UNCOUNTABLE STANDARD MODELS
OF ZFC + V $\neq$ L

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Dedicated to the memory of A. Mostowski

A well-known result of Cohen ([1], p.109) asserts that in ZF + V = L one can prove that there are no uncountable standard models of ZFC + "There is a non-constructible real". It is natural to ask what the situation is for uncountable standard models of ZFC + "There is a non-constructible set". In this paper we shall prove the following

THEOREM. ZFC + "There exists a natural model R_α of ZFC" $\vdash$ "There exist standard models of ZFC + V $\neq$ L of all cardinalities $< \alpha$."

This theorem has the following consequences. Let ZFI = ZFC + "There exists an inaccessible cardinal".

COROLLARY 1. ZFI $\vdash$ "There is a standard model of ZFC + V $\neq$ L of any cardinality less than the first inaccessible cardinal".

Let KMC be Kelley-Morse set theory with choice. Since it is known [5] that in KMC one can prove the existence of arbitrarily large natural models of ZFC , it follows immediately from the theorem that

COROLLARY 2. KMC $\vdash$ "There is a standard model of ZFC + $V \neq L$ of any cardinality".

The proof of the theorem uses the technique of Boolean-valued models of set theory as presented, e.g. in [2]. For the theory of Boolean algebras we refer the reader to [6].

As usual, we write ZF for Zermelo-Fraenkel set theory, ZFC for ZF + axiom of choice, $V = L$ for the axiom of constructibility and $V \neq L$ for its negation.

By a <u>standard</u> model of ZF we understand a model of the form $\mathfrak{M} = \langle M, \in/M \rangle$, where M is a <u>transitive</u> set and $\in/M = \{\langle x,y\rangle \in M^2 : x \in y\}$. If $\mathfrak{M}$ is a standard model of ZFC and B is a complete Boolean algebra in $\mathfrak{M}$, we write, as usual $\mathfrak{M}^{(B)}$ for the B-extension of $\mathfrak{M}$ and $\|\sigma\|$ for the B-value of any sentence σ of set theory (which may contain names for elements of $\mathfrak{M}^{(B)}$). Well-known is the fact that $\|\sigma\| = 1$ for any theorem σ of ZFC. We recall that there is a canonical map $x \mapsto \hat{x}$ of $\mathfrak{M}$ into $\mathfrak{M}^{(B)}$. We shall also need the following fact ([2], Lemma 50).

LEMMA 1. For each formula $\varphi(x)$ of set theory (which may contain names for elements of $\mathfrak{M}^{(B)}$) there is $t \in \mathfrak{M}^{(B)}$ such that:

$$\|\exists x \varphi(x)\| = \|\varphi(t)\| .$$

Let B be a complete Boolean algebra; a subset P of B is said to be <u>dense</u> if $0 \notin P$ and $\forall x \in B[x \neq 0 \Rightarrow \exists p \in P\ (p \leq x)]$. If $\varkappa$ is a cardinal, P is said to satisfy the <u>$\varkappa$-descending chain condition</u> ($\varkappa$-dcc) if for each $\alpha < \varkappa$ and each descending α-sequence $p_0 \geq p_1 \geq \ldots \geq p_\xi \geq \ldots$ $(\xi<\alpha)$ from P there is $p \in P$ such that $p \leq p_\xi$ for all $\xi < \alpha$.

LEMMA 2. Suppose that B contains a dense subset satisfying the $\varkappa$-dcc, and let $\{A_\xi : \xi < \varkappa\}$ be a family of subsets of B such that $\bigvee A_\xi = 1$ for each $\xi < \varkappa$. Then there is an ultrafilter U in B such that $U \cap A_\xi \neq \emptyset$ for all $\xi < \varkappa$.

<u>Proof</u>. Let J be a set sufficiently large so that each A_ξ can be enumerated as $\{a_{\xi j} : j \in J\}$. We show that there is $f \in J^\varkappa$ such that, for each $\alpha < \varkappa$,

$$(1) \qquad \bigwedge_{\xi<\alpha} a_{\xi f(\xi)} \neq 0 .$$

We define f by recursion as follows. Let $\alpha < \varkappa$ and suppose that for each $\xi < \alpha$ we have selected $p_\xi \in P$ and $f(\xi) \in J$ in such a way that

$$(2) \qquad p_\xi \leq a_{\xi f(\xi)} \quad \text{for all} \quad \xi < \alpha$$

$$(3) \qquad \eta \leq \xi < \alpha \Rightarrow p_\eta \geq p_\xi .$$

We show how to obtain p_α and $f(\alpha)$. Since P satisfies the $\varkappa$-dcc, there is $p \in P$ such that $p \leq p_\xi$ for all $\xi < \alpha$. We have

