts to bolts. This gives three types: **N, B,** and **S.** Then given the instruction *Use* ___ *to attach* ___ *to* ___ , it is implicit that the blanks be filled in with (names of) components of types **S, N,** and **B** respectively.

This case points up nicely the difference between types and sets. For example, we can consider the concrete *set* or *collection* of, say, nuts, of a given toolkit. If (as is so often the case) a few of these go missing, the *set* of nuts is changed, but not the associated *type.* The set, that is, the concrete embodiment of the type, is subject to variation, but the type is not.

Here is another illustration. We can take two toolkits and *amalgamate* them. Now while it would be possible to do this simply by jumbling their contents together — that is, to take their set-theoretical union — , in order for the result to remain a toolkit in the specified sense *typing must be respected*: the nuts, bolts and wrench bits of each toolkit must be conjoined only with the nuts, bolts and wrench bits, respectively, of the other.

Sets and types, while different, are nevertheless clearly related. Indeed, since each type gives rise to the set of entities of that type, a set may be regarded as a *type considered in extension.* Clearly different types can give rise to the same set: for example those old chestnuts the types "rational featherless biped" and "human being" are distinct, but their associated sets are (terrestrially, at least) identical.

Now sets may also be regarded as *properties* considered in extension. This, conjoined with the fact that sets may be regarded as types considered in extension, might suggest that types are just properties. But that would be a mistake.For while the domain of properties is closed under the usual logical operations, these operations are not (in general) defined for types at all. For example, we may consider "natural number" and "circle" as types, but we do not similarly admit "natural number or circle" as defining a type.

The realm of types is, in fact, now customarily taken to be closed not under logical operations but under what may be termed *mathematical* operations. These may include the Cartesian product $\times$, the exponential or function type operation $\to$, and the power type operation $\mathbf{P}$. Here, for types $\mathbf{A}$, $\mathbf{B}$, $\mathbf{A} \times \mathbf{B}$ is the common type of pairs (a, b) with a of type $\mathbf{A}$ and b of type $\mathbf{B}$, $\mathbf{PA}$ is the type of sets of entities of type $\mathbf{A}$, and $\mathbf{A} \to \mathbf{B}$ is the type of correspondences (functions) between entities of types $\mathbf{A}$, $\mathbf{B}$, respectively.

The idea that a set may be regarded as a type considered in extension suggests a way of representing (simple) type theory in set theory, viz., just interpret types as sets, and function types $\mathbf{A} \to \mathbf{B}$ as the set of functions from (the set corresponding to) $\mathbf{A}$ to (the set corresponding to) $\mathbf{B}$.

The representation of set theory within type theory is an altogether subtler affair. The problem here is to represent within type theory the membership relation, which is defined for all pairs of sets. An elegant way of doing this has been elaborated by Miquel [2001] using the notions of pointed graph and bisimilarity.

A *pointed graph* is a triple $\mathfrak{A} = (A, R, a)$ with A a set, R a binary relation on A, and $a \in A$. A *bisimilarity* between two pointed graphs $\mathfrak{A} = (A, R, a)$ and $\mathfrak{B} = (B, S, b)$ is a relation $\approx$ between A and B satisfying the conditions:

- $\forall x x' \in A \forall y \in B[x' R x \wedge x \approx y \Rightarrow \exists y' \in B(y' S y \wedge x' \approx y')]$
- $\forall y y' \in B \forall x \in A[y' S y \wedge x \approx y \Rightarrow \exists x' \in A(x' R x \wedge x' \approx y)]$
- $a \approx b$

Two pointed graphs $\mathfrak{A}, \mathfrak{B}$ are *bisimilar*, written $\mathfrak{A} \sim \mathfrak{B}$, if there is a bisimilarity between them. It is readily checked that bisimilarity is an equivalence relation.

We shall require the *transitive closure* $\tau(A)$ of a set A: this is the set whose elements are A itself, the members of A, the members of members of A, etc. Now let $\mathfrak{A}^*$ be the pointed graph $(\tau(A), \in |\tau(A), A)$. A straightforward induction on rank argument shows that, for any sets A, B,

$$\mathfrak{A}^* \sim \mathfrak{B}^* \Leftrightarrow A = B.$$

It follows that

$$A \in B \Leftrightarrow \exists X(X \in B \wedge \mathfrak{X}^* \iota \mathfrak{A}^*). \qquad (*)$$

This last equivalence,[17] which expresses set membership in terms of bisimilarity of pointed graphs — *shifted bisimilarity* — furnishes the basis for the representation of sets as types. The idea is to interpret sets as pointed graphs, and then to observe that pointed graphs can be represented as types.

[17] While this equivalence was actually proved only for *well-founded* sets, its purpose is to suggest the form of the type-theoretic representation of the membership relationship for *arbitrary* (i.e., not necessarily well-founded) sets.

In type-theoretic terms, a pointed graph may be defined as a triple $(\mathbf{A}, R, a)$ with $\mathbf{A}$ a type, R a term of type $\mathbf{A} \to (\mathbf{A} \to \Omega)$, and a a term of type $\mathbf{A}$. The conditions for a bisimilarity between pointed graphs given above — and so also the representation of set membership in terms of shifted bisimilarity — are then readily translated into the language of type theory. This yields a faithful interpretation of set theory[18] in a certain version of type theory[19] under which sets correspond to pointed graphs and the membership relation corresponds to shifted bisimilarity.

5 CATEGORIES AND TYPES

When Eilenberg and Mac Lane first introduced (in 1945) the concepts of category and functor their concerns were far removed from the theory of types. Yet category theory and type theory have proved to have deep connections. In fact categories can *themselves* be viewed as type theories of a certain kind; this fact alone indicates that type theory is much more closely related to category theory than it is to set theory. In thinking of a category as a type theory, the objects of a category are regarded as *types* (or sorts) and the arrows as *mappings* between the corresponding types. Roughly speaking, a category may be thought of a type theory shorn of its syntax. In the 1970s Lambek[20] established that, viewed in this way, cartesian closed categories correspond to the typed λ-calculus. Later Seely [1984] proved that locally Cartesian closed categories correspond to Martin-Löf, or predicative, type theories. Lambek and Dana Scott independently observed that *C-monoids*, i.e., categories with products and exponentials and a single, nonterminal object correspond to the *untyped* λ-calculus. The analogy between type theories and categories has since led to what Jacobs [1999] terms a "type-theoretic boom", with much input from, and applications to, computer science.

5.1 Cartesian closed categoires and the typed λ-calculus

As already mentioned, Lambek has shown that the typed λ-calculus corresponds to a cartesian closed category. Here we give a brief sketch of how this correspondence is established.

A *typed λ-calculus* is a theory formulated within a certain kind of formal language which we shall call a *λ-language*. A language Λ of this kind is equipped with the following *basic symbols*:

- a *type symbol* $\mathbf{1}$
- *ground types* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ (possibly none of these)
- *function symbols* $\mathbf{f}, \mathbf{g}, \mathbf{h}, \ldots$ (possibly none of these)

[18]I.e., intuitionistic Zermelo (or even Zermelo-Fraenkel) set theory.
[19]To be precise, polymorphic dependent type theory (see below) with a type of propositions.
[20]See the reference entries under Lambek and Lambek and Scott.

- *variables* $x_A, y_A, z_A, \ldots$ of each type $\mathbf{A}$, where a *type* is as defined below
- unique entity $\bigstar$ of type **1.**

The *types* of Λ are defined recursively as follows:

- **1** and each ground type are types
- $\mathbf{A}_1 \times \ldots \times \mathbf{A}_n$ is a type whenever $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are, where, if $n = 1$, $\mathbf{A}_1 \times \ldots \times \mathbf{A}_n$ is $\mathbf{A}_1$, while if $n = 0$, $\mathbf{A}_1 \times \ldots \mathbf{A}_n$ is $\mathbf{1}$ (*product types*)
- $\mathbf{B}^A$ is a type whenever $\mathbf{A}, \mathbf{B}$ are.

Each function symbol $\mathbf{f}$ is assigned a *signature* of the form $\mathbf{A} \to \mathbf{B}$ where $\mathbf{A}, \mathbf{B}$ are types. We normally write $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ to indicate this.

Terms of Λ and their associated *types* are defined recursively as follows. We write $\tau : \mathbf{A}$ to indicate that the term τ has type $\mathbf{A}$.

Term: **type**	Proviso
$\bigstar : \mathbf{1}$	
$x_A : \mathbf{A}$ (we shall usually omit the subscript)	
$\mathbf{f}(\tau) : \mathbf{b}$	$\mathbf{f} : \mathbf{A} \to \mathbf{B} \quad \tau : \mathbf{A}$
$\langle \tau_1, \ldots, \tau_n \rangle : \mathbf{A}_1 \times \ldots \times \mathbf{A}_n$, where $\langle \tau_1, \ldots, \tau_n \rangle$ is τ_1 if $n = 1$, and $\bigstar$ if $n = 0$.	$\tau_1 : \mathbf{A}_1, \ldots, \tau_n : \mathbf{A}_n$
$(\tau)_i : \mathbf{A}_i$ where $(\tau)_i$ is τ if $n = 1$	$\tau : \mathbf{A}_1 \times \ldots \times \mathbf{A}_n, 1 \leq i \leq n$
$\tau'\sigma : \mathbf{B}$	$\tau : \mathbf{B}^A \quad \sigma : \mathbf{A}$
$\lambda_x \tau : \mathbf{B}^A$	$x : \mathbf{A} \quad \tau : \mathbf{B}$

We write $\tau(x/\sigma)$ or $\tau(\sigma)$ for the result of substituting σ at each free occurrence of x in τ, where an occurrence of x is *free* if it does not appear within a term of the form $\lambda_x \rho$. A term σ is *substitutable* for a variable x in a term τ if no variable free in σ becomes bound when so substituted.

An *equation* in Λ is an expression of the form $\sigma = \tau$, where σ and τ are terms of the same type. A *theory* in Λ (or simply a λ-*theory*) is a set T of equations satisfying the following conditions:

- $\sigma = \sigma \in T \quad \sigma = \tau \in T \Rightarrow \tau = \sigma \in T$
  $\sigma = \tau \in T \;\&\; \tau = \upsilon \in T \Rightarrow \sigma = \upsilon \in T$
- $\sigma = \upsilon \in T \Rightarrow \tau'\sigma = \tau'\upsilon \in T \quad \sigma = \tau \in T \Rightarrow \lambda_x \sigma = \lambda_x \tau \in T$
- T contains all equations of the following forms:
  $\tau = *$ for $\tau : \mathbf{1}$

  $$(\langle \tau_1, \ldots, \tau_n \rangle)_i = \tau_i \quad \tau = \langle (\tau)_1, \ldots, (\tau)_n \rangle$$
  $$(\lambda_x \tau)'\sigma = \tau(x/\sigma) \text{ for all } \sigma \text{ substitutable for } x \text{ in } \tau$$

$$\lambda_x(\tau' x) = \tau \text{ for all } \tau : \mathbf{B}^A, \text{ provided } x : \mathbf{A} \text{ is not free in } \tau.$$

Now λ-theories comprise the objects of a category $\lambda - \mathcal{T}h$ whose arrows are the so-called *translations* between such theories. Let us define a *translation* $\mathbf{K}$ of a λ-language $\mathcal{L}$ into a λ-language $\mathcal{L}'$ to be a map which assigns to each type $\mathbf{A}$ of $\mathcal{L}$ a type $\mathbf{KA}$ of $\mathcal{L}'$ and to each function symbol $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ a function symbol of $\mathcal{L}'$ of signature $\mathbf{KA} \to \mathbf{KB}$ in such a way that

$$\mathbf{K1} = 1,\ \mathbf{K}(\mathbf{A}_1 \times \ldots \times \mathbf{A}_n) = \mathbf{KA}_1 \times \ldots \mathbf{KA}_n,\ \mathbf{K}(\mathbf{B}^A) = \mathbf{KB}^{\mathbf{KA}}.$$

Any translation $\mathbf{K} : \mathcal{L} \to \mathcal{L}'$ may be extended to the terms of $\mathcal{L}$ in the evident recursive way — i.e., by defining $\mathbf{K}x_A = x_{\mathbf{K}A}, \mathbf{K}* = *, \mathbf{K}(\mathbf{f}(\tau)) = \mathbf{Kf}(\mathbf{K}_\tau), \mathbf{K}(\lambda_x \tau) = \lambda_{\mathbf{K}x}\mathbf{K}\tau$, etc. — so that if $\tau : \mathbf{A}$, then $\mathbf{K}\tau : \mathbf{KA}$. We shall sometimes write α_K for $\mathbf{K}\alpha$. If T, T' are λ-theories in $\mathcal{L}, \mathcal{L}'$ respectively, a translation $\mathbf{K} : \mathcal{L} \to \mathcal{L}'$ is a *translation of* T *into* T', and is written $\mathbf{K} : T \to T'$ if, for any equation $\sigma = \tau$ in T, the equation $\mathbf{L}\sigma = \mathbf{K}\tau$ is in T'.

As we shall see, λ-theories correspond to Cartesian closed categories in such a way as to make the category λ-$\mathcal{T}h$ equivalent to the category $\mathcal{C}art$ of Cartesian closed categories. To establish this, we first introduce the notion of an *interpretation* I of a λ-language $\mathcal{L}$ in a cartesian closed category $\mathcal{C}$. This is defined to be an assignment

- to each type $\mathbf{A}$, of a $\mathcal{C}$-object $\mathbf{A}_I$ such that:
  $(\mathbf{A}_1 \times \ldots \times \mathbf{A}_n)_I = (\mathbf{A}_1)_I \times \ldots \times (\mathbf{A}_n)_I$,
  $(\mathbf{B}^A)_I = (\mathbf{B}_I)^{A_I}$
  $\mathbf{1}_I = 1$, the terminal object of $\mathcal{C}$,
- to each function symbol $\mathbf{f} : \mathbf{A} \to \mathbf{B}$, a $\mathcal{C}$-arrow $\mathbf{f}_I : A_I \to B_I$.

We shall sometimes write $A_{\mathcal{C}}$ or just A for $\mathbf{A}_I$.

