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Weak Compactness in Restricted Second-order Languages

by

J. L. BELL

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A *restricted second-order* language is obtained by taking a first-order language and adding a collection of one place predicate (set) variables which are then interpreted as ranging over sets of individuals of less than some prescribed power(s). Let L be such a language, and let us say that a cardinal κ is *weakly L-compact* if a set of sentences of L of power $\leqslant \kappa$ has a model whenever every subset of power $< \kappa$ has a model. In this paper we investigate the properties of weakly L-compact inaccessible cardinals: our first result (Theorem 1) is that, if L has set variables ranging over sets of individuals of power $< \aleph_1$, and if κ is an inaccessible $\leqslant$ the first measurable cardinal, then κ is weakly L-compact only if κ is weakly compact in the usual sense. We also obtain (in Theorem 2) a somewhat weaker conclusion when $\kappa \geqslant$ the first measurable cardinal.

DEFINITION 1. Let a be an ordinal. W_α is the restricted second-order language with set variables $V_0, V_1, \ldots$ ranging over sets of individuals of power $< \aleph_\alpha$. W_α^* is the restricted second-order language with set variables $V_0^{(\xi)}, V_1^{(\xi)}, \ldots$ for each $\xi < a$ ranging over sets of individuals of power $< \aleph_\xi$.

Evidently W_α^* is translatable into $L_{\omega_\alpha \omega_\alpha}$ and W_α is translatable into $L_{\omega_{\alpha+1} \omega_\alpha}$, where $L_{\kappa\lambda}$ is the infinitary language which allows conjunctions and disjunctions of sequences of formulas of length $< \kappa$, and quantifications of sequences of variables of length $< \lambda$.

DEFINITION 2. A cardinal κ is said to be *weakly* W_α- (W_α^*-) *compact* if, whenever Σ is a set of sentences of W_α (W_α^*) of power $\leqslant \kappa$ such that each subset of Σ of power $< \kappa$ has a model, then Σ itself has a model.

DEFINITION 3. Let $\mathcal{A}$ and $\mathcal{B}$ be structures. An elementary embedding f of $\mathcal{A}$ in $\mathcal{B}$ is said to be a W_α- (W_α^*-) *elementary* embedding if, for each formula Φ of W_α (W_α^*) *without* free second-order variables and each sequence $\tilde{a}$ of elements of A, we have $\mathcal{A} \models \Phi[\tilde{a}]$ if and only if $B \models \Phi[f(\tilde{a})]$.

DEFINITION 4. If ξ is an ordinal and κ is a cardinal the predicate $\mathrm{In}(\xi, \kappa)$ is defined by recursion as follows:

$$\mathrm{In}(0, \kappa) \Leftrightarrow \kappa \quad \text{is inaccessible},$$

$$\mathrm{In}(\xi+1, \kappa) \Leftrightarrow \mathrm{In}(\xi, \kappa) \quad \& \quad (\forall a < \kappa)(\exists \beta < \kappa)[a < \beta \quad \& \quad \mathrm{In}(\xi, \beta)]$$

for limit λ, $\mathrm{In}(\lambda, \kappa) \Leftrightarrow (\forall \xi < \lambda)\, \mathrm{In}(\xi, \kappa)$.

DEFINITION 5. μ_0 is the first (uncountable) measurable cardinal.

We shall use the following result, more or less implicit in Reinhardt [1], which was brought to our attention by George Wilmers.

RESULT 1. Let κ be a cardinal and f an elementary embedding of $\langle R_\kappa, \in \rangle$ into a transitive structure $\langle A, \in \rangle$. If f moves some ordinal, then the first such ordinal is a measurable cardinal ($> \omega$).

Our first theorem asserts that weak W_α-compactness for some $a \geq 1$ entails weak compactness, at least for inaccessible cardinals $\leq \mu_0$.

THEOREM 1. *If $\kappa \leq \mu_0$, κ is inaccessible and weakly W_α-compact for some $a \geq 1$, then κ is weakly compact.*

Proof. We prove that under these conditions κ is Π^1_1-indescribable; by Silver [2] it is sufficient to show that

(*) if $U \subseteq R_\kappa$, then $\langle R_\kappa, \in, U \rangle$ has an elementary extension $\langle A, \in, V \rangle$, where A is transitive and $\kappa \in A$.