$$0 \neq p = p \wedge 1 = p \wedge \bigvee_{j\in J} a_{\alpha j} = \bigvee_{j\in J} p \wedge a_{\alpha j} ,$$

so there must be $j \in J$ such that $p \wedge a_{\alpha j} \neq 0$, and hence, since P is dense, $q \in P$ such that $q \leq p \wedge a_{\alpha j}$. We take $f(\alpha)$ to be such a $j \in J$, and p_α to be such a $q \in P$. It is now clear that (2) and (3) hold with "$\xi < \alpha$" replaced by "$\xi \leq \alpha$" and so by recursion we obtain p_α and $f(\alpha)$ to satisfy (2) and (3) for all $\alpha < \varkappa$. If $\alpha < \varkappa$, $\langle p_\xi : \xi < \alpha \rangle$ is a descending α-sequence in P and so there is (by dcc) a $p \in P$ such that $p \leq p_\xi$ for all $\xi < \alpha$. But then, by (2), we immediately obtain (1).

To complete the proof we observe that, by (1), the set $\{a_{\alpha f(\alpha)} : \alpha < \varkappa\}$ has the finite intersection property and hence can be extended to an ultrafilter in B. This ultrafilter clearly meets the requirements of the Lemma. ■

An ultrafilter U in B is said to <u>preserve</u> the family of joins $\bigvee A_\alpha$ $(\alpha < \varkappa)$, where $\{A_\alpha : \alpha < \varkappa\}$ is family of subsets of B, provided that for each $\alpha < \varkappa$,

$$\bigvee A_\alpha \in U \Rightarrow U \cap A_\alpha \neq \emptyset .$$

Lemma 2 gives the following generalization, for complete Boolean algebras, of the well-known Rasiowa-Sikorski lemma:

COROLLARY. Suppose that B contains a dense subset satisfying the $\varkappa$-dcc. Then for each family $\{A_\alpha : \alpha < \varkappa\}$ of subsets of B there is an ultrafilter in B which preserves the family of joins $\bigvee A_\alpha$ $(\alpha < \varkappa)$.

<u>Proof</u>. Put $a_\alpha = \bigvee A_\alpha$ and apply Lemma 2 to the family $\{A_\alpha \cup \{a_\alpha^*\} : \alpha < \varkappa\}$, where a_α^* is the complement of a_α in B. ■

<u>Remark</u>. I am grateful to Professor Vopěnka and others at the conference for suggesting the present version of this Corollary, which is stronger than my original version.

Now let $\varkappa$ be a <u>regular</u> cardinal and let $X_\varkappa$ be the space $2^\varkappa$ endowed with the $\varkappa$-topology, i.e. the topology whose basic open sets are of the form

$$U(\alpha, f) = \{g \in X_\varkappa : g(\xi) = f(\xi) \quad \text{for} \quad \xi \leq \alpha\}$$

where $f \in X_\varkappa$ and $\alpha < \varkappa$. We denote by $B_\varkappa$ the complete Boolean algebra of regular open subsets of $X_\varkappa$. ($B_\varkappa$ is the algebra which, in the corresponding Boolean extension, adds a new member to $\mathfrak{P}\varkappa$ but leaves $\mathfrak{P}\alpha$ undisturbed for all $\alpha < \varkappa$.)

It is clear that the family of all sets $U(\alpha, f)$ is dense in $B_\varkappa$ and that this family satisfies the $\varkappa$-dcc (since $\varkappa$ is regular). Hence, by the Corollary to Lemma 2 we have

LEMMA 3. If $\varkappa$ is a regular cardinal, then for each family $\{A_\alpha : \alpha < \varkappa\}$ of subsets of $B_\varkappa$ there is an ultrafilter in $B_\varkappa$ which preserves the family of joins $\bigvee A_\alpha$ $(\alpha < \varkappa)$.

We now turn to

<u>Proof of the Theorem</u>. Let R_α be a natural model of ZFC. By [4], α is a limit cardinal, and so by the downward Löwenheim-Skolem theorem it will be enough to show that there is a standard model of ZFC + V ≠ L for each <u>regular</u> cardinal $< \alpha$. So let $\mathfrak{M} = \langle R_\alpha, \in/R_\alpha \rangle$ and let $\varkappa$ be a regular cardinal $< \alpha$. Put $B = B_\varkappa$. Then B is a complete Boolean algebra in $\mathfrak{M}$ and so we can form the B-extension $\mathfrak{M}^{(B)}$ of $\mathfrak{M}$.