We extend I to terms of $\mathcal{L}$ as follows. If $\tau : \mathbf{B}$, write x for $(x_1, \ldots, x_n)$, any sequence of variables containing all variables of τ (and call such sequences *adequate* for τ. Define the $\mathcal{C}$-arrow.

$$[\![\tau]\!]_x : A_1 \times \ldots A_n \to B$$

recursively as follows:

$$\begin{aligned}
&[\![*]\!]_x = A_1 \times \ldots \times A_n \longrightarrow 1 \\
&[\![x_i]\!]_x = \pi_i : A_1 \times \ldots \times A_n \longrightarrow A_i \\
&[\![\mathbf{f}(\tau)]\!]_x = \mathbf{f}_I \circ [\![\tau]\!]_x \\
&[\![\langle \tau_1, \ldots, \tau_n \rangle]\!]_x = \langle [\![\tau_1]\!]_x, \ldots, [\![\tau_n]\!]_x \rangle \\
&[\![(\tau)_i]\!]_x = \pi_i \circ [\![\tau]\!]_x \\
&[\![\tau'\sigma]\!]_x = ev \circ \langle [\![\tau]\!]_x, [\![\sigma]\!]_x \rangle \\
&[\![\lambda_y \tau]\!]_x = \widehat{[\![\tau]\!]_{xy}}
\end{aligned}$$

Here $\widehat{\ }$ denotes exponential transpose and ev the appropriate evaluation arrow in the definition of the exponential.[21]

Next, one shows that any λ-theory T determines a Cartesian closed category $\mathcal{C}(T)$. The objects of $\mathcal{C}(T)$ are taken to be the types of the language of T. The arrows of $\mathcal{C}(T)$ are pairs of the form (x, τ), where τ is a term with no free variables other than x, two such pairs (x, τ) and (y, σ) being identified whenever the equation $\tau = \sigma$ is a member of T. The identity arrow on a type $\mathbf{A}$ is the pair (x_A, x_A). The composite of $(x, \tau) : \mathbf{A} \to \mathbf{B}$ and $(y, \sigma) : \mathbf{B} \to \mathbf{C}$ is given by the pair $(x, \sigma(y/\tau))$. It is now readily checked that $\mathcal{C}(T)$ is a Cartesian closed category with terminal object the type $\mathbf{1}$ and in which products and exponentials are given by the analogous operations on types. Moreover,

$$
\begin{gathered}
1_A = (x_A, *) \\
\pi_i = (z, (z)_i) \text{ (with } z : \mathbf{A}_1 \times .. \times \mathbf{A}_n) \\
\langle (z, \sigma), (z, \tau) \rangle = (z, \langle \sigma, \tau \rangle) \\
\widehat{(z, \tau)} = (x, \lambda_y \tau(\langle x, y \rangle) \text{ with } z : \mathbf{A} \times \mathbf{B}, x : \mathbf{A}, y : \mathbf{B} \\
ev_{C,A} = (y, (y)_2'(y)_1) \text{ with } y : \mathbf{C}^A \times \mathbf{A}.
\end{gathered}
$$

There is a canonical interpretation of $\mathcal{L}$ in $\mathcal{C}(T)$ — which we denote by the same expression $\mathcal{C}(T)$ — given by

$$\mathbf{A}_{\mathcal{C}(T)} = \mathbf{A} \qquad \mathbf{f}_{\mathcal{C}(T)} = (x, \mathbf{f}(x)).$$

A straightforward induction shows that, for any term τ with free variables $\mathbf{x} = (x_1, ..., x_n)$ of types $\mathbf{A}_1, ..., \mathbf{A}_n$, we have

$$[\![\tau]\!]_x = (z, \tau)$$

where z is a variable of type $\mathbf{A}_1 \times ... \times \mathbf{A}_n$.

Inversely, each Cartesian closed category $\mathcal{E}$ determines a λ-theory $Th(\mathcal{E})$ in the following way. First, one defines the *internal language* $\mathcal{L}_{\mathcal{E}}$ of $\mathcal{E}$: the ground type symbols of $\mathcal{L}_{\mathcal{E}}$ are taken to match the objects of $\mathcal{E}$ other than 1, that is, for each $\mathcal{E}$-object A other than 1 we assume given a ground type $\mathbf{A}$. Next, we with each type symbol $\mathbf{A}$ of $\mathcal{L}_{\mathcal{E}}$ we associate the $\mathcal{E}$-object $\mathbf{A}_{\mathcal{E}}$ defined by

$$\mathbf{A}_{\mathcal{E}} = A \text{ for ground types } \mathbf{A},\ (\mathbf{A} \times \mathbf{B})_{\mathcal{E}} = \mathbf{A}_{\mathcal{E}} \times \mathbf{B}_{\mathcal{E}},\ (\mathbf{B}^A)_{\mathcal{E}} = \mathbf{B}_{\mathcal{E}}^{A}\mathcal{E}.$$

As function symbols in $\mathcal{L}_E$ we take triples of the form $(f, \mathbf{A}, \mathbf{B}) = \mathbf{f}$ with $f : \mathbf{A}_{\mathcal{E}} \to \mathbf{B}_{\mathcal{E}}$ in $\mathcal{E}$. The signature of $\mathbf{f}$ is $\mathbf{A} \to \mathbf{B}$.

The *natural interpretation* — denoted by $\mathcal{E}$ — of $\mathcal{L}_{\mathcal{E}}$ in $\mathcal{E}$ is determined by the assignments

$$\mathbf{A}_{\mathcal{E}} = A \text{ for ground types } \mathbf{A},\ (f,\ \mathbf{A},\ \mathbf{B}) = f.$$

[21] For these notions see the Appendix.

The λ-*theory* Th($\mathcal{E}$) *of* $\mathcal{E}$ is now defined in the following way. Given a pair of terms σ, τ of the same type in $\mathcal{L}_{\mathcal{E}}$, write $\mathbf{x}$ for the sequence of variables occurring free in either σ or τ. The theory $Th(\mathcal{E})$ consists of all equations $\sigma = \tau$ for which the arrows $([\![\sigma]\!]_{\mathbf{x}})_{\mathcal{E}}$ and $([\![\tau]\!]_{\mathbf{x}})_{\mathcal{E}}$ coincide.

It can then be shown that the canonical functor $F : \mathcal{E} \to \mathcal{C}(Th(\mathcal{E}))$ defined by

$$FA = \mathbf{A} \text{ for each } \mathcal{E}\text{-object } A$$
$$Ff = (x, \mathbf{f}(x)) \text{ for each } \mathcal{E}\text{-arrow } f : A \to B$$

is an *isomorphism of categories.*

Similarly, for each λ-theory T there is a canonical translation $G : T \to Th(\mathcal{C}(T))$ given by

$$G\mathbf{A} = \mathbf{A} \text{ for each type } \mathbf{A}$$
$$G\mathbf{f} = ((x, \mathbf{f}(x), \mathbf{A}, \mathbf{B})) \text{ for } \mathbf{f} : \mathbf{A} \to \mathbf{B}.$$

This translation is clearly an isomorphism in the category λ-$\mathcal{T}h$. Now let $\mathcal{C}art$ be the category whose objects are Cartesian closed categories and whose arrows are functors between these preserving terminal objects, products, and exponentials. The mappings $\mathcal{C}$ and $\mathcal{T}h$ act as functors λ-$\mathcal{T}h \to \mathcal{C}art$ and $\mathcal{C}art \to \lambda$-$\mathcal{T}h$ respectively; in view of the fact that, for any objects $\mathcal{E}$ of $\mathcal{C}art$, T of λ-$\mathcal{T}h$, $\mathcal{C}(\mathrm{Th}(\mathcal{E})) \cong \mathcal{E}$ and $\mathrm{Th}(\mathcal{C}(T)) \cong T$, these functors are equivalences. Accordingly λ-$\mathcal{T}h$ and $\mathcal{C}art$ are equivalent categories. This is one exact sense in which a formal theory is completely representable in categorical terms.

5.2 Logical Languages and Local Set Theories

What sort of category corresponds to the simple theory of types[22] itself? The answer to this question — a(n *elementary*)*topos* — was provided in 1969 by Lawvere and Tierney,[23] although it was not at first grasped that their work in fact could in fact be seen as answering that question. However, it soon became clear that toposes are associated with a natural version of simple type theory, based on intuitionistic logic, which we shall call *intuitionistic type theory.* The system to be described here, *local set theory,* is a modification, due to Zangwill [1977], of that of Joyal and Boileau, later published as their [1981]. It combines the most convenient features of simple type theory and set theory. A full account is given in Bell [1988].

A local set theory is a type-theoretic system built on the same primitive symbols $=, \in, \{\ :\ \}$ as classical set theory, in which the set-theoretic operations of forming products and powers of types can be performed, and which in addition contains a "truth value" type acting as the range of values of "propositional functions"

[22]It should be stressed that here by the "simple theory of types" is meant the theory of types in Church's sense which includes a type of propositions. Without a type of propositions the simple theory of types is essentially just the typed λ-calculus.

[23]See Lawvere [1971, 1972].

on types. A local set theory is determined by specifying a collection of *axioms* formulated within a *local language* defined as follows.

A *local language* $\mathcal{L}$ has the following *basic symbols:*

- $\mathbf{1}$ (*unit type*) $\mathbf{\Omega}$ (*truth value type*)
- **S, T, U,...** (*ground types*: possibly none of these)
- **f, g, h,...** (*function symbols*: possibly none of these)
- $x_A, y_A, z_A, \ldots$ (*variables of each type* **A**, where a *type* is as defined below)
- $\bigstar$ (*unique entity of type* **1**)

The *types* of $\mathcal{L}$ are defined recursively as follows:

- $\mathbf{1}$, Ω are types
- any ground type is a type
- $\mathbf{A}_1 \times \ldots \times \mathbf{A}_n$ is a type whenever $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are, where, if $n = 1$, $\mathbf{A}_1 \times \ldots \times \mathbf{A}_n$ is $\mathbf{A}_1$, while if $n = 0$, $\mathbf{A}_1 \times \ldots \times \mathbf{A}_n$ is $\mathbf{1}$ (*product types*)
- **PA** is a type whenever **A** is (*power types*)

Each function symbol **f** is assigned a *signature* of the form $\mathbf{A} \to \mathbf{B}$, where $\mathbf{A}, \mathbf{B}$ are types; this is indicated by writing $\mathbf{f} : \mathbf{A} \to \mathbf{B}$.

Terms of $\mathcal{L}$ and their associated *types* are defined recursively as follows. We write $\tau : \mathbf{A}$ to indicate that the term τ has type **A**.

Term: **type**	Proviso
$\bigstar : \mathbf{1}$	
$x_A : \mathbf{A}$	
$\mathbf{f}(\tau) : \mathbf{B}$	$\mathbf{f}\colon \mathbf{A} \to \mathbf{B}$ $\quad \tau : \mathbf{A}$
$\langle \tau_1, ..., \tau_n \rangle : \mathbf{A}_1 \times ... \times A_n$, where $\langle \tau_1, ..., \tau_n \rangle$ is τ_1 if $n = 1$, and $\bigstar$ if $n = 0$.	$\tau_1 : \mathbf{A}_1, \ldots, \tau_n\colon \mathbf{A}_n$
$(\tau)_i\colon \mathbf{A}_i$ where $(\tau)_i$ is τ if n= 1	$\tau : \mathbf{A}_1 \times \ldots \times \mathbf{A}_n$, $1 \leq i \leq n$
$\{x_A\colon \alpha\}\colon \mathbf{PA}$	$\alpha : \Omega$
$\sigma = \tau\colon \Omega$	σ, τ of same type
$\sigma \in \tau$	$\sigma : \mathbf{A}$, $\tau\colon \mathbf{PA}$ for some type **A**

Terms of type Ω are called *formulas, propositions,* or *truth values.* Notational conventions we shall adopt include:

$\omega, \omega', \omega''$	variables of type Ω
α, β, γ	formulas
$x, y, z, \ldots$	$x_A, y_A, z_A \cdots$
$\tau(x/\sigma)$ or $\tau(\sigma)$	result of substituting σ at each free occurrence of x in τ: an occurrence of x is *free* if it does not appear within $\{x : \alpha\}$
$\alpha \leftrightarrow \beta$	$\alpha = \beta$
$\Gamma : \alpha$	sequent notation: Γ a finite set of formulas
$: \alpha$	$\emptyset : \alpha$

A term is *closed* if it contains no free variables; a closed term of type Ω is called a *sentence.*

The *basic axioms* for $\mathcal{L}$ are as follows:

Unity: $x_1 = *$

Equality: $x = y, \alpha(z/x) : \alpha(z/y)$ (x, y free for z in α)

Products: $(\langle x_1, \ldots, x_n \rangle)_i = x_i$

$x = \langle (x)_1, \ldots, (x)_n \rangle$

Comprehension: $x \in \{x : \alpha\} \leftrightarrow \alpha$

The *rules of inference* for $\mathcal{L}$ are:

Thinning: $\dfrac{\Gamma : \alpha}{\beta, \Gamma : \alpha}$

Restricted Cut: $\dfrac{\Gamma : \alpha \quad \alpha, \Gamma : \beta}{\Gamma : \beta}$ (any free variable of α free in Γ or β)

Substitution: $\dfrac{\Gamma : \alpha}{\Gamma(x/\tau) : \alpha(x/\tau)}$ (τ free for x in Γ and α)

Extensionality: $\dfrac{\Gamma : x \in \sigma \leftrightarrow x \in \tau}{\Gamma : \sigma = \tau}$ (x not free in Γ, σ, τ)

Equivalence: $\dfrac{\alpha, \Gamma : \beta \qquad \beta, \Gamma : \alpha}{\Gamma : \alpha \leftrightarrow \beta}$

These axioms and rules of inference yield a system of *natural deduction* in $\mathcal{L}$. If S *is* any collection of sequents in $\mathcal{L}$, we say that the sequent $\Gamma : \alpha$ is *deducible from* S, and write $\Gamma \vdash_S \alpha$ provided there is a deduction of $\Gamma : \alpha$ using the basic axioms, the sequents in S, and the rules of inference. We shall also write $\Gamma \vdash \alpha$ for $\Gamma \vdash_\emptyset \alpha$ and $\vdash_S \alpha$ for $\emptyset \vdash_S \alpha$.

A *local set theory* in $\mathcal{L}$ is a collection S of sequents closed under deducibility from S. Any collection of sequents S *generates* the local set theory $S*$ comprising all the sequents deducible from S. The local set theory in $\mathcal{L}$ generated by $\emptyset$ is called *pure* local set theory in $\mathcal{L}$.

5.3 Logic in a Local Set Theory

The *logical operations* in $\mathcal{L}$ are defined as follows:

Logical Operation	**Definition**
$\top$ (true)	$\bigstar = \bigstar$
$\alpha \wedge \beta$	$\langle \alpha, \beta \rangle = \langle \top, \top \rangle$
$\alpha \to \beta$	$(\alpha \wedge \beta) \leftrightarrow \alpha$
$\forall x \alpha$	$\{x : \alpha\} = \{x : \top\}$
$\bot$ (false)	$\forall \omega . \omega$
$\neg \alpha$	$\alpha \to \bot$
$\alpha \vee \beta$	$\forall \omega [(\alpha \to \omega \wedge \beta \to \omega) \to \omega]$
$\exists x \alpha$	$\forall \omega [\forall x (\alpha \to \omega) \to \omega]$

We also write $x \neq y$ for $\neg(x = y), x \notin y$ for $\neg(x \in y)$, and $\exists! x \alpha$ for $\exists x[\alpha \wedge \forall y(\alpha(x/y) \to x = y)]$.

It can now be shown that the logical operations on formulas just defined satisfy the axioms and rules of free intuitionistic logic. We present just a few of the relevant derivations.