Let $\{c_a : a \in R_\kappa\}$ be a set of individual constants to match the members of R_κ and let c be a constant distinct from all the c_a. Now let Σ be the union of the W_α theory of $\langle R_\kappa, \in, U, \{a : a \in R_\kappa\} \rangle$ with the set of sentences which say that c is an ordinal and it is bigger than c_ξ for each $\xi < \kappa$. Since κ is inaccessible, card $(R_\kappa) = \kappa$ so that card $(\Sigma) = \kappa$. Each subset of Σ of power $< \kappa$ clearly has a model, namely $\langle R_\kappa, \in, U \rangle$ with the constants interpreted appropriately, so, since κ is weakly W_α-compact, Σ itself has a model, $\mathfrak{B}$ say. Since the well-foundedness of $\in$ can be expressed by a sentence of W_α, and this sentence holds in $\langle R_\kappa, \in, U \rangle$, it follows that $\mathfrak{B}$ is well-founded. And since $\mathfrak{B}$ is also a model of the axiom of extensionality, there is a collapsing isomorphism g of $\mathfrak{B}$ onto a transitive structure $\langle A, \in, V \rangle$. If f is the composition of g with the elementary embedding of $\langle R_\kappa, \in, U \rangle$ into $\mathfrak{B}$ which exists by construction, then f is an elementary embedding of $\langle R_\kappa, \in, U \rangle$ into $\langle A, \in, V \rangle$. Since $\kappa \leq \mu_0$, Result 1 implies that f must leave all ordinals $< \kappa$ fixed so that f is the identity on R_κ. The denotation of c in $\langle A, \in, V \rangle$ is an ordinal with at least κ predecessors so, since A is transitive, $\kappa \in A$. This proves (*). The proof of the theorem is completed by observing that each Π^1_1-indescribable cardinal is weakly compact.

Clearly Theorem 1 holds when "W_α" is replaced by "W^*_α" and "$a \geq 1$" by "$a \geq 2$".

Our next result gives some information about the properties of weakly W^*_α-compact cardinals when $\kappa > \mu_0$.

THEOREM 2. *If $\kappa > \mu_0$, κ is inaccessible and weakly W_α^*-compact for some $\alpha \geq \kappa$, then* $\text{In}(\mu_0, \kappa)$.

The proof of this theorem breaks down into several lemmas.

LEMMA 1. *If A is transitive and $\xi, \kappa \in A$ then*

$$\text{In}(\xi, \kappa) \Rightarrow \langle A, \in \rangle \models \text{In}[\xi, \kappa].$$

Proof. Easy, by transfinite induction on ξ.

LEMMA 2. *Under the conditions of Theorem 2, there is a transitive set A such that $\kappa \in A$ and an elementary embedding $f: \langle R_\kappa, \in \rangle \to \langle A, \in \rangle$ such that $f``\,\kappa \subseteq \kappa$.*

Proof. The proof is similar to that of Theorem 1, only a trifle more delicate. Again let $\{c_a : a \in R_\kappa\}$ be a set of individual constants to match the members of R_κ, and let c be a constant distinct from all the c_a. Let Σ be the union of the W_κ^* theory of $\langle R_\kappa, \in, \{a : a \in R_\kappa\} \rangle$ with the set of sentences asserting that c is an ordinal and that it is bigger than c_ξ for each $\xi < \kappa$. Since κ is inaccessible, card $(R_\kappa) = \kappa$ and so card $(\Sigma) = \kappa$. As in the proof of Theorem 1, since κ is weakly W_α^*-compact for some $\alpha \geq \kappa$, and *a fortiori* weakly W_κ^*-compact, there is a model $\mathfrak{B}$ of Σ and a collapsing isomorphism g of $\mathfrak{B}$ onto a transitive structure $\langle A, \in \rangle = \mathfrak{A}$. Now $g(c^{(\mathfrak{B})})$ is an ordinal in $\mathfrak{A}$, and, for each $\xi < \kappa$, $c_\xi^{(\mathfrak{B})} <^{(\mathfrak{B})} c^{(\mathfrak{B})}$. Therefore $g(c_\xi^{(\mathfrak{B})}) < g(c^{(\mathfrak{B})})$ for each $\xi < \kappa$ so that $g(c^{(\mathfrak{B})})$ has at least κ predecessors. Hence $g(c^{(\mathfrak{B})}) \geq \kappa$; since A is transitive, $\kappa \in A$. It is clear from the construction of $\mathfrak{B}$ that the map $h: R_\kappa \to B$ defined by setting $h(a) = c_a^{(\mathfrak{B})}$ for each $a \in R_\kappa$ is a W_κ^*-elementary embedding of $\langle R_\kappa, \in \rangle$ into $\mathfrak{B}$, so that $f = g \circ h$ is a W_κ^*-elementary embedding of $\langle R_\kappa, \in \rangle$ into $\langle A, \in \rangle$. We finally show that $f``\,\kappa \subseteq \kappa$. If $\xi < \kappa$, then