Using Lemma 1, for each formula $\varphi(v_0,\ldots,v_n)$ of the language of set theory (<u>without</u> parameters from $\mathfrak{M}^{(B)}$) we let

$$f_\varphi : (\mathfrak{M}^{(B)})^n \to \mathfrak{M}^{(B)}$$

be a Skolem function for $\varphi(v_0,\ldots,v_n)$ in $\mathfrak{M}^{(B)}$, i.e. such that, for all $x_1,\ldots,x_n \in \mathfrak{M}^{(B)}$

(1) $\|\exists v_0 \varphi(v_0,x_1,\ldots,x_n)\| = \|\varphi(f_\varphi(x_1,\ldots,x_n), x_1,\ldots,x_n)\|$.

Let $\mathcal{A} \subseteq \mathfrak{M}^{(B)}$ be the closure of the set $\{\hat{\xi} : \xi < \varkappa\}$ under the f_φ. Then $\mathcal{A}$ has cardinality $\varkappa$ and, using (1) we have

(2) for any formula $\varphi(v_0,\ldots,v_n)$ and any $a_1,\ldots,a_n \in \mathcal{A}$, there is $a_0 \in \mathcal{A}$ such that $\|\exists v_0 \varphi(v_0,a_1,\ldots,a_n)\| = \|\varphi(a_0,a_1,\ldots,a_n)\|$.

Let $\mathrm{Ord}(x)$ be the formula "x is an ordinal". It is well-known that, for any $x \in \mathfrak{M}^{(B)}$, we have $\|\mathrm{Ord}(x)\| = \bigvee_{\xi<\alpha} \|x = \hat{\xi}\|$. Using Lemma 3, let U be an ultrafilter in B which preserves the joins

(3) $$\|\mathrm{Ord}(a)\| = \bigvee_{\xi<\alpha} \|a = \hat{\xi}\| \qquad (a \in \mathcal{A})$$

Let $\mathcal{A}/U$ be the quotient of $\mathfrak{M}^{(B)}$ by U, i.e.

$$\mathcal{A}/U = \langle \{a^U : a \in \mathcal{A}\}, \in_U \rangle$$

where a^U is the equivalence class of $a \in \mathcal{A}$ under the relation $\sim_U$ defined by $a \sim_U a' \iff \|a = a'\| \in U$ and $\in_U$ is defined by $a^U \in_U a'^U \iff \|a \in a'\| \in U$. Using (2), it is easy to show by induction on complexity of formulas that for any formula $\varphi(v_0,\ldots,v_n)$ of set theory and any $a_0,\ldots,a_n \in \mathcal{A}$,

$$\mathcal{A}/U \models \varphi[a_0^U,\ldots,a_n^U] \iff \|\varphi(a_0,\ldots,a_n)\| \in U .$$

It follows that $\mathcal{A}/U$ is a model of ZFC . Also, the ξ^U for $\xi < \varkappa$ are all distinct, so $\mathcal{A}/U$ has cardinality $\varkappa$. Since B is atomless, we have $\|V \neq L\| = 1$, so $\mathcal{A}/U$ is also a model of $V \neq L$. Finally, since U preserves the joins (3), it quickly follows that the map $\xi \mapsto \xi^U$ is order-preserving from (true) ordinals onto the ordinals of $\mathcal{A}/U$, so that the ordinals of $\mathcal{A}/U$ are well-ordered. The usual rank argument now implies that ϵ_U is a well-founded relation, so that $\mathcal{A}/U$ is isomorphic to a standard model which meets the requirements of the theorem. This completes the proof. ■

CONCLUDING REMARKS

1. Since $B_\varkappa$ is known to preserve cardinals, it is not hard to see that for a definable cardinal $\varkappa$ (e.g. $\aleph_0, \aleph_1, \ldots, \aleph_\omega$, etc.) the proof of the theorem yields a standard model $\mathcal{N}$ of cardinality $\varkappa^+$ such that

$$\mathcal{N} \models \text{ZFC} + \mathfrak{P}\varkappa \subseteq L + \mathfrak{P}\varkappa^+ \not\subseteq L \ .$$

Notice that in any theory consistent with $ZF + V = L$ one <u>cannot</u> prove the existence of a standard model $\mathcal{N}$ of cardinality $\varkappa^+$ such that $\mathcal{N} \models \text{ZFC} + \mathfrak{P}\varkappa \not\subseteq L$, because in $ZF + V = L$ one can prove that, for any such model, $\mathcal{N} \models \mathfrak{P}\varkappa \subseteq L$.