We write

$$\frac{\Gamma_1 : \alpha_1, \ldots, \Gamma_n : \alpha_n}{\Delta : \beta}$$

for deducibility of $\Delta : \beta$ from $\Gamma_1 : \alpha_1, \ldots, \Gamma_n : \alpha_n$.

$$\vdash x = x.$$

Derivation: $: (x)_1 = x$.

$$\alpha \vdash \alpha$$

Derivation: $$\cfrac{\cfrac{\omega, \omega = \omega : \omega \qquad \overline{: \omega : \omega}}{\omega : \omega}}{\alpha : \alpha}$$

$$\frac{\Gamma : a \quad \Gamma : \beta}{\Gamma : \alpha \wedge \beta}$$

Derivation:

$$\dfrac{\dfrac{}{\beta:\beta=\top} \qquad \dfrac{\dfrac{}{\alpha:\alpha=\top} \qquad \dfrac{}{\alpha=\top,\beta=\top:\alpha\wedge\beta}}{\alpha,\beta=\top:\alpha\wedge\beta}}{\dfrac{\Gamma:\alpha \qquad \Gamma,\beta:\alpha\wedge\beta}{\Gamma:\alpha\wedge\beta}}$$

$$\dfrac{\Gamma:\alpha\leftrightarrow\beta}{\gamma:\{x:\alpha\}=\{x:\beta\}}$$

Derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma:\alpha\leftrightarrow\beta}{\dfrac{/}{\alpha,\Gamma:\beta} \qquad \dfrac{\backslash}{\beta,\Gamma:\alpha}}}{x\in\{x:\alpha\},\Gamma:x\in\{x:\beta\} \quad x\in\{x:\beta\},\Gamma:x\in\{x:\alpha\}}}{\dfrac{\Gamma:x\in\{x:\alpha\}\leftrightarrow x\in\{x:\beta\}}{\Gamma:\{x:a\}=\{x:\beta}}$$

In any local set theory the following *Modified Cut Rule* holds:

(i) $$\dfrac{\Gamma:\alpha \qquad \alpha,\Gamma:\beta}{\exists x_1(x_1=x_1),\ldots,\exists x_n(x_n=x_n),\Gamma:\beta}$$
where $x_1,\ldots,x_n$ are the free variables of α not occurring freely in Γ or β.

(ii) $$\dfrac{\Gamma:\alpha \qquad \alpha,\Gamma:\beta}{\Gamma:\beta}$$
provided that, whenever **A** is the type of a free variable of α with no free occurrences in Γ or β, there is a closed[24] term of type **A.**

5.4 Set Theory in a Local Language

We can now introduce the concept of *set* in a local language. A *set-like* term is a term of power type; a *closed* set-like term is called an ($\mathcal{L}-$) *set.* We shall use upper case italic letters $X, Y, Z, ...$ for sets, as well as standard abbreviations such as $\forall x \in X.\alpha$ for $\forall x(x \in X \to \alpha)$. The set theoretic *operations* and *relations* are defined as follows. Note that in the definitions of $\subseteq$, $\cap$, and $\cup$, X and Y *must be of the same type*:

[24] A term is *closed* if it contains no free variables.

Operation	**Definition**
$\{x \in X : \alpha\}$	$\{x : x \in X \wedge \alpha\}$
$X \subseteq Y$	$\forall x \in X . x \in Y$
$X \cap Y$	$\{x : x \in X \wedge x \in Y\}$
$X \cup Y$	$\{x : x \in X \vee x \in Y\}$
$x \notin X$	$\neg(x \in X)$
U_A or A	$\{x_A : \top\}$
$\emptyset_A$ or $\emptyset$	$\{x_A : \bot\}$
$E - X$	$\{x : x \in E \wedge x \notin X\}$
PX	$\{u : u \subseteq X\}$
$\bigcap U$ $(U : \mathbf{PPA})$	$\{x : \forall u \in U . x \in u\}$
$\bigcup U$ $(U : \mathbf{PPA})$	$\{x : \exists u \in U . x \in u\}$
$\bigcap_{i \in I} X_i$	$\{x : \forall i \in I . x \in X_i\}$
$\bigcup_{i \in I} X_i$	$\{x : \exists i \in I . x \in X_i\}$
$\{\tau_1, \ldots, \tau_n\}$	$\{x : x = \tau_1 \vee \ldots \vee x = \tau_n\}$
$\{\tau : \alpha\}$	$\{z : \exists x_1 \ldots \exists x_n (z = \tau \wedge \alpha)\}$
$X \times Y$	$\{\langle x, y\rangle : x \in X \wedge y \in Y\}$
$X + Y$	$\{\langle \{x\}, \emptyset\rangle : x \in X\} \cup \{\langle \emptyset, \{y\}\rangle : y \in Y\}$
$Fun(X, Y)$	$\{u : u \subseteq X \times Y \wedge \forall x \in X \exists! y \in Y . \langle x, y\rangle \in u\}$

The standard facts concerning the set-theoretic operations and relations now follow as straightforward consequences of their definitions.

Given a term τ such that

$$\langle x_1, ..., x_n\rangle \in X \vdash_S \tau \in Y$$

we write $(\langle x_1, ..., x_n\rangle \mapsto \tau)$ or simply $\mathbf{x} \mapsto \tau$ for

$$\{\langle x_1, ..., x_n\rangle, \tau : \langle x_1, ..., x_n\rangle \in X\}.$$

Clearly we have

$$\vdash_S (\langle x_1, ..., x_n\rangle \mapsto \tau)) \in Fun(X, Y),$$

and so we may think of $(\langle x_1, ..., x_n\rangle \mapsto \tau)$ as the function from X to Y determined by τ.

We now show that each local set theory determines a topos. Let S be a local set theory in a local language $\mathcal{L}$. Define the relation $\approx_S$ on the collection of all $\mathcal{L}$-sets by

$$X \approx_S Y \text{ iff } \vdash_S X = Y.$$

This is an equivalence relation. An *S-set* is an equivalence class $[X]_S$ — which we normally identify with X — of $\mathcal{L}$-sets under the relation $\approx_S$. An *S-map f:* $X \to Y$ is a triple (f, X, Y) — normally identified with f — of S-sets such that $\vdash_S f \in Y^X$. X and Y are, respectively, the *domain* $\mathrm{dom}(f)$ and the *codomain* cod(f) of f. It is now readily shown that the collection of all $S-$sets and maps forms a category $\mathcal{C}(S)$, the *category of S-sets,* in which the composite of two maps $f : X \to Y$and $g : Y \to Z$ is given by

$$g \circ f = \{\langle x, z\rangle : \exists y(\langle x, y\rangle \in f \wedge \langle y, z\rangle \in g)\}.$$

In fact, $\mathcal{C}(S)$ is a *topos*, the *topos of sets determined by* S. It has terminal object $U_{\mathbf{1}}$, the product of two objects (S-sets) X, Y is the S-set $X \times Y$, with projections given by

$$\pi_1 = (\langle x, y\rangle \mapsto x) : X \times Y \to X, \quad \pi_2 = (\langle x, y\rangle \mapsto x) : X \times Y \to Y,$$

its truth-value object is (Ω, t), where $t : 1 \to \Omega$ is the S-map $\{\langle *, \top\rangle\}$, and the power object of an object X is $(PX,\ e_X)$, where $e_X : X \times PX \to \Omega$ is the S-map $\langle x, z\rangle \mapsto x \in z$. All this is proved in much the same way as for classical set theory.

It can also be shown that properties of maps in $\mathcal{C}(S)$ are characterized in just the same way as in ordinary set theory, e.g. a map is monic iff it is one-to-one, epic iff it is onto, and an isomorphism iff it is both. This fact will be used implicitly in the sequel.

5.5 Interpreting a Local Language in a Topos: The Soundness and Completeness Theorems

Let $\mathcal{L}$ be a local language and $\mathcal{E}$ a topos. An *interpretation* I of $\mathcal{L}$ in $\mathcal{E}$ is an assignment:

- to each type $\mathbf{A}$, of an $\mathcal{E}$-object $\mathbf{A}_I$ such that:
  $(\mathbf{A}_1 \times \ldots \times \mathbf{A}_n)_I = (\mathbf{A}_1)_I \times \ldots \times (\mathbf{A}_n)_I$.
  $(\mathbf{PA})_I = P\mathbf{A}_I$,
  $\mathbf{1}_I = 1$, the terminal object of $\mathcal{E}$,
  $\Omega_I = \Omega$, the truth-value object of $\mathcal{E}$.
- to each function symbol $\mathbf{f} : \mathbf{A} \to \mathbf{B}$, an $\mathcal{E}$-arrow $\mathbf{f}_I : \mathbf{A}_I \to \mathbf{B}_I$.

We shall sometimes write $A_{\mathcal{E}}$ or just A for $\mathbf{A}_I$.

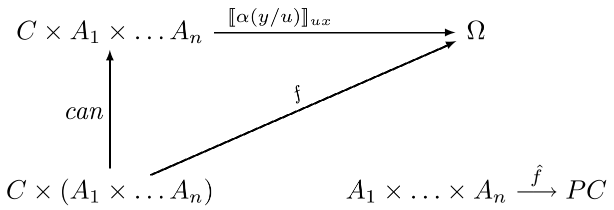
We extend I to terms of $\mathcal{L}$ as follows. If $\tau : \mathbf{B}$, write x for $(x_1, \ldots, x_n)$, any sequence of variables containing all variables of τ (and call such sequences *adequate* for τ). Define the $\mathcal{E}$-arrow

$$[\![\tau]\!]_x : A_1 \times \ldots \times A_n \longrightarrow B$$

recursively as follows:

$$\begin{array}{l}
[\![*]\!]_x = A_1 \times \ldots \times \longrightarrow 1 \\
[\![x_i]\!]_x = \pi_i : A_1 \times \ldots \times A_n \longrightarrow A_i \\
[\![\mathbf{f}(\tau)]\!]_x = \mathbf{f}_1 \circ [\![\tau]\!]_x \\
[\![\langle \tau_1, \ldots, \tau_n\rangle]\!]_x = \langle [\![\tau_1]\!]_x, \ldots, [\![\tau_n]\!]_x\rangle \\
[\![(\tau)_i]\!]_x = \pi_i \circ [\![\tau]\!]_x \\
[\![\{y : \alpha\}]\!]_x = ([\![\alpha(y/u)]\!]_{ux} \circ\ \mathit{can})\hat{}
\end{array}$$

where in this last clause u differs from $x_1, \ldots, x_n$, is free for y in α, y is of type $\mathbf{C}$, (so that B is of type $\mathbf{PC}$), *can* is the canonical isomorphism $C \times (A_1 \times ... \times A_n) \cong C \times A_1 \times ... \times A_n$, and $\hat{f}$ is as defined for power objects.[25] To understand why, consider the diagrams



In set theory, $\hat{f}(a_1, \ldots, a_n) = \{y \in C : \alpha(y, a_1, ..., a_n)\}$, so we take $[\![\{y : \alpha\}]\!]_x$ to be $\hat{f}$.

Finally,

$$\begin{array}{lll} [\![\sigma = \tau]\!]_x = & eq_c \circ [\![\langle\sigma, \tau\rangle]\!]_x & \text{(with } \sigma, \tau : \mathbf{c}) \\ [\![\sigma \in \tau]\!]_x = & e_c \circ [\![\langle\sigma, \tau\rangle]\!]_x & \text{(with } \sigma : \mathbf{C}, \tau : \mathbf{PC} \text{ and where } e_c \text{ is as defined for power objects)} \end{array}$$

If $\tau : \mathbf{B}$ is closed, then $\mathbf{x}$ may be taken to be the empty sequence $\emptyset$. In this case we write $[\![\tau]\!]$ for $\langle\tau\rangle_\emptyset$; this is an arrow $1 \to B$.

We note that

$$[\![\top]\!]_x = [\![\bigstar = \bigstar]\!]_x = eq \circ \langle[\![*]\!]_x, [\![*]\!]_x\rangle = T.$$

For any finite set $\Gamma = \{\alpha_1, \ldots, \alpha_m\}$ of formulas write

$$[\![\Gamma]\!]_{I,x} \text{ for } [\![\alpha_1]\!]_{I,x} \wedge \ldots \wedge [\![\alpha_m]\!]_{I,x} \text{ if } m \geq 1 \text{ or } T \text{ if } m = 0.$$

Given a formula α, let $\mathbf{x} = (x_1, ..., x_n)$ list all the free variables in $\Gamma \cup \{\alpha\}$; write

$$\Gamma \vDash_1 \alpha \text{ for } [\![\Gamma]\!]_{I,\mathbf{x}} \leq [\![\alpha]\!]_{I,\mathbf{x}}.$$

$\Gamma \vDash_I \alpha$ is read: "$\Gamma : \alpha$ is *valid* under the interpretation I in $\mathcal{E}$." If S is a local set theory, we say that I is a *model* of S if every sequent of S is valid under I. Notice that

$$\vDash_I \beta \text{ iff } [\![\beta]\!]_{\mathbf{x}} = T.$$

So if I is an interpretation in a *degenerate* topos, i.e., a topos possessing just one object up to isomorphism, then $\vDash_i \alpha$ for all α.

We write:

$$\begin{array}{lll} \Gamma \vDash \alpha & \text{for} & \Gamma \vDash_I \alpha \text{ for every } I \\ \Gamma \vDash_S \alpha & \text{for} & \Gamma \vDash_I \alpha \text{ for every model } I \text{ of } S. \end{array}$$

---

[25] See Appendix for definitions of all categorical notions used in the sequel.

It can be shown (laboriously) that the basic axioms and rules of inference of any local set theory are valid under every interpretation. This yields the

Soundness Theorem.

$$\Gamma \vdash \alpha \Rightarrow \Gamma \vDash \alpha \qquad \Gamma \vdash_S \alpha \Rightarrow \Gamma \vDash_S \alpha.$$

A local set theory S is said to be *consistent* if it is not the case that $\vdash_S \bot$. The soundness theorem yields the

Corollary. *Any pure local set theory is consistent.*

Proof. Set up an interpretation I of $\mathcal{L}$ in the topos $\mathcal{F}inset$ of finite sets as follows: $\mathbf{1}_I = 1, \Omega_I = \{0,1\} = 2$, for any ground type $\mathbf{A}, \mathbf{A}_I$ is any nonempty finite set. Extend I to arbitrary types in the obvious way. Finally $\mathbf{f}_I : \mathbf{A}_I \to \ \mathbf{B}_I$ is to be any map from $\mathbf{A}_I$ to $\mathbf{B}_I$.