$$\langle R_\kappa, \in \rangle \models \exists V_0^{(\gamma+1)} \forall u \left(u \in v \leftrightarrow V_0^{(\gamma+1)}(u)\right) [\xi],$$

where $\omega_\gamma = \text{card}(\xi) < \kappa$. Hence since f is a W_κ^*-elementary embedding,

$$\mathfrak{A} \models \exists V_0^{(\gamma+1)} \forall u \left(u \in v \leftrightarrow V_0^{(\gamma+1)}(u)\right) [f(\xi)].$$

Since A is transitive, it follows that card $\left(f(\xi)\right) \leq \omega_\gamma < \kappa$ so that $f(\xi) < \kappa$. This completes the proof of Lemma 2.

Proof of Theorem 2. By Lemma 2, under the conditions of the Theorem, there is a transitive set A such that $\kappa \in A$ and an elementary embedding, f, of $\langle R_\kappa, \in \rangle$ into $\langle A, \in \rangle$ such that $f``\,\kappa \subseteq \kappa$. We now show, by induction on ξ, that $\xi < \mu_0 \Rightarrow \text{In}(\xi, \kappa)$, from which we infer that $\text{In}(\mu_0, \kappa)$. We know that $\text{In}(0, \kappa)$ and the induction step for limit ξ is trivial. Suppose then that $\xi < \mu_0$ and $\text{In}(\xi, \kappa)$. By Lemma 1, $\langle A, \in \rangle \models \text{In}[\xi, \kappa]$ and, for each $\beta < \kappa$ we know that $f(\beta) < \kappa$.

Therefore, since $\kappa \in A$, we have

$$\langle A, \in \rangle \models \exists u \left(v_1 < u \quad \& \quad \text{In}(v_0, u)\right) [\xi, f(\beta)].$$

Since $\xi < \mu_0$, Result 1 implies that $f(\xi) = \xi$ and *a fortiori* $\xi \in f``\,R_\kappa$. Because $\langle f``\,R_\kappa, \in \rangle \prec \langle A, \in \rangle$ it follows that

$$\langle f``\,R_\kappa, \in \rangle \models \exists u \left(v_1 < u \quad \& \quad \text{In}(v_0, u)\right) [\xi, f(\beta)].$$

Hence, since $\langle f\text{``} R_\kappa, \in\rangle \cong \langle R_\kappa, \in\rangle$ we have

$$\langle R_\kappa, \in\rangle \models \exists u \left(v_1 < u \quad \& \quad \mathrm{In}\,(v_0, u)\right) [\xi, \beta].$$

But In is obviously absolute with respect to natural models, so we infer that

$$\exists \gamma < \kappa\, [\beta < \gamma \quad \& \quad \mathrm{In}\,(\xi, \gamma)]$$

which implies that In $(\xi+1, \kappa)$, completing the induction step and the proof.

Note. In view of the well-known fact that the first weakly compact inaccessible is smaller than the first measurable, it follows from Theorem 1 that the first weakly W_1-compact inaccessible cardinal coincides with the first weakly compact inaccessible. In fact it is clear from this theorem that the properties of weak compactness and weak W_1-compactness coincide for inaccessible cardinals $\leqslant$ the first measurable cardinal. Whether this result holds without restriction on the size of the cardinal is still an open question, as in the weaker problem of whether every weakly W_κ^*-compact inaccessible cardinal κ is weakly compact.

In conclusion, I would like to express my gratitude to George Wilmers for his stimulating observations on the contents of this paper and in particular for communicating Result 1 to me, and to Professor A. Mostowski, who very generously read a preliminary draft of this paper, and made several valuable suggestions.

DEPARTMENT OF MATHEMATICS, THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCES

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