2. Both P. Vopěnka and J. Paris have pointed out that the assumption in the theorem that there exists a natural model of ZFC can be substantially weakened (thereby yielding, of course, a weaker conclusion). In fact one can prove the following

(*) ZFC + "There exists an uncountable standard model of ZFC" $\vdash$ "There exists an uncountable standard model of ZFC + $V \neq L$".

The proof of (*) can be based on the following Lemma (which I have recently noticed resembles a result implicit in [3]):

LEMMA. Let $\varkappa$ be a regular uncountable cardinal and let $\mathfrak{M}$ be a standard model of ZFC such that (i) $|\mathfrak{M}| = \varkappa$, (ii) $\varkappa \in \mathfrak{M}$ and (iii) $\{x \subseteq \varkappa : |x| < \varkappa\} \subseteq \mathfrak{M}$. Then there is a standard model $\mathcal{N}$ of ZFC such that $\mathfrak{M} \subseteq \mathcal{N}$ and $\mathcal{N} \models \mathfrak{P}\varkappa \not\subseteq L$.

<u>Proof</u>. (Sketch). Let $B = B_\varkappa^{(\mathfrak{M})}$, i.e. the Boolean algebra B constructed in $\mathfrak{M}$. Since every subset of $\varkappa$ of cardinality $< \varkappa$ is in $\mathfrak{M}$, it quickly follows that B has a dense subset satisfying the $\varkappa$-dcc (consider the set of $U(\alpha, f)$ constructed in $\mathfrak{M}$). Hence, by the Corollary to Lemma 2 and the fact that $|\mathfrak{M}| = \varkappa$, there is an

$\mathfrak{M}$-generic ultrafilter U in B. Then $\mathcal{N} = \mathfrak{M}[U]$ meets the requirements of the lemma.

Now we can prove (*) á la Vopěnka and Paris. Suppose that there is an uncountable standard model $\mathfrak{M}$ of ZFC. If $\mathfrak{M} \models V \neq L$ then we are done, so assume $\mathfrak{M} \models V = L$. There are now two cases to consider.

<u>Case (a)</u>: $\omega_1 \in \mathfrak{M}$. We work in L until further notice, with the proviso that ω_1 is always the <u>true</u> ω_1, <u>not</u> $\omega_1^{(L)}$. By the Löwenheim-Skolem theorem we may assume $|\mathfrak{M}| = \omega_1$. It is now easy to see that (inside L), conditions (i) through (iii) of the above Lemma are satisfied by $\mathfrak{M}$ (with $\varkappa = \omega_1$). Therefore, applying the Lemma inside L, there is a standard model $\mathcal{N}$ of ZFC + $V \neq L$ such that $\mathfrak{M} \subseteq \mathcal{N}$, so that $\omega_1 \in \mathcal{N}$. But the property of being a standard model of ZFC + $V \neq L$ is L-absolute, so, emerging form L into the real world, $\mathcal{N}$ is truly a standard model of ZFC + $V \neq L$. Since $\omega_1 \in \mathcal{N}$, we have $|\mathcal{N}| \geq \omega_1$ and (*) follows.

<u>Case (b)</u>: $\omega_1 \notin \mathfrak{M}$. By the downward Löwenheim-Skolem theorem we may assume $|\mathfrak{M}| = \omega_1$. It is clear that every member of $\mathfrak{M}$ is countable, since if x were an uncountable member of $\mathfrak{M}$ it could (by AC in $\mathfrak{M}$) be put into one-one correspondence with an ordinal of $\mathfrak{M}$ which would have to be uncountable, contradicting the assumption that $\omega_1 \notin \mathfrak{M}$. It follows that there are only countably many subsets of ω in $\mathfrak{M}$, and so by the usual forcing argument we can find a generic extension $\mathcal{N}$ of $\mathfrak{M}$ which is a standard model of ZFC + $V \neq L$.

Thus in either case we have the conclusion of (*), completing the proof.

Notice that an argument similar to that used in case (a) also proves the following:

ZFC + "There exists an (uncountable) model of ZFC containing a regular uncountable cardinal $\varkappa$ " $\vdash$ "There exists a standard model of ZFC + $V \neq L$ of cardinality $\varkappa$ ".

<u>Acknowledgments</u>. I am grateful to several participants at the conference, in particular P.Vopěnka and W.Guzicki, for their stimulating observations, I would also like to thank Jeff Paris for his valuable comments on an earlier draft of this paper, Kenneth Kunen

for his timely assistance in proving (*), and George Wilmers for providing general aid.

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