If $\vdash \bot$, then $\vdash \alpha$, so $\vDash_I \alpha$ for any formula α. In particular $\vDash_I u = v$, where u, v are variables of type $\mathbf{P1}$. Hence $[\![u]\!]_{I,uv} = [\![v]\!]_{I,uv}$, that is, the two projections $P1 \times P1 \to P1$would have to be identical, a contradiction. ∎

Given a local set theory S in a language $\mathcal{L}$, define the *canonical interpretation* $C(S)$ of $\mathcal{L}$ in $\mathcal{C}(S)$ by:

$$\mathbf{A}_{C(S)} = U_A \ \mathbf{f}_{C(S)} = (x \mapsto \mathbf{f}(x)) : U_A \to U_B \text{ for } \mathbf{f} : \mathbf{A} \to \mathbf{B}.$$

A straightforward induction establishes

$$[\![\tau]\!]_{C(S),\mathbf{x}}, = (\mathbf{x} \mapsto \tau).$$

This yields

$$\Gamma \vDash_{C(S)} \alpha \Leftrightarrow \Gamma \vdash_S \alpha. \qquad (*)$$

For:

$$\begin{aligned} \vDash_{C(S)} \alpha \ &\Leftrightarrow \ [\![\alpha]\!]_{C(S)\mathbf{x}} = T \\ &\Leftrightarrow \ (\mathbf{x} \mapsto \alpha) = (\mathbf{x} \mapsto \top) \\ &\Leftrightarrow \ \vdash_S \alpha = \top \\ &\Leftrightarrow \ \vdash_S \alpha. \end{aligned}$$

Since $\Gamma \vdash_S \alpha \Leftrightarrow \ \vdash_S \gamma \to \alpha$, where γ is the conjunction of all the formulas in Γ, the special case yields the general one.

Equivalence (*) may be read as asserting that $\mathcal{C}(S)$ is a *canonical model* of S. This fact yields the

Completeness Theorem.

$$\Gamma \vDash \alpha \Rightarrow \Gamma \vdash \alpha \qquad \Gamma \vDash_S \alpha \Rightarrow \Gamma \vdash_S \alpha$$

Proof. We know that $C(S)$ is a model of S. Therefore, using (*),

$$\Gamma \vDash_S \alpha \Rightarrow \Gamma \vdash_{C(S)} \alpha \Rightarrow \Gamma \vdash_S \alpha.$$

∎

5.6 *Every Topos is Linguistic: The Equivalence Theorem*

A topos of the form $\mathcal{C}(S)$ is called a *linguistic* topos. It can be shown that every topos is, in a certain sense, equivalent to a linguistic one. This fact was apparently established independently by a number of mathematicians: it appears for example, in Fourman [1977], Zangwill [1977], and Boileau and Joyal [1981].

Given a topos $\mathcal{E}$, we exhibit a theory $Th(\mathcal{E})$ and an equivalence $\mathcal{E} \simeq \mathcal{C}(Th(\mathcal{E}))$.

The local language $\mathcal{L}_E$ associated with $\mathcal{E}$ — also called the *internal language* of $\mathcal{E}$ — is defined as follows. The ground type symbols of $\mathcal{L}_{\mathcal{E}}$ are taken to match the objects of $\mathcal{E}$ other than its terminal and truth-value objects, that is, for each $\mathcal{E}$-object A (other than 1, Ω) we assume given a ground type $\mathbf{A}$ in $\mathcal{L}_{\mathcal{E}}$. Next, we define for each type symbol $\mathbf{A}$ an $\mathcal{E}$-object $\mathbf{A}_{\mathcal{E}}$ by

$$\mathbf{A}_{\mathcal{E}} = \mathbf{A} \text{ for ground types } \mathbf{A},$$
$$(\mathbf{A} \times \mathbf{B})_{\mathcal{E}} = \mathbf{A}_{\mathcal{E}} \times \mathbf{B}_{\mathcal{E}}{}^{26}$$
$$(\mathbf{PA})_{\mathcal{E}} = P(\mathbf{A}_{\mathcal{E}}).$$

The function symbols of $\mathcal{L}$ are triples $(f, \mathbf{A}, \mathbf{B}) = \mathbf{f}$ with $f : \mathbf{A}_{\mathcal{E}} \to \mathbf{B}_{\mathcal{E}}$ in $\mathcal{E}$. The signature of $\mathbf{f}$ is $\mathbf{A} \to \mathbf{B}$.[27]

The *natural interpretation* — denoted by $\mathcal{E}$ — of $\mathcal{L}_E$ in $\mathcal{E}$ is determined by the assignments

$$\mathbf{A}_{\mathcal{E}} = \mathbf{A} \text{ for each ground type } \mathbf{A} \qquad (f, \mathbf{A}, \mathbf{B})_{\mathcal{E}} = \mathbf{f}.$$

The local set theory $Th(\mathcal{E})$, the *theory of* $\mathcal{E}$, is the theory in $\mathcal{L}_{\mathcal{E}}$ generated by the collection of all sequents $\Gamma : \alpha$ such that $\Gamma \vDash_{\mathcal{E}} \alpha$ under the natural interpretation of $\mathcal{L}_{\mathcal{E}}$ in $\mathcal{E}$. Then we have

$$\Gamma \vdash_{Th(\mathcal{E})} \alpha \Leftrightarrow \Gamma \vDash_{\mathcal{E}} \alpha.$$

For if $\Gamma \vdash_{Th(\mathcal{E})} \alpha$ then by Soundness $\Gamma \vDash_{Th(\mathcal{E})} \alpha$, i.e., $\Gamma : \alpha$ is valid in every model of $Th(\mathcal{E})$. But by definition $\mathcal{E}$ is a model of $Th(\mathcal{E})$.

It can be shown that the canonical functor $F : \mathcal{E} \to C(Th(\mathcal{E}))$ defined by

$$\begin{aligned} FA &= U_{\mathbf{A}} \text{ for each } \mathcal{E}\text{-object } A \\ Ff &= (x \mapsto \mathbf{f}(x) : U_{\mathbf{A}} \to U_{\mathbf{B}} \text{ for each } \mathcal{E}\text{-arrow } f : A \to B \end{aligned}$$

is an equivalence of categories. This is the **Equivalence Theorem.**

Finally, we state two more facts about $Th(\mathcal{E})$.

A local set theory S in a language $\mathcal{L}$ is said to be

[26] Note that, if we write C for $A \times B$, then while $\mathbf{C}$ is a ground type, $\mathbf{A} \times \mathbf{B}$ is a product type. Nevertheless $\mathbf{C}_{\mathcal{E}} = (\mathbf{A} \times \mathbf{B})_{\mathcal{E}}$.

[27] Note the following: if $f : A \times B \to D$, in $\mathcal{E}$, then, writing C for $A \times B$ as in the footnote above, $(f, \mathbf{C}, \mathbf{D})$ and $(f, \mathbf{A} \times \mathbf{B}, \mathbf{D})$ are both function symbols of $\mathcal{L}_{\mathcal{E}}$ associated with f. But the former has signature $\mathbf{C} \to \mathbf{D}$, while the latter has the different signature $\mathbf{A} \times \mathbf{B} \to \mathbf{D}$.

- *well-termed* if whenever $\vdash_S \exists! x\alpha$, there is a term τ of $\mathcal{L}$ whose free variables are those of α with x deleted such that $\vdash_S \alpha(x/\tau)$;
- *well-typed* if for any S-set X there is a type symbol $\mathbf{A}$ of $\mathcal{L}$ such that $U_A \cong X$ in $\mathcal{C}(S)$.

Then, *for any topos $\mathcal{E}$, $Th(\mathcal{E})$ is well-termed and well-typed.*

5.7 Translations of Local Set Theories

A *translation* $\mathbf{K} : \mathcal{L} \to \mathcal{L}'$ of a local language $\mathcal{L}$ into a local language $\mathcal{L}'$ is a map which assigns to each type $\mathbf{A}$ of $\mathcal{L}$ a type $\mathbf{KA}$ of $\mathcal{L}'$ and to each function symbol $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ of $\mathcal{L}$ a function symbol **Kf:** $\mathbf{KA} \to \mathbf{KB}$ of $\mathcal{L}'$ in such a way that

$$\mathbf{K1} = \mathbf{1},\ \mathbf{K}\Omega = \Omega,\ \mathbf{K}(\mathbf{A})_1 \times \ldots \times \mathbf{A}_n) = \mathbf{KA}_1 \times \ldots \times \mathbf{KA}_n,\ \mathbf{K}(\mathbf{PA}) = \mathbf{PKA}.$$

Any translation $\mathbf{K} : \mathcal{L} \to \mathcal{L}'$ may be extended to the terms of $\mathcal{L}$ in the evident recursive way — i.e., by defining $\mathbf{K}\bigstar = \bigstar$, $\mathbf{K}(\mathbf{f}(\tau)) = \mathbf{Kf}(\mathbf{k}\tau)$, $\mathbf{K}(\sigma \in \tau) = \mathbf{K}\sigma \in \mathbf{K}\tau$, etc — so that if $\tau : \mathbf{A}$, then $\mathbf{K}\tau : \mathbf{KA}$. We shall sometimes write α_K for $\mathbf{K}\alpha$.

If S, S' are local set theories in $\mathcal{L}, \mathcal{L}'$ respectively, a translation $\mathbf{K} : \mathcal{L} \to \mathcal{L}'$ is a *translation of* S into S', and is written $\mathbf{K} : S \to S'$ if, for any sequent $\Gamma : \alpha$ of $\mathcal{L}$,

$$\Gamma \vdash_S \alpha \Rightarrow \mathbf{K}\Gamma \vdash_{S'} \mathbf{K}\alpha, \qquad (*)$$

where if $\Gamma = \{\alpha_1, ..., \alpha_n\}, \mathbf{K}\Gamma = \{K\alpha_1, ..., K\alpha_n\}$. If the reverse implication to (*) also holds, $\mathbf{K}$ is called a *conservative translation* of S into S'. If S' is an extension of S and the identity translation of S into S' is conservative, S' is called a *conservative extension of* S.

There is a natural correspondence between models of S in a topos $\mathcal{E}$ and translations of S into $Th(\mathcal{E})$: in particular the *identity translation* $Th(\mathcal{E}) \to Th(\mathcal{E})$ corresponds to the *natural interpretation* of $Th(\mathcal{E})$ in $\mathcal{E}$.

Now let $\mathcal{E}, \mathcal{E}'$ be toposes with specified terminal objects, products, projection arrows, truth-value objects, power objects and evaluation arrows: a functor $F : \mathcal{E} \to \mathcal{E}'$ which preserves all these is called a *logical functor.* It is easily seen that the canonical functor $\mathcal{E} \to \mathcal{C}(Th(\mathcal{E}))$ is logical.

If $\mathbf{K} : S \to S'$ is a translation, then for terms σ, τ of $\mathcal{L}, \vdash_S \sigma = \tau$ implies $\vdash_{S'} \mathbf{K}\sigma = \mathbf{K}\tau$, so that $\mathbf{K}$ induces a map C_K from the class of S-sets to the class of S'-sets via

$$C_{\mathbf{K}}([\sigma]_S) = [\mathbf{K}\sigma]_S.$$

$C_{\mathbf{K}}$ is actually a *logical functor* $C(S) \to C(S')$. Writing $\mathcal{L}$oc for the category of local set theories and translations, and $\mathcal{T}$op for the category of toposes and logical functors, C is a functor $\mathcal{L}$oc $\to \mathcal{T}$op. And reciprocally any logical functor $F : \mathcal{E} \to \mathcal{E}'$ induces a translation *Th*$(F) : Th(\mathcal{E}) \to$ *Th*$(\mathcal{E}\prime)$ in the natural way, so yielding a functor *Th*: $\mathcal{L}$oc $\to \mathcal{T}$op. C and *Th* are "almost" inverse, making $\mathcal{L}$oc and $\mathcal{T}$op "almost" equivalent.

Given a local set theory S in a language $\mathcal{L}$, define a translation $\mathbf{K} : \mathcal{L} \to \mathcal{L}_{\mathcal{L}(S)}$ by

$$\mathbf{KA} = U_A,\ \mathbf{Kf} = (f,\ \mathbf{A},\ \mathbf{B}) \text{ if } \mathbf{f} : \mathbf{A} \to \mathbf{B}.$$

An easy induction on the formation of terms shows that, for any term τ of $\mathcal{L}$, $[\![\tau]\!]_{\mathcal{C}(S)},\ x = [\![\mathbf{K}\tau]\!]_{\mathcal{C}(S)},\ \mathbf{K}_x$. It follows from this that $\mathbf{K}$ is a *conservative* translation of S into $Th(\mathcal{C}(S))$. For $\Gamma \vdash_S \alpha \Leftrightarrow \Gamma \vdash_{\mathcal{C}(S)} \alpha \Leftrightarrow \mathbf{K}\Gamma \vdash_{\mathcal{C}(S)} \mathbf{K}\alpha \Leftrightarrow \mathbf{K}\Gamma \vdash_{Th(\mathcal{C}(S))} \mathbf{K}\alpha$.

Accordingly *any local set theory can be conservatively embedded in one which is well-termed and well-typed.*

A particularly important instance of translation is the *adjunction of indeterminates* to a local set theory. Let us define a *constant* of type $\mathbf{A}$ in a local language $\mathcal{L}$ to be a term of the form $\mathbf{f}(\bigstar)$, where $\mathbf{f} : 1 \to \mathbf{A}$. Write $\mathcal{L}(\mathbf{c})$ for the language obtained from $\mathcal{L}$ by adding a new function symbol $\mathbf{c} : 1 \to \mathbf{A}$ and write c for $\mathbf{c}(\bigstar)$. Given a local set theory S in $\mathcal{L}$, and a formula α of $\mathcal{L}$ with exactly one free variable x of type $\mathbf{A}$, write $S(\alpha)$ for the theory in $\mathcal{L}(\mathbf{c})$ generated by S together with all sequents of the form: $\beta(x/c)$ where $\alpha \vdash_S \beta$. Since clearly $\vdash_{S(\alpha)} \alpha(x/c)$, it follows that $\vdash_{S(\alpha)} \exists x\alpha$. Also, an analysis of derivations establishes that proofs in $S(\alpha)$ are related to proofs in S by the condition:

$$\Gamma(x/c) \vdash_{S(\alpha)} \gamma(x/c) \Leftrightarrow \alpha, \Gamma \vdash_S \gamma. \qquad (*)$$

for any sequent $\Gamma : \gamma$ of $\mathcal{L}$.

It is not hard to show that:

$$\vdash_S \exists x\alpha \quad \Leftrightarrow \quad S(\alpha) \text{ is a conservative extension of } S$$
$$\vdash_S \neg\exists x\alpha \quad \Leftrightarrow \quad S(\alpha) \text{ is inconsistent.}$$

In $S(\alpha), c$ behaves as an *indeterminate* of sort α in the sense that it can be arbitrarily assigned any value satisfying α. To be precise, the following can be proved:

Let S^* be a local set theory in a local language $\mathcal{L}^*$ and let $\mathbf{K} : S \to S^*$. Then for any constant c^* of $\mathcal{L}^*$ of type $\mathbf{KA}$ such tha t$\vdash_{S^*} \alpha_{\mathbf{K}}(c^*)$, there is a unique translation $\mathbf{K}^* : S(\alpha) \to S$ extending $\mathbf{K}$ such that $\mathbf{K}^*(c) = c^*$.

If I is an S-set and α the formula $x \in I$, we write S_I for $S(\alpha)$ and call it the theory obtained from $S$by *adjoining an indeterminate element of I*. It follows from (*) above that, for any formula γ of $\mathcal{L}$ in which i is free for x,

$$\vdash_{S(I)} \gamma(x/c) \Leftrightarrow \vdash_S \forall i \in I\gamma(x/i).$$

If α is the formula $x = x$ with $x : \mathbf{A}$, then $S(\alpha)$ is written $S(\mathbf{A})$ and called the theory obtained from S by *adjoining an indeterminate of type* $\mathbf{A}$. In particular, let S_0 be the pure local set theory in the local language $\mathcal{L}_0$ with no ground types or function symbols. Evidently S_0 is an *initial object* in the category $\mathcal{L}$oc: there is a unique translation of $\mathcal{L}_0$ into any given local set theory S. (Similarly, the topos $C(S_0)$ is an initial object in the category top.) Now consider the theory $S_0(\mathbf{A})$, where $\mathbf{A}$ is a type symbol of $\mathcal{L}_0$: $\mathbf{A}$ may be considered a type symbol of *any* local language $\mathcal{L}$. If d is a constant of type $\mathbf{A}$ in $\mathcal{L}$, and S a local set theory in $\mathcal{L}$, there is then a unique translation $\mathbf{K} : S_0(\mathbf{A}) \to S$ mapping c to d. So $S_0(\mathbf{A})$ may be considered *the universal theory of an indeterminate of type* $\mathbf{A}$.

5.8 Classicality and the Choice Principle

Let S be a local set theory in a language $\mathcal{L}$. We make the following

Definitions.

- S is *classical* if $\vdash_S \forall\omega(\omega \vee \neg\omega)$. This is the full law of excluded middle for S.

- S is *sententially classical* if $\vdash_S \alpha \vee \neg\alpha$ for any sentence α. This is a weakened form of the law of excluded middle.

- For each $S-$set $X : \mathbf{PA}$ let $\Delta(X)$ be the set of closed terms τ such that $\vdash_S \tau \in X$. X is *standard* if for any formula α with at most the variable $x : \mathbf{A}$ free the following is valid:

$$\frac{\vdash_S \alpha(x/\tau) \text{ for all } \tau \text{ in } \Delta(X)}{\vdash_S \forall x \in X\alpha}$$

  S is *standard* if every S-set is so.

- S is *witnessed* if for any type symbol $\mathbf{A}$ of $\mathcal{L}$ and any formula α with at most the variable $x : \mathbf{A}$ free the following is valid:

$$\frac{\vdash_S \exists x\alpha}{\vdash_S \alpha(x/\tau) \text{ for some closed term } \tau : \mathbf{A}}$$

- S is *choice* if, for any S-sets X, Y and any formula α with at most the variables x, y free the following rule (the *choice rule*) is valid:

$$\frac{\vdash_S \forall x \in X\exists y \in Y\alpha(x, y)}{\vdash_S \forall x \in X\alpha(x, fx) \text{ for some } f : X \to Y}$$

- S is *internally choice* if under the conditions of the previous definition

$$\forall x \in X\exists y \in Y\alpha(x, y) \vdash_S \exists f \in \mathit{Fun}(X, Y)\forall x \in X\exists y \in Y[\alpha(x, y) \wedge \langle x, y\rangle \in f].$$

- An S-set X is *discrete* if

$$\vdash_S \forall x \in X\forall y \in X.x = y \vee x \neq y.$$

- A *complement* for an S-set $X : \mathbf{PA}$ is an S-set $Y : \mathbf{PA}$ such that

$$\vdash_S X \cup Y = A \wedge X \cap Y = \emptyset.$$

An S-set that has a complement is said to be *complemented*.

Now for some facts concerning these notions. In formulating our arguments we shall assume that our background metatheory is constructive, in that no use of the metalogical law of excluded middle will be made.

1. *Any of the following conditions is equivalent to the classicality of S:*

(i) $\vdash_S \Omega = \{\top, \bot\}$

(ii) $\vdash_S \neg\neg\omega \to \omega$

(iii) *any S-set is complemented,*

(iv) *any S-set is discrete,*

(v) Ω *is discrete,*

(vi) $\vdash_S 2 = \{0, 1\}$ *is well-ordered under the usual ordering,*

Proof. (iii) If S is classical, clearly $\{x : x \notin X\}$ is a complement for X. Conversely, if $\{\top\}$ has a complement U, then

$$\vdash_S \omega \in U \to \neg(\omega = \top) \to \neg\omega \to \omega = \bot.$$

Hence $\vdash_S U = \{\bot\}$, whence $\vdash_S \Omega = \{\top\} \cup U = \{\top, \bot\}$.
(v) If Ω is discrete, then $\vdash_S \omega = \top \vee \neg(\omega = \top)$, so $\vdash_S \omega \vee \neg\omega$.
(vi) If S is classical, then 2 is trivially well-ordered under the usual well-ordering. Conversely, if 2 is well-ordered, take any formula α, and define $X = \{x \in 2 : x = 1 \vee \alpha\}$. Then X has a least element, a, say. Clearly $\vdash_S a = 0 \leftrightarrow \alpha$, so, since $\vdash_S a = 0 \vee a = 1$, we get $\vdash_S a = 1 \leftrightarrow \neg\alpha$, and hence $\vdash_S \alpha \vee \neg\alpha$. ■

2. *For well-termed S, S choice $\Leftrightarrow$ S internally choice and witnessed.*

Proof. Suppose S is choice. If $\vdash_S \exists x\alpha$, let $u : 1$ and define $\beta(u, x) \equiv \alpha(x)$. Then $\vdash_S \forall u \in 1 \exists x \in X \beta(u, x)$. Now choice yields an S-map $f : 1 \to X$ such that $\vdash_S \forall u \in 1 \beta(u, f(u))$ i.e., $\vdash_S \beta(\bigstar, f\bigstar)$ or $\vdash_S \alpha(f\bigstar)$. By well-termedness, $f\bigstar$ may be taken to be a closed term τ, and we then have $\vdash_S \alpha(\tau)$. So S is witnessed.

To derive internal choice from choice, we argue as follows: let

$$X^* = \{x \in X : \exists y \in Y \alpha(x, y)\}.$$

Then $\vdash_S \forall x \in X^* \exists y \in Y \alpha(x, y)$. Accordingly choice yields a map $f : X^* \to Y$ such that $\vdash_S \forall x \in X^* \alpha(x, fx)$, i.e. $\vdash_S \forall x \in X^* \exists y \in Y[\langle x, y \rangle \in f \wedge \alpha(x, y)]$. Now

$$\forall x \in X \exists y \in Y \alpha(x, y)] \vdash_S X = X^* \vdash_S f \in \mathit{Fun}(X, Y)$$

so

$$\forall x \in X \exists y \in Y \alpha(x, y)] \vdash_S \forall x \in X \exists y \in Y[\langle x, y\rangle \in f \wedge \alpha(x, y)].$$

Hence

$$\forall x \in X \exists y \in Y \alpha(x, y) \vdash_S \exists f \in Fun(X, Y) \forall x \in X \exists y \in Y[\alpha(x, y) \wedge \langle x, y\rangle \in f],$$

as required. The converse is easy. ■

3. *If S is well-termed and well-typed, then S is choice $\Leftrightarrow S_X (= S(X))$ is witnessed for every S-set X.*

Proof. Suppose S is choice and $\vdash_{S(X)} \exists y \alpha(y)$. We may assume that X is of the form U_A, in which case α is of the form $\beta(x/c, y)$ with $x : \mathbf{A}$. From $\vdash_{S(X)} \exists y \beta(x/c, y)$ we infer $\vdash_S \forall x \exists y \beta(x/c, y)$. So using choice in S and the well-termedness of S we obtain a term $\tau(x)$ such that $\vdash_S \forall x \beta(x, \tau(x))$. Hence $\vdash_{S(X)} \beta(c, \tau(c))$, i.e., $\vdash_{S(X)} \alpha(\tau(c))$. Therefore S_X is witnessed.

Conversely, suppose S_X is witnessed for every S-set X, and that $\vdash_S \forall x \in X \exists y \in Y \alpha(x, y)]$. Then $\vdash_{S(X)} \exists y \in Y \alpha(c, y)]$, so there is a closed $\mathcal{L}_X$-term τ such that $\vdash_{S(X)} \tau \in Y \wedge \alpha(c, \tau)$. But τ is $\sigma(x/c)$ for some $\mathcal{L}$-term $\sigma(x)$. Thus $\vdash_{S(X)} \sigma(c) \in Y \wedge \alpha(c, \sigma(c))$, whence $\vdash_S \forall x \in X[\sigma(x) \in Y \wedge \alpha(x, \sigma)]$. Defining $f = (x \mapsto \sigma) : X \to Y$ then gives $\vdash_S \forall x \in X \alpha(x, fx)]$ as required. ■

4. **Diaconescu's Theorem**. *S choice $\Rightarrow$ S classical.*

   This is the version, for local set theories, of Diaconescu's [1975] result[28] that any topos in which the axiom of choice holds is Boolean.

Proof. Step 1. *S choice $\Rightarrow S_I$ choice for any S-set I.*
Proof of step 1. Suppose that S is choice, and

$$\vdash_{S(I)} \forall x \in X(c) \exists y \in Y(c) \alpha(x, y, c).$$

Then

$$\vdash_S \forall x \in X(i) \exists y \in Y(i) \alpha(x, y, i).$$

Define

$$X^* = \{\langle x, i\rangle : x \in X(i) \wedge i \in I\}, Y^* = \bigcup_{i \in I} Y(i).$$

$$\beta(u, i) \equiv \exists x \in X(i) \exists i \in I[u = \langle x, i\rangle \wedge \alpha(x, y, i) \wedge y \in Y(i)].$$

Then

$$\vdash_S \forall u \in X^* \exists y \in Y^* \beta(u, y).$$

So choice yields $f^* : X^* \to Y^*$ such that

$$\vdash_S \forall u \in X^* \beta(u, f^* u).$$

[28] First presented in purely logical form in Goodman and Myhill [1978].

i.e.,

$$\vdash_S \forall i \in I \forall x \in X(i)[\alpha(x, f^*(\langle x, i\rangle, i) \wedge f^*(\langle x, i\rangle) \in Y(i)],$$

whence

$$\vdash_S \forall x \in X(c)[\alpha(x, f^*(\langle x, c\rangle, c) \wedge f^*(\langle x, c\rangle) \in Y(c)],$$

Now define $f = (x \mapsto f^*(\langle x, c\rangle))$. Then $f : X(c) \to Y(c)$ in S_I and

$$\vdash_{S(I)} \forall x \in X(c)\alpha(x, fx, c).$$

This completes the proof of step 1.

Step 2. *S choice $\Rightarrow$ S sententially classical.*
Proof of step 2. Define $2 = \{0, 1\}$ and let $X = \{u \subseteq 2 : \exists y.y \in u\}$. Then

$$\vdash_S \forall u \in X \exists y \in 2.y \in u.$$

So by choice there is $f : X \to 2$ such that

$$\vdash_S \forall u \in X.fu \in u.$$

Now let α be any sentence; define

$$U = \{x \in 2 : x = 0 \vee \alpha\}, V = \{x \in 2 : x = 1 \vee \alpha\},$$

Then $\vdash_S U \in X \wedge V \in X$, so, writing $a = fU, b = fV$, we have

$$\vdash_S [a = 0 \vee \alpha] \wedge [b = 1 \vee \alpha],$$

whence

$$\vdash_S [a = 0 \wedge b = 1] \vee \alpha,$$

so that

$$\vdash_S a \neq b \vee \alpha. \qquad (*)$$

But $\alpha \vdash_S U = V \vdash_S a = b$, so that $a \neq b \vdash_S \neg\alpha$. It follows from this and (*) that

$$\vdash_S \alpha \vee \neg\alpha,$$

as claimed. This establishes step 2.

Moral of step 2: if pair sets have choice functions, then logic is classical.

Step 3 (obvious). *S classical $\Leftrightarrow$ S_Ω sententially classical.*
Finally, to prove Diaconescu's theorem, we observe that
S choice $\Rightarrow$ S_Ω choice $\Rightarrow$ S_Ω sententially classical $\Rightarrow$ S classical. ■

5.9 Characterisation of $\mathcal{S}et$

We remind the reader that we are assuming that our background metatheory is constructive. For definiteness we will take that metatheory to be intuitionistic set theory **IST.** Now consider the category $\mathcal{S}et$ of sets in **IST**. Its objects are all sets and its arrows all maps between sets. $\mathcal{S}et$ is a topos in with truth-value object $P1$. We seek a characterization of $\mathcal{S}et$ in terms of its associated local set theory, that is, *a characterization of the category of sets in type-theoretic terms.* This is to be carried out in a constructive manner.

To achieve this, we first define a local set theory S to be *full* if for each set I in the background metatheory there is a type symbol $\widehat{I}$ of the language $\mathcal{L}$ of S together with a collection $\{\widehat{i} : i \in I\}$ of closed terms each of type $\widehat{I}$ satisfying the following:

(i) $\vdash_S \widehat{i} = \widehat{j} \;\Rightarrow\; i = j.$

(ii) For any I-indexed family $\{\tau_i : i \in I\}$ of closed terms of common type $\mathbf{A}$, there is a term $\tau(x) : \mathbf{A}, x : \widehat{I}$ such that

$$\vdash_S \tau_i = \tau(\widehat{i}) \text{ for all } i \in I,$$

and, for any term $\sigma(x) : \mathbf{A}, x : \widehat{I}$, if $\vdash_S \tau_i = \sigma(\widehat{i})$ for all $i \in I$, then

$$\vdash_S \tau = \sigma.$$

We establish the *generalization principle for hatted type symbols:*
The following is valid for any formula $\alpha(x)$ with $x : \widehat{I}$:

$$\frac{\vdash_S \alpha(\widehat{i}) \text{ for all } i \in I}{\vdash_S \forall x \alpha}$$

and similarly for more free variables. In particular, $\widehat{I}$ is standard.

Proof. Assume the premises. Then for any $i \in I$ we have $\vdash_S \alpha(\widehat{i}) = \top$ and it follows from the uniqueness condition that $\vdash_S \alpha(x) = \top$, whence have $\vdash_S \forall x\alpha$. ■

Note the following: (i) if S is well-termed and well-typed, then $\mathcal{C}(S)$ is full iff the *I-indexed sum of* 1 exists in $\mathcal{C}(S)$ for any (intuitive) set I. Here the I-indexed sum of an object X in a category $\mathcal{C}$ is an object $I \otimes X$ together with arrows $\sigma_i : X \to I \otimes X (i \in I)$ such that, for any arrows $f_i : X \to A$, there is a unique arrow $f : I \otimes X \to A$ such that $f_i = f \circ \sigma_i$ for all $i \in I$. In **Set**, $I \otimes X$ is the disjoint union of I copies of X, so that, in particular, $I \otimes 1$ is essentially I itself.

(ii) S is standard iff 1 is a *generator* in $\mathcal{C}(S)$. Here an object A of a category $\mathcal{C}$ is a generator if, for any pair of $\mathcal{C}$-arrows $f, g : X \to Y$, $f = g$ whenever $f \circ h = g \circ h$ for all arrows $h : A \to X$.

Conditions (i) and (ii) are both satisfied by $\mathcal{S}et$.

We can now prove the

Theorem. *The following conditions on a well-termed, well-typed consistent local set theory S are equivalent:*

(i) *S is full and standard,*

(ii) $\mathcal{C}(S) \simeq \mathcal{S}$et.

Proof. Implication (ii) $\Rightarrow$ (i) has been discussed above. Now assume that S is full. Since S is well-termed, for any S-map $f : X \to Y$ we can write $f(\tau)$ for each closed term τ such that $\vdash_S \tau \in X$.

We define functors $\Delta : \mathcal{C}(S) \to \mathcal{S}et, \hat{} : \mathcal{S}et \to \mathcal{C}(S)$, which, under the specified conditions, we show to define an equivalence.

First, $\Delta(X)$ is the set of closed terms τ such that $\vdash_S \tau \in X$, where we identify σ, τ if that $\vdash_S \sigma = \tau$. Given $f : X \to Y$, we define $\Delta(f)$ to be the map $(\tau \mapsto f(\tau)) : \Delta(X) \to \Delta(Y)$.

Next, given I in $\mathcal{S}$et, we define $\widehat{I}$ to be the S-set $U_{\widehat{I}}$. Given $f : I \to J$, there is a term $f(x) : \widehat{J}$ with $x : \widehat{I}$ such that that $\vdash_S \widehat{f}(\widehat{i}) = \widehat{f(i)}$ for all $i \in I$. We define $\widehat{f} : \widehat{I} \to \widehat{J}$ to be the S-map $x \mapsto \widehat{f}(x)$.

For any set I and any S-set X, we have natural maps $\eta_I : I \to \Delta(\widehat{I})$ and $\varepsilon : \widehat{\Delta(X)} \to X$ defined as follows:

$$\eta_I(i) = \widehat{i} \text{ for } i \in I; \vdash_S \varepsilon(\widehat{\tau}) = \tau \text{ for all } \tau \in \Delta(X).$$

Clearly η is monic. The same is true of ε since for $\sigma, \tau \in \Delta(X)$,

$$\vdash_S \varepsilon(\widehat{\sigma}) = \varepsilon(\widehat{\tau}) \to \sigma = \tau,$$

whence

$$\vdash_S \forall x \forall y[\varepsilon(x) = \varepsilon(y) \to x = y]$$

by generalization for hatted type symbols.

Now suppose that S is *standard*. We claim that then ε is epic and hence an isomorphism. For we have, for all $\tau \in \Delta(X), \vdash_S \varepsilon(\widehat{\tau}) = \tau$, whence $\vdash_S \exists y \varepsilon(y) = \tau$. Since X is standard, we infer that

$$\vdash_S \forall x \in X \exists y \varepsilon(y) = x,$$

so that ε is onto, hence epic.

Using the fact that ε is an isomorphism we can now show that η is epic, and hence also an isomorphism. To do this we require the readily established fact that, for $f : I \to J, \widehat{f} : \widehat{I} \to \widehat{J}$ is a map in S, and if $\widehat{f}$ is epic, then so is f.

Now consider $\widehat{\eta} : \widehat{I} \to \widehat{\Delta\widehat{I}}$. We note that

$$\varepsilon \circ \widehat{\eta} = 1_{\widehat{I}}. \qquad (*)$$

For if $i \in I$, then

$$\vdash_S \varepsilon(\widehat{\eta}(\widehat{i})) = \varepsilon(\widehat{\eta i}) = \eta i = \widehat{i}$$

It follows by generalization that

$$\vdash_S \forall x \in \widehat{I}\ \varepsilon(\eta x) = x,$$

whence (*).

Since ε is an isomorphism, it follows easily from (*) that $\widehat{\eta}$ is an isomorphism, hence also epic. Accordingly η is itself epic, and hence also an isomorphism.

We conclude that $(\Delta, \hat{\ })$ define an equivalence between $\mathcal{C}(S) \to \mathcal{S}\text{et}$, as required. ■

6 NEW FORMS OF TYPE THEORY AND THE DOCTRINE OF "PROPOSITIONS AS TYPES"

Type theory took a remarkable turn in the 1980s with the emergence of the *propositions as types* doctrine. Underlying this doctrine is the idealist notion, traceable to Kant, and central to Brouwerian intuitionism, that the meaning of a proposition does not derive from an absolute standard of truth external to the mind, but resides rather in the evidence for its assertability in the form of a mental construction or proof. Thus the central thesis of the "propositions-as-types" doctrine is that each proposition is to be *identified* with the type, set, or assemblage of its proofs.[29] As a result, such proof types, or sets of proofs, have to be accounted the *only* types, or sets. Strikingly, then, in the "propositions as types" doctrine, a type, or set, simply *is* the type, or set, of proofs of a proposition, and, reciprocally, a proposition *is* just the type, or set, of its proofs. These are truly radical identifications.

In a simple type theory or a local set theory, each type is independent of other types and is thus, so to speak, absolute or static; this holds in particular of the type of propositions or truth values. Now formulas or propositional functions in general manifest variation, since their values vary over, or depend on, the domain(s) of their free variables. Because of this they cannot be accurately represented as static types. This limitation makes it impossible for a simple type theory to realize faithfully the "propositions as types" doctrine. In order to achieve this it is necessary to develop a theory of "variable" or *dependent* types, wherein types can depend on, or "vary over" other types. In a dependent type theory, type symbols may take the form $\mathbf{B}(x)$, with x a variable of a given type $\mathbf{A} : \mathbf{B}(x)$ is then a type dependent on or varying over the type $\mathbf{A}$.

[29] This idea was advanced by Curry and Feys [1958] and later by Howard [1980]. As the *Curry-Howard correspondence* it has come to play an important role in theoretical computer science.

Such a theory — *constructive dependent type theory (CDTT)* — was introduced[30] by Per Martin-Löf [1975; 1982; 1984]. His theory, which has subsequently undergone much development, is also (as its name indicates) the first truly constructive theory of types, in the sense of being both predicative (so in particular it lacks a type of propositions) and based on intuitionistic logic. In introducing it Martin-Löf's purpose was to provide, as he put it in [1975] "a full scale system for formalizing intuitionistic mathematics as developed, for example, in the book by Bishop".[31] Martin-Löf's system provides a complete embodiment of the "propositions-as-types" doctrine.[32] Here is Martin-Löf himself on the latter in [1975]:

> Every mathematical object is of a certain kind or type. Better, a mathematical object is always given together with its type, that is it is not just an object: it is an object of a certain type. ... A type is defined by prescribing what we have to do in order to construct an object of that type... Put differently, a type is well-defined if we understand...what it means to be an object of that type. ... Note that it is required, neither that we should be able to generate somehow all the objects of a given type, nor that we should so to say know all of them individually. It is only a question of understanding what it means to be an arbitrary object of the type in question.
>
> A proposition is defined by prescribing how we are allowed to prove it, and a proposition holds or is true intuitionistically if there is a proof of it. ... Conversely, each type determines a proposition, namely, the proposition that the type in question is nonempty. This is the proposition which we prove by exhibiting an object of the type in question. On this analysis, there appears to be no fundamental difference between propositions and types. Rather, the difference is one of point of view: in the case of a proposition, we are not so much interested in what its proofs are as in whether it has a proof, that is, whether it is true or false, whereas, in the case of a type, we are of course interested in what its objects are and not only in whether it is empty or nonempty.

A key element in Martin-Löf's formulation of type theory is the distinction, which goes back to Frege, between *propositions* and *judgments.* Propositions (which, as we have seen, in Martin-Löf's systems are identified with types) are syntactical objects on which mathematical operations can be performed and which

[30]Dependent types were actually first studied in the late 1960s by de Bruijn and his colleagues at the University of Eindhoven in connection with the AUTOMATH project. CDTT has been employed as a basis for various computational devices employed for the verification of mathematical theories and of software and hardware systems in computer science.

[31]I.e. Bishop [1967].

[32]Martin-Löf's original calculus contained a type of all types. This assumption was shown to be inconsistent by Girard [1972]. Martin-Löf accordingly dropped this assumption in later versions of his theory.

bear certain formal relationships to other syntactical objects called proofs. Propositions and proofs are, so to speak, *objective* constituents of the system. Judgments, on the other hand, typically involve the *idealist* notion of "understanding" or "grasping the meaning of". Thus, for example, while *2 + 2 = 4* is a proposition, "*2 + 2 = 4* is a proposition" and "*2 + 2 = 4* is a true proposition" are judgments.

Martin-Löf also follows Frege in taking the rules of inference of logic to concern judgments rather than propositions. Thus, for example, the correct form of the rule of →-elimination is not

$$\frac{A \quad A \to B}{B}$$

but

$$\frac{A \text{ true} \quad A \to B \text{ true}}{B \text{ true}}.$$

That is, the rule does not say that the proposition B follows from the propositions A and $A \to B$, but that the *truth* of the proposition B follows from the *truth* of the proposition A conjoined with that of $A \to B$. In general, judgments may be characterized as expressions which appear at the conclusions of rules of inference.

Another important respect in which Martin-Löf follows Frege is in his insistence that judgments and formal rules be accompanied by full explanations of their *meaning*. (This is to be contrasted with the usual model-theoretic semantics which is really nothing more than a translation of one object-language into another.) In particular, the judgment *A is a proposition* may be made only when one knows what a (canonical) proof of A is, and the judgment *A is a true proposition* only when one knows how to find such a proof. Judgments, and the notion of truth, are thus seen to be mind-dependent.

Martin-Löf's various systems abound in subtle distinctions. For example, in addition to the distinction between proposition and judgment, there is a parallel distinction between *type* (or set) and *category*[33] (or species). In order to be able to judge that $A$is a category one must be able to tell what kind of objects fall under it, and when they are equal. To be in a position to make the further judgment that a category is a type, or set, one must be able to specify what its "canonical" or typical, elements are. In judging something to be a set, one must possess sufficient information concerning the its elements to enable quantification over it to make sense. Thus, for example, the natural numbers form a set $\mathbf{N}$, with canonical elements given by: 0 is a canonical element of $\mathbf{N}$, and if n is a canonical element of $\mathbf{N}$, then $n + 1$ is a canonical element of $\mathbf{N}$. On the other hand the collection of subsets of $\mathbf{N}$ forms a category, but not a set.

The "propositions-as-types" doctrine (which for convenience we abbreviate to PAT) gives rise to a correspondence between logical operators and operations on (dependent) types . Let us follow Tait's [1994] exposition of the idea in set-theoretic terms. To begin with, consider two propositions/types/sets $\mathbf{A}$ and $\mathbf{B}$.

[33]In this usage, of course, to be distinguished from the term as employed in its mathematical sense throughout the present article.

What should be required of a proof f of the implication $\mathbf{A} \to \mathbf{B}$? Just that, given any proof x of $\mathbf{A}$, f should yield a proof of $\mathbf{B}$, that is, f should be a function from $\mathbf{A}$ to $\mathbf{B}$. In other words, the proposition $A \to B$ is just the type of functions from $\mathbf{A}$ to $\mathbf{B}$:

$$\mathbf{A} \to \mathbf{B} = \mathbf{B}^A$$

Similarly, all that should be required of a proof c of the conjunction $\mathbf{A} \wedge \mathbf{B}$ is that it should yield proofs x and y of $\mathbf{A}$ and $\mathbf{B}$, respectively. From this point of view $\mathbf{A} \wedge \mathbf{B}$ is accordingly just the type $\mathbf{A} \times \mathbf{B}$ of all pairs (x, y), with x of type $\mathbf{A}$ (we write this as $x : \mathbf{A}$) and $y : \mathbf{B}$.

A proof of the disjunction $\mathbf{A} \vee \mathbf{B}$ is either a proof of $\mathbf{A}$ or a proof of $\mathbf{B}$ together with the information as to which of $\mathbf{A}$ or $\mathbf{B}$ it is a proof. That is, if we introduce the type $\mathbf{2}$ with the two distinct elements 0 and 1, a proof of $\mathbf{A} \vee \mathbf{B}$ may be identified as a pair (c, n) in which either c is a proof of $\mathbf{A}$ and n is 0, or c is a proof of $\mathbf{B}$ and n is 1. This means that $\mathbf{A} \vee \mathbf{B}$ should be construed as the disjoint union $\mathbf{A} + \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$.

The true proposition $\top$ may be identified with the one element type $\mathbf{1} = \{0\} : 0$ thus counts as the unique proof of $\top$. The false proposition $\bot$ is taken to be a proposition which lacks a proof altogether: accordingly $\bot$ is identified with the empty set $\emptyset$. The negation $\neg\mathbf{A}$ of a proposition $\mathbf{A}$ is defined as $\mathbf{A} \to \bot$, which therefore becomes identified with the set $\mathbf{A}^{\emptyset}$.

As we have already said, a proposition A is deemed to be true if it (i.e, the associated type $\mathbf{A}$) has an element, that is, if there is a function $\mathbf{1} \to \mathbf{A}$.Accordingly the *law of excluded middle* for a proposition A becomes the assertion that there is a function $\mathbf{1} \to \mathbf{A} + \emptyset^A$.

If a and b are objects of type $\mathbf{A}$, we introduce the *identity proposition* or *type* $a =_A b$ expressing that a and b are identical objects of type A. This proposition is true, that is, the associated type has an element, if and only if a and b are identical. In that case $\mathrm{id}(a)$ will denote an object of type a.

In *PAT* one must, as in Martin-Löf's system, distinguish sharply between *propositions*, which have proofs, and *judgements*, which do not. For example $0 =_2 0$ is a proposition, while "0 is of type $\mathbf{2}$" is a judgement. Rather than being true or false, a judgement is either assertable, or nonsensical.

In order to deal with the quantifiers we require operations defined on families of types, that is, types $\Phi(x)$ depending on objects x of some type $\mathbf{A}$. By analogy with the case $\mathbf{A} \to \mathbf{B}$, a proof f of the proposition $\forall x : \mathbf{A}\Phi(x)$, that is, an object of type $\forall x : \mathbf{A}\Phi(x)$, should associate with each $x : \mathbf{A}$ a proof of $\Phi(x)$. So f is just a function with domain A such that, for each $x : \mathbf{A}$, fx is of type $\Phi(x)$. That is, $\forall x : \mathbf{A}\Phi(x)$ is the *product* $\Pi x : \mathbf{A}\Phi(x)$ of the $\Phi(x)$s. We use the λ-notation in writing f as $\lambda x f x$.

A proof of the proposition $\exists x : \mathbf{A}\Phi(x)$, that is, an object of type $\exists x : \mathbf{A}\Phi(x)$, should determine an object $x : \mathbf{A}$ and a proof y of $\Phi(x)$, and *vice-versa.* So a proof of this proposition is just a pair (x, y) with $x : \mathbf{A}$ and $y : \Phi(x)$. Therefore $\exists x : \mathbf{A}\Phi(x)$ is the *disjoint union*, or *coproduct* $\coprod x : \mathbf{A}\Phi(x)$ of the $\Phi(x)$s.

To translate all this into the language of CDTT,[34] one uses the following concordance:

Logical Operation	Set-theoretic Operation	Type-theoretic Operation
$\wedge$	$\times$	$\times$
$\vee$	$+$	two-term dependent sum
$\rightarrow$	set exponentiation	type exponentiation
$\forall x$	Cartesian product $\prod_{i \in I}$	dependent product $\Pi x{:}\mathbf{A}$
$\exists x$	disjoint sum $\coprod_{i \in I}$	dependent sum $\coprod x{:}\mathbf{A}$

Of especial interest is the status of the *axiom of choice* in this framework. Again following Tait, we introduce the functions σ, π, π' of types $\forall x : \mathbf{A}(\Phi(x) \rightarrow \exists x : \mathbf{A}\Phi(x)), \exists x : \mathbf{A}\Phi(x) \rightarrow \mathbf{A}$, and $\forall y : (\exists x\Phi(x)).\Phi(\pi(y))$ as follows. If $b : \mathbf{A}$ and $c : \Phi(b)$, then σbc is (b, c). If $d : \exists x : \mathbf{A}\Phi(x)$, then d is of the form (b, c) and in that case $\pi(d) = b$ and $\pi'(d) = c$. These yield the equations

$$\pi(\sigma bc) = b\pi'(\sigma bc) = c\sigma(\pi d)(\pi\prime d) = d.$$

We may take the axiom of choice as the proposition

$$\forall x : \mathbf{A}\exists y : \mathbf{B}\Phi(x, y)) \rightarrow \exists f : B^A\forall x : \mathbf{A}\Phi(x, fx)). \qquad (AC)$$

Remarkably, AC is correct under *PAT*, that is, *provable* in *CDTT* , as the following argument shows. Let u be a proof of the antecedent $\forall x : \mathbf{A}\exists y : \mathbf{B}\Phi(x, y))$. Then, for any $x : \mathbf{A}, \pi(ux)$ is of type $\mathbf{B}$ and $\pi'(ux)$ is a proof of $\Phi(x, \pi ux)$. So $s(u) = \lambda x.\pi(ux)$ is of type $\mathbf{B}^A$ and $t(u) = \lambda x.\pi'(ux)$ is a proof of $\forall x : \mathbf{A}\Phi(x, s(u)x)$. Accordingly $\lambda u.\sigma s(u)t(u)$ is a proof of $\forall x : \mathbf{A}\exists y : \mathbf{B}\Phi(x, y)) \rightarrow \exists x : \mathbf{B}^A\forall x : \mathbf{A}\Phi(x, fx))$.

Put informally, what this shows is that in *CDTT* the consequent of (AC) means *nothing more than its antecedent.* Indeed, in many versions of constructive mathematics the assertability of an alternation of quantifiers $\forall x\exists yR(x, y)$ means *precisely* that one is given a function f for which $R(x, fx)$ holds for all x.

We note that in ordinary set theory this argument establishes the *isomorphism* of the sets $\prod x : A \coprod y : \mathbf{B}\Phi(x, y))$ and $\coprod f : \mathbf{B}^A \prod x : \mathbf{A}\Phi(x, fx))$, but not the validity of the usual axiom of choice. In set theory AC is not represented by this isomorphism, but is rather (equivalent to) the equality in which $\prod$ is replaced by $\bigcap$ and $\coprod$ by $\bigcup$, namely

$$\bigcap_{x \in A} \bigcup_{y \in B} \Phi(x, y) = \bigcup_{f \in B^A} \bigcap_{x \in A} \Phi(x, fx).$$

[34]For a complete specification of the operations and rules of DCTT, see Chapter 10 of Jacobs [1999] or Gambino and Aczel [2005].

While in *PAT* AC is provable, and so *a fortiori* has no "untoward" logical consequences, in intuitionistic set theory, or in the internal language of a topos this is far from being the case, for, as Diaconescu's theorem shows, in the latter AC implies the law of excluded middle. In other words, AC interpreted à la "propositions as types" is tautologous,[35] while construed set-, or topos-theoretically it is anything but, since so construed its affirmation yields classical logic. This prompts the question: what modification needs to be made to the "propositions-as-types" paradigm so as to yield the topos-theoretic interpretation of AC? An illuminating answer to this question has been given by Maietti [2005] through the use of so-called *monotypes* (or mono-objects), that is, (dependent) types containing at most one entity or having at most one proof. In $\mathcal{S}et$, mono objects are *singletons*, that is, sets containing at most one element.

Monotypes correspond to monic maps. This can be illustrated concretely by considering the categories $\mathcal{I}ndset$ of *indexed* sets and $\mathcal{S}et^{\rightarrow}$ of *bivariant* sets. The objects of $\mathcal{I}ndset$ are indexed sets of the form $M = \{\langle i, M_i\rangle : i \in I\}$ and those of $\mathcal{S}et^{\rightarrow}$ maps $A \rightarrow B$ in $\mathcal{S}et$, with appropriately defined arrows in each case. It can be shown that these two categories are equivalent. If we think of (the objects of) $\mathcal{S}et$ as representing simple or static types, then (the objects of) $\mathcal{I}ndset$, and hence also of $\mathcal{S}et^{\rightarrow}$, represent dependent or variable types. It is easily seen that a monotype, or object, in $\mathcal{I}ndset$, is precisely an object M for which each M_i has at most one element. Moreover, under the equivalence between $\mathcal{I}ndset$ and $\mathcal{S}et^{\rightarrow}$, such an object corresponds to a monic map- object in $\mathcal{S}et^{\rightarrow}$.

Now consider $\mathcal{S}et^{\rightarrow}$ as a topos. Under the topos-theoretic interpretation in $\mathcal{S}et^{\rightarrow}$, formulas correspond to monic arrows, which in turn correspond to mono-objects in $\mathcal{I}ndset$. Carrying this over entirely to $\mathcal{I}ndset$ yields the sought modification of the "propositions-as-types" paradigm to bring it into line with the topos-theoretic interpretation of formulas, namely, to take formulas or propositions to correspond to *mono*-objects, rather than to *arbitrary* objects. Let us call this the "formulas-as-monotypes" interpretation.

Finally let us reconsider AC under the "formulas-as-monotypes" interpretation within $\mathcal{S}et$. It will be convenient to rephrase AC as the assertion

$$\forall i \in I \exists j \in J M_{ij} \leftrightarrow \exists f \in J^I \forall i \in I M_{if(i)} \qquad (*)$$

where $\langle M_{ij} : i \in I, j \in J\rangle$ is any doubly indexed family of propositions (or sets). In the "propositions as types" interpretation, where (*) corresponds to the existence of an isomorphism between $\prod_{i\in I} \coprod_{j\in J} M_{ij}$ and $\coprod_{f\in J^I} \prod_{i\in I} M_{if(i)}$. On the other hand, AC

[35]Precisely as Ramsey (*v. supra*) asserted, but in this case for quite different reasons. Ramsey construed, and accepted the truth of, the axiom of choice as asserting the objective existence of choice functions, given extensionally and so independently of the manner in which they might be described. But the intensional nature of constructive mathematics, and, in particular, of DCTT decrees that nothing is given completely independently of its description. This leads to a strong construal of the quantifiers which, as we have observed, trivializes the axiom of choice by rendering the antecedent of the implication constituting it essentially equivalent to the consequent. It is remarkable that the axiom of choice can be considered tautological both from an extensional and from an intensional point of view.

interpreted in the usual way, that is, using the rules of topos semantics, can be presented in the form of the distributive law

$$\bigcap_{i\in I}\bigcup_{j\in J} M_{ij} = \bigcup_{f\in J^I}\bigcap_{i\in I} M_{if(i)}. \qquad (**)$$

In the "propositions-as-types" interpretation (as applied to $\mathcal{S}et$), the universal quantifier $\forall i \in I$ corresponds to the product $\prod_{i\in I}$ and the existential quantifier $\exists i \in I$ to the coproduct, or disjoint sum, $\coprod_{i\in I}$. Now in the "formulas-as-monotypes" interpretation, wherein formulas correspond to singletons, $\forall i \in I$ continues to correspond to $\prod_{i\in I}$, since the product of singletons is still a singleton. But the interpretation of $\exists i \in I$ is changed. In fact, the interpretation of $\exists i \in IM_i$ (with each M_i a singleton) now becomes $[\coprod_{i\in I} M_i]$, where for each set $X, [X] = \{u : u = 0 \wedge \exists x.x \in X\}$ is the *canonical singleton* associated with X.

It follows that, under the "formulas-as-monotypes" interpretation, the proposition $\forall i \in I\exists j \in JM_{ij}$ is interpreted as the singleton

$$\prod_{i\in I}[\coprod_{j\in J} M_{ij}] \qquad (1)$$

and the proposition $\exists f \in J^I\forall i \in IM_{if(i)}$ as the singleton

$$[\coprod_{f\in J^I}\prod_{i\in I} M_{if(i)}]. \qquad (2)$$

Under the "formulas-as-monotypes" interpretation AC would be construed as asserting the existence of an isomorphism between (1) and (2).

Now it is readily seen that to give an element of (1) amounts to no more than affirming that, for every $i \in I$, $\bigcup_{j\in J} M_{ij}$ is nonempty. But to give an element of (2) amounts to specifying maps $f \in J^I$ and g with domain I such that $\forall i \in Ig(i) \in M_{if(i)}$. It follows that to assert the existence of an isomorphism between (1) and (2), that is, to assert AC under the "formulas-as-monotypes" interpretation, is tantamount to asserting AC in the form (**), so leading in turn to classical logic. This is in sharp contrast with AC under the "propositions-as-types" interpretation, where, as we have seen, its assertion is automatically correct and so has no nonconstructive consequences.

In the "propositions as types" interpretation each logical operation corresponds to a categorical operation: $\wedge$ to $\times, \vee$ to $+, \rightarrow$ to exponentiation, $\perp$ to an initial element 0, $\neg$ to exponentiation by 0, $\forall$ to product and $\exists$ to coproduct. This suggests that CDTT should be interpretable in suitable categories, just as local set theories are interpretable in toposes. For some time it was conjectured that the appropriate categories in this respect were the so-called *locally Cartesian closed* categories:

these are finitely complete categories $\mathcal{E}$ such that, for each object A of $\mathcal{E}$, the "slice" category $\mathcal{E}/A$ is Cartesian closed. (Any topos is locally Cartesian closed.) It was already known that many of the mathematical constructions within a topos could be carried out within a locally Cartesian closed category. In such a category, the notion of "variable set", for example an "A-indexed set" , is represented by an arrow $B \to A$ of $\mathcal{E}$, that is, by an object of $\mathcal{E}/A$. In the interpretation of CDTT that was explicitly worked out by Seely [1984], types are interpreted as objects, and terms as arrows, of a locally Cartesian closed category $\mathcal{E}$. As Seely points out, this interpretation engenders a systematic ambiguity among the notions of type, predicate and term, and between object and proof: indeed a term of type A is an arrow into A, which is in turn a predicate over A, and an arrow $1 \to A$ may be regarded either as an object of type A or as a proof of the proposition A. (This ambiguity is, of course, shared by the "propositions as types" interpretation.) In precise analogy with the correspondence between local set theories and toposes, Seely establishes a correspondence between theories formulated in Martin-Löf's system — constructive dependent type theories — and locally Cartesian closed categories. The relationship between type theories and categories is investigated in a general setting in Jacobs [1999].

It is natural to ask what form of *set theory* is interpretable in CDTT. This has been worked out by Peter Aczel[36] and has become come known as *constructive set theory* (CST). This is a system of intuitionistic set theory in which the set-theoretic operations in reflect those on types in CDTT. n particular CST will admit products and exponentials of sets. But, given the predicative nature of CDTT, CST must also be predicative. This means, in particular, that in CST there can be no set of propositions, and no power sets. A particularly illuminating form of CST[37] employs the distinction between sets and classes, as in Gödel-Bernays set theory. In this version of CST, any predicate determines a class, but only certain classes are sets. Here a class may be thought of as something like a species or category in Martin-Löf's sense, that is, a range of objects all of which share a given property, and whose identity relations are fully determinate, but whose "extent" is not sufficiently fixed to admit quantification over it. A set, on the other hand, is a class with a fully determinate extent, hence supporting quantification. In CST, the class PA of all subsets — the power class — of a set A is almost never a set (the only exception being the power class of the empty set). On the other hand the exponential A^B of two sets A, B — the class of maps from B to A — is always a set, so in particular 2^A is always a set. In classical set theory 2^A is essentially the power set of A, which cannot be the case in CST. This means that in CST 2 cannot represent, as it does in classical set theory, the set of propositions or truth values. In classical set theory 2 is P1, the power set of a one-element set, but in CST P1, while of course a class, is not a set. It can be shown that classical set theory is equivalent to CST augmented by the law of excluded middle and the assertion that P1 is a set.

[36]See bibliographic entries under Aczel, Aczel and Gambino, and Aczel and Rathjen.
[37]See Aczel and Rathjen [2201].

Finally, a word on other forms of type theory that have emerged in the past few decades. *Polymorphic* type theory admits the presence of *type variables*, so that a type variable **X** may occur inside a type **A**(**X**). The assignment of "types" to *type* variables (or terms) is achieved by the introduction of a new order of entities called *kinds.* Just as each ordinary term is assigned a type, so each type term is assigned a kind. In particular, there is a kind called **Type** which represents the "kind of all types".[38] Any type **A** then satisfies the judgment **A: Type**, and **Type** itself satisfies the judgment **Type: Kind.** In addition to **Type**, the presence of arbitrarily arbitrary "subkinds" of **Type** may be assumed. It is also assumed that **Kind** is closed under the operations corresponding to cartesian product and exponentiation, and that **Type** is closed under these latter and those corresponding to product and sum over type variables of an arbitrarily given kind.

Polymorphic and dependent type theories have been combined in various ways. One form is the *Calculus of Constructions* of Coquand and Huet [1988]. This has been extended to what Jacobs [1999] calls *Full Higher Order Dependent Type Theory*, or the *full theory* for short. In dependent type theory types can "depend on" types; in polymorphic type theory types can "depend on" kinds. In the full theory, the dependence is extended to the remaining possibilities: those of kinds on types and kinds on kinds. An account of this, and other amalgamations of polymorphic and dependent type theories, may be found in Jacobs [1999].

APPENDIX: BASIC CONCEPTS OF CATEGORY THEORY

A *category* $\mathcal{C}$ is determined by first specifying two classes $Ob(\mathcal{C}), Arr(\mathcal{C})$ — the collections of $\mathcal{C}$-*objects* and $\mathcal{C}$-*arrows*. These collections are subject to the following axioms:

- Each $\mathcal{C}$-arrow f is assigned a pair of $\mathcal{C}$-objects $\text{dom}(f)$, $\text{cod}(f)$ called the *domain* and *codomain* of f, respectively. To indicate the fact that $\mathcal{C}$-objects X and Y are respectively the domain and codomain of f we write $f : X \to Y$ or $X \xrightarrow{f} Y$. The collection of $\mathcal{C}$-arrows with domain X and codomain Y is written $\mathcal{C}(X, Y)$.
- Each $\mathcal{C}$-object X is assigned a $\mathcal{C}$-arrow $1_X : X \to X$ called the *identity arrow* on X.
- Each pair f, g of $\mathcal{C}$-arrows such that $\text{cod}(f) = \text{dom}(g)$ is assigned an arrow $g \circ f$: $\text{dom}(f) \to \text{cod}(g)$ called the *composite* of f and g. Thus if $f : X \to Y$ and $g : Y \to Z$ then $g \circ f : X \to Z$. We also write $X \xrightarrow{f} Y \xrightarrow{g} Z$ for $g \circ f$. Arrows f, g satisfying $\text{cod}(f) = \text{dom}(g)$ are called *composable*.
- *Associativity law.* For composable arrows (f, g) and (g, h), we have $h \circ (g \circ f) = h \circ (g \circ f)$.

[38]This is, of course, not the same as asserting that there is a *type* of all types, for to do so would be to rekindle Girard's paradox (see note 25 above).

- *Identity law.* For any arrow $f : X \to Y$, we have $f \circ 1_X = f = 1_Y \circ f$.

As a basic example of a category, we have the category $\mathcal{S}et$ of sets whose objects are all sets and whose arrows are all maps between sets (strictly, triples (f, A, B) with domain$(f) = A$ and range$(f) \subseteq B$.) Other examples of categories are the category of groups, with objects all groups and arrows all group homomorphisms and the category of topological spaces with objects all topological spaces and arrows all continuous maps. Categories with just one object may be identified with *monoids*, that is, algebraic structures with an associative multiplication and an identity element.

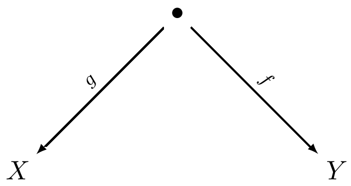
A *subcategory* $\mathcal{C}$ of a category $\mathcal{D}$ is any category whose class of objects and arrows is included in the class of objects and arrows of $\mathcal{D}$, respectively, and which is closed under domain, codomain, identities, and composition. If, further, for any objects C, C' of $\mathcal{C}$, we have $\mathcal{C}(C, C') = \mathcal{D}(C, C')$, we shall say that $\mathcal{C}$ is a *full* subcategory of $\mathcal{D}$.

Basic Category-theoretic Definitions

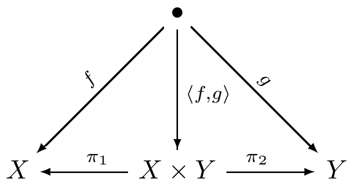
Commutative diagram (in category)	Diagram of objects and arrows such that the arrow obtained by composing the arrows of any connected path depends only on the endpoints of the path.
Initial object	Object 0 such that, for any object X, there is a unique arrow $0 \to X$ (e.g., $\emptyset$ in $\mathcal{S}et$)
Terminal object	Object 1 such that, for any object X, there is a unique arrow $X \to 1$ (e.g. any singleton in $\mathcal{S}et$)
Element of an object X	Arrow $1 \to X$
Monic arrow $X \rightarrowtail Y$	Arrow $f : X \to Y$ such that, for any arrows $g, h : Z \to X, f \circ g = f \circ h \Rightarrow g = h$ (in $\mathcal{S}et$, *one-one* map)
Epic arrow $X \twoheadrightarrow Y$	Arrow $f : X \to Y$ such that, for any arrows $g, h : Y \to Z, g \circ f = h \circ f \Rightarrow g = h$ (in $\mathcal{S}et$, *onto* map)
Isomorphism $X \cong Y$	Arrow $f : X \to Y$ for which there is $g : Y \to X$ such that $g \circ f = 1_X, f \circ g = 1_Y$

Product of objects X, Y

Object $X \times Y$ with arrows (projections) $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$ such that any diagram

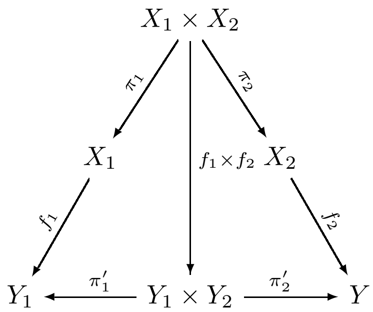


can be uniquely completed to a commutative diagram



Product of arrows $f_1 : X_1 \to Y_1 f_2 : X_2 \to Y_2$

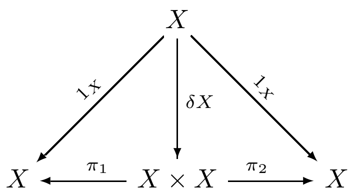
Unique arrow $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ making the diagram



commute. I.e., $f_1 \times f_2 = \langle f_1 \circ \pi_1, f_2 \circ \pi_2 \rangle$.

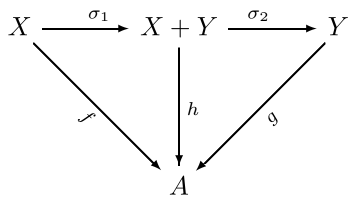
Diagonal arrow on object X

Unique arrow $\delta_X : X \to X \times X$ making the diagram



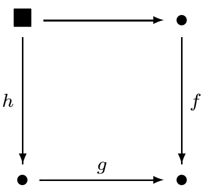
commute. I.e., $\delta_X = \langle 1_X, 1_X \rangle$.

Coproduct of objects X, Y — Object $X+Y$ together with a pair of arrows $X \xrightarrow{\sigma_1} X+Y \xleftarrow{\sigma_2} Y$ such that for any pair of arrows $X \xrightarrow{f} A \xleftarrow{g} Y$, there is a unique arrow $X+Y \xrightarrow{h} A$ such that the diagram

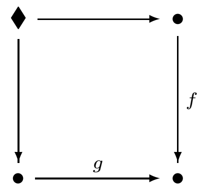


commutes.

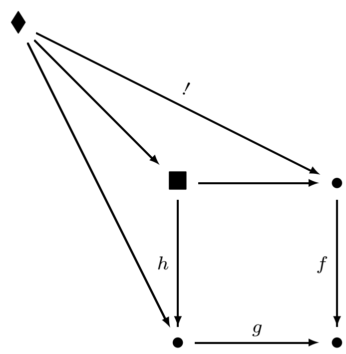
Pullback diagram or *square* — Commutative diagram of the form



such that for any commutative diagram



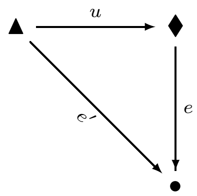
there is a unique ♦ $\xrightarrow{!}$ ■ such that



commutes. h is called the *pullback* of f (along g).

Equalizer of pair of arrows $\bullet \underset{g}{\overset{f}{\rightrightarrows}} \blacksquare$

Arrow $\blacklozenge \xrightarrow{e} \bullet$ such that $f \circ e = g \circ e$ and, for any arrow $\blacktriangle \xrightarrow{e'} \bullet$ such that $f \circ e' = g \circ e'$ there is a unique $\blacktriangle \xrightarrow{u} \blacklozenge$ such that

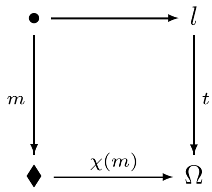


commutes.

Subobject of an object X — Pair (m, Y), with m a monic arrow $Y \rightarrowtail X$

Inclusion of subobjects — For $m : Y \rightarrowtail X, n : Z \rightarrowtail X, m \subseteq n$ iff there is $f : Y \to Z$ such that $n \circ f = m$.

Truth value object or *subobject* classifier — Object Ω together with arrow $t : 1 \to \Omega$ such that every monic $m : \bullet \rightarrowtail \blacklozenge$ (i.e., subobject of $\blacklozenge$) can be uniquely extended to a pullback diagram of the form



$\chi(m)$ is the *characteristic arrow* of m.

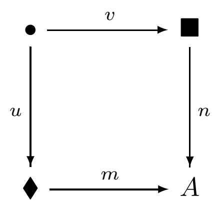
Conversely, any diagram of the form $\blacklozenge \xrightarrow{u} \Omega \xleftarrow{t} 1$ must have a pullback.

The pullback of u will be written $\overline{u}$. It is necessarily monic.

The next few definitions assume that the ambient category has a terminal object, products and a subobject classifier. It can be shown that in any such category, every monic arrow has a pullback.

Partial ordering $\leq$ *on arrows to* Ω. — For $u, v : A \to \Omega, u \leq v$ iff $\overline{u} \subseteq \overline{v}$.

Intersection of subobjects — For monics m, n with common codomain A, form the pullback square

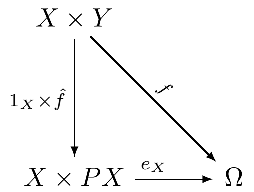


Then the map $m \circ u (= n \circ v)$ is the intersection $m \cap n$ of m and n.

Meet of two maps to Ω. — For $u, v : A \to \Omega, u \wedge v = \chi(\overline{\overline{u} \cap \overline{v}})$.

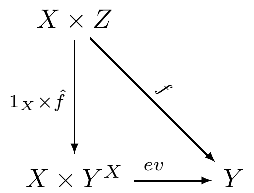
Equality arrow $A \times A \to \Omega$ — $eq_A = \chi(\delta_A)$

Power object of an object X. — An object PX together with an arrow ("evaluation") $e_X : X \times PX \to \Omega$ such that, for any $f : X \times PX \to \Omega$, there is a unique arrow $\widehat{f} : Y \to PX$ such that



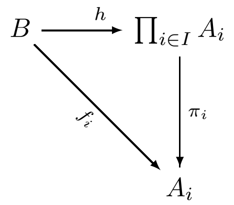
commutes. (In $\mathcal{S}et$, PX is the power set of X and e_X the characteristic function of the membership relation between X and PX.)

Exponential object of objects Y, X — An object Y^X, together with an arrow *ev*: $X \times Y^X \to Y$ such that, for any arrow $f : X \times Z \to Y$ there is a unique arrow $\widehat{f} : Z \to Y^X$ — the *exponential transpose* of f — such that the diagram



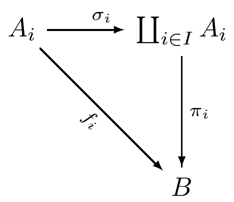
commutes. In $\mathcal{S}et$, Y^X is the set of all maps $X \to Y$ and *ev* is the map that sends (x, f) to $f(x)$.

Product of indexed set $\{A_i : i \in I\}$ of objects — Object $\prod_{i\in I} A_i$ together with arrows $\prod_{i\in I} A_i \xrightarrow{\pi_i} A_i (i \in I)$ such that, for any arrows $f_i : B \to A_i (i \in I)$ there is a *unique* arrow $h : B \longrightarrow \prod_{i\in I} A_i$ such that, for each $i \in I$, the diagram



commutes.

Coproduct of indexed set $\{A_i : i \in I\}$ of objects — Object $\coprod_{i\in I} A_i$ together with arrows $A_i \xrightarrow{\sigma_i} \coprod_{i\in I} A_i$ $(i \in I)$ such that, for any arrows $f_i : A_i \to B(i \in I)$ there is a *unique* arrow $h : \coprod_{i\in I} A_i \longrightarrow B$ such that, for each $i \in I$, the diagram

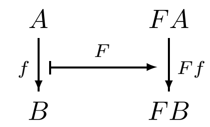


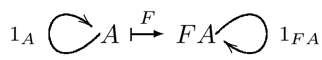
commutes.

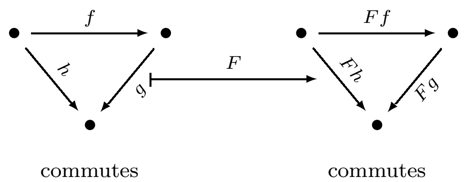
A category is *cartesian closed* if it has a terminal object, as well as products and exponentials of arbitrary pairs of its objects. It is *finitely complete* if it has a terminal object, products of arbitrary pairs of its objects, and equalizers. A *topos* is a category possessing a terminal object, products, a truth-value object, and power objects. It can be shown that every topos is cartesian closed and finitely complete (so that this notion of topos is equivalent to that originally given by Lawvere and Tierney). The category $\mathcal{S}et$ of sets is a topos.

More on products in a category. A *product* of objects $A_1, ..., A_n$in a category $\mathcal{C}$ is an object $A_1 \times ... \times A_n$ together with arrows $\pi_i : A_1 \times ... \times A_n \to A_i$ for $i = 1, ..., n$, such that, for any arrows $f_i : B \to A_i, i = 1, ..., n$, there is a unique arrow, denoted by $\langle f_1, ..., f_n\rangle : B \to A_1 \times ... \times A_n$ such that $\pi_i \circ \langle f_1, ..., f_n\rangle = f_i, i = 1, ..., n$. Note that, when $n = 0, A_1 \times ... \times A_n$ is the terminal object 1. The category is said to *have finite products* if $A_1 \times ... \times A_n$ exists for all $A_1, ..., A_n$. If $\mathcal{C}$ has binary products, it has finite products, since we may take $A_1 \times ... \times A_n$ to be $A_1 \times (A_2 \times (... \times A_n)...)$. It is easily seen that the product operation is, up to isomorphism, commutative and associative. The relevant isomorphisms are called *canonical* isomorphisms.

A *functor* $F : \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ is a map that "preserves commutative diagrams", that is, assigns to each $\mathcal{C}$-object A a $\mathcal{D}$-object FA and to each $\mathcal{C}$-arrow $f : A \to B$ a $\mathcal{D}$-arrow $Ff : FA \to FB$ in such a way that:







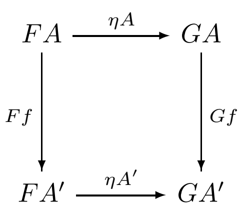
A functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence* if it is "an isomorphism up to isomorphism", that is, if it is

- *faithful*: $Ff = Fg \Rightarrow f = g$.
- *full*: for any $h : FA \to FB$ there is an $f : A \to B$ such that $h = Ff$.
- *dense*: for any $\mathcal{D}$-object B there is a $\mathcal{C}$-object A such that $B \cong FA$.

Two categories are *equivalent,* written $\simeq$, if there is an equivalence between them. Equivalence is the appropriate notion of "identity of form" for categories.

Given functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* between F and G is a map η from the objects of C to the arrows of D satisfying the following conditions.

- For each object A of $\mathcal{C}, \eta A$ is an arrow $FA \to GA$ in $\mathcal{D}$
- For each arrow $f : A \to A'$ in $\mathcal{C}$,



Finally, two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and are said to be *adjoint* to one another if, for any objects A of $\mathcal{C}$, B of $\mathcal{D}$, there is a "natural" bijection between arrows $A \to GB$ in $\mathcal{C}$ and arrows $FA \to B$ in $\mathcal{D}$. To be precise, for each such pair A, B we must be given a bijection $\varphi_{AB} : \mathcal{C}(A, GB) \to \mathcal{D}(FA, B)$ satisfying the "naturality" conditions

- for each $f : A \to A'$ and $h : A' \to GB, \varphi_{AB}(h \circ f) = \varphi_{A\prime B}(h) \circ Ff$
- for each $g : B \to B\prime$ and $h : A \to GB\prime, \varphi_{AB}(Gg \circ h) = g \circ \varphi_{AB\prime}(h)$.

Under these conditions F is said to be *left adjoint* to G, and G *right adjoint* to F.